

Pensieve header: Normally Ordered Exponentials at 0-Co in the t variables.

"NO" for "Normal Order".

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 $\mathbb{E} /: \mathbb{E}[Q1\_]\mathbb{E}[Q2\_]\ := \text{Simp@}\mathbb{E}[Q1 + Q2]$ 
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 $\text{Simp}[\omega\_.\mathbb{E}[Q\_]]\ := \text{ExpandDenominator}[\omega]\mathbb{E}[\omega\text{Expand@Cancel}[Q/\omega]];$   
 $\omega1\_.\mathbb{E}[Q1\_]\equiv\omega2\_.\mathbb{E}[Q2\_]\ := \text{Together}[\omega1 - \omega2] == \theta \wedge \text{Together}[Q1 - Q2] == \theta;$   
 $\text{brule} = \{\mathbf{b} \rightarrow \text{Log}[\mathbf{t}], \mathbf{b}_L \rightarrow \text{Log}[\mathbf{t}_L]\};$ 
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 $\text{NO}[\mathbf{u}_i, \mathbf{c}_j, \mathbf{k}_-][\omega\_.\mathbb{E}[Q\_]]\ := \text{Simp@Module}[\mathbf{u}_i \rightarrow \theta, \mathbf{c}_j \rightarrow \theta, \mathbf{k}_- \rightarrow \theta, \mathbf{Q} / . \mathbf{c}_j | \mathbf{u}_i \rightarrow \theta, \mathbf{v}],$   
 $\mathbf{v} = (1 + (\mathbf{t}_k - 1) \delta)^{-1};$   
 $\mathbf{v} \omega \mathbb{E}[(1 - \mathbf{t}_k) \mathbf{v} \alpha \beta + \mathbf{v} \beta \mathbf{u}_k + \mathbf{v} \delta \mathbf{u}_k \mathbf{w}_k + \mathbf{v} \alpha \mathbf{w}_k + \Theta];$   
 $\text{NO}[\mathbf{w}_i, \mathbf{c}_j, \mathbf{k}_-][\omega\_.\mathbb{E}[Q\_]]\ := \text{Simp@Module}[\mathbf{w}_i \rightarrow \theta, \mathbf{c}_j \rightarrow \theta, \mathbf{k}_- \rightarrow \theta, \mathbf{Q} / . \mathbf{c}_j | \mathbf{w}_i \rightarrow \theta, \mathbf{v}],$   
 $\mathbf{v} = (1 + (\mathbf{t}_k - 1) \delta)^{-1};$   
 $\mathbf{v} \omega \mathbb{E}[(1 - \mathbf{t}_k) \mathbf{v} \alpha \beta + \mathbf{v} \beta \mathbf{u}_k + \mathbf{v} \delta \mathbf{u}_k \mathbf{w}_k + \mathbf{v} \alpha \mathbf{w}_k + \Theta];$   
 $\text{NO}[\mathbf{w}_i, \mathbf{u}_j, \mathbf{k}_-][\omega\_.\mathbb{E}[Q\_]]\ := \text{Simp@Module}[\mathbf{w}_i \rightarrow \theta, \mathbf{u}_j \rightarrow \theta, \mathbf{k}_- \rightarrow \theta, \mathbf{Q} / . \mathbf{w}_i | \mathbf{u}_j \rightarrow \theta, \mathbf{v}],$   
 $\mathbf{v} = (1 + (\mathbf{t}_k - 1) \delta)^{-1};$   
 $\mathbf{v} \omega \mathbb{E}[(1 - \mathbf{t}_k) \mathbf{v} \alpha \beta + \mathbf{v} \beta \mathbf{u}_k + \mathbf{v} \delta \mathbf{u}_k \mathbf{w}_k + \mathbf{v} \alpha \mathbf{w}_k + \Theta];$ 
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 $\mathbf{m}[\mathbf{i}_-, \mathbf{j}_-, \mathbf{k}_-][Z\_]\ := \text{Simp@Module}[\mathbf{a}, \mathbf{Z} / . \{\mathbf{b}_i | \mathbf{j} \rightarrow \mathbf{b}_k, \mathbf{t}_i | \mathbf{j} \rightarrow \mathbf{t}_k\} // \text{NO}[\mathbf{w}_i, \mathbf{c}_j, \mathbf{a}] // \text{NO}[\mathbf{u}_i, \mathbf{c}_a, \mathbf{a}] // \text{NO}[\mathbf{w}_a, \mathbf{u}_j, \mathbf{a}]] / .$   
 $\{\mathbf{c}_i \rightarrow \mathbf{c}_k, \mathbf{w}_j \rightarrow \mathbf{w}_k, \mathbf{y}_{-a} \rightarrow \mathbf{y}_k\}$ 
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 $\sigma[\mathbf{i}_-, \mathbf{j}_-][\mathcal{E}_-]\ := \mathcal{E} / . \{\mathbf{b}_i \rightarrow \mathbf{b}_j, \mathbf{t}_i \rightarrow \mathbf{t}_j, \mathbf{c}_i \rightarrow \mathbf{c}_j, \mathbf{u}_i \rightarrow \mathbf{u}_j, \mathbf{w}_i \rightarrow \mathbf{w}_j\}$ 
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 $\text{Rp}[\mathbf{i}_-, \mathbf{j}_-]\ := \mathbb{E}[\mathbf{b}_i \mathbf{c}_j + \mathbf{u}_i \mathbf{w}_j];$   
 $\text{Rm}[\mathbf{i}_-, \mathbf{j}_-]\ := \mathbb{E}[-\mathbf{b}_i \mathbf{c}_j - \mathbf{t}_i^{-1} \mathbf{u}_i \mathbf{w}_j];$ 
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 $\text{NO}[\mathbf{u}_1, \mathbf{c}_1, 1][\mathbb{E}[\mathbf{b}_2 \mathbf{c}_1 + \mathbf{w}_2 \mathbf{u}_1 + \mathbf{u}_3 \mathbf{w}_4]]$ 
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 $\mathbb{E}[\mathbf{b}_2 \mathbf{c}_1 + \frac{\mathbf{u}_1 \mathbf{w}_2}{\mathbf{t}_2} + \mathbf{u}_3 \mathbf{w}_4]$ 
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 $\mathbf{Q0} = \mathbb{E}[\text{Sum}[\beta_i \mathbf{c}_i, \{\mathbf{i}, 3\}] + \text{Sum}[\alpha_{i,j} \mathbf{u}_i \mathbf{w}_j, \{\mathbf{i}, 3\}, \{\mathbf{j}, 3\}]]$ 
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 $\mathbb{E}[\mathbf{c}_1 \beta_1 + \mathbf{c}_2 \beta_2 + \mathbf{c}_3 \beta_3 + \mathbf{u}_1 \mathbf{w}_1 \alpha_{1,1} + \mathbf{u}_1 \mathbf{w}_2 \alpha_{1,2} + \mathbf{u}_1 \mathbf{w}_3 \alpha_{1,3} +$   
 $\mathbf{u}_2 \mathbf{w}_1 \alpha_{2,1} + \mathbf{u}_2 \mathbf{w}_2 \alpha_{2,2} + \mathbf{u}_2 \mathbf{w}_3 \alpha_{2,3} + \mathbf{u}_3 \mathbf{w}_1 \alpha_{3,1} + \mathbf{u}_3 \mathbf{w}_2 \alpha_{3,2} + \mathbf{u}_3 \mathbf{w}_3 \alpha_{3,3}]$ 
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**Simplify[t1 == t2]**

True

**Rp[1, 2] Rm[4, 3]**

$$\mathbb{E} \left[ b_1 c_2 - b_4 c_3 + u_1 w_2 - \frac{u_4 w_3}{t_4} \right]$$

**Rp[1, 2] Rm[4, 3] // m[1, 4, 1] // m[2, 3, 2]**

$\mathbb{E}[\emptyset]$

**Rm[1, 2] Rp[4, 3] // m[4, 1, 4] // m[2, 3, 2]**

$\mathbb{E}[\emptyset]$

**t1 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[3, 5, a] // m[1, 6, b] // m[2, 4, c]**

$$\mathbb{E} [ b_a c_b + b_a c_c + b_b c_c + u_a w_b + u_a w_c + u_b w_c ]$$

**t2 = Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 3, a] // m[2, 5, b] // m[4, 6, c]**

$$\mathbb{E} [ b_a c_b + b_a c_c + b_b c_c + u_a w_b + u_a w_c + u_b w_c ]$$

**t1 == t2**

True

**t3 = Rm[12, 1] Rm[2, 7] Rm[8, 3] Rm[4, 11] Rp[16, 5] Rp[6, 13] Rp[14, 9] Rp[10, 15]**

$$\mathbb{E} \left[ -b_{12} c_1 - b_8 c_3 + b_{16} c_5 - b_2 c_7 + b_{14} c_9 - b_4 c_{11} + b_6 c_{13} + b_{10} c_{15} - \frac{u_{12} w_1}{t_{12}} - \frac{u_8 w_3}{t_8} + u_{16} w_5 - \frac{u_2 w_7}{t_2} + u_{14} w_9 - \frac{u_4 w_{11}}{t_4} + u_6 w_{13} + u_{10} w_{15} \right]$$

**t3 // m[1, 2, 1]**

$$\mathbb{E} \left[ -b_{12} c_1 - b_8 c_3 + b_{16} c_5 - b_1 c_7 + b_{14} c_9 - b_4 c_{11} + b_6 c_{13} + b_{10} c_{15} - \frac{u_{12} w_1}{t_{12}} - \frac{u_8 w_3}{t_8} + u_{16} w_5 - \frac{u_1 w_7}{t_1} - \frac{u_{12} w_7}{t_{12}} + \frac{u_{12} w_7}{t_1 t_{12}} + u_{14} w_9 - \frac{u_4 w_{11}}{t_4} + u_6 w_{13} + u_{10} w_{15} \right]$$

**t3 // m[1, 2, 1] // m[1, 3, 1]**

$$\mathbb{E} \left[ -b_8 c_1 - b_{12} c_1 + b_{16} c_5 - b_1 c_7 + b_{14} c_9 - b_4 c_{11} + b_6 c_{13} + b_{10} c_{15} - \frac{u_8 w_1}{t_8} - \frac{u_{12} w_1}{t_8 t_{12}} + u_{16} w_5 - \frac{t_8 u_1 w_7}{t_1} - \frac{u_{12} w_7}{t_{12}} + \frac{u_{12} w_7}{t_1 t_{12}} + u_{14} w_9 - \frac{u_4 w_{11}}{t_4} + u_6 w_{13} + u_{10} w_{15} \right]$$

**t4 = t3;**

**Do[t4 = t4 // m[1, kk, 1], {kk, 2, 16}]; Simp@t4**

\$Aborted

**Simplify[t4 /. \_E -> 1]**

$$(t t_1^4) / (t^2 - 3 t (1 + 2 t) t_1 + (1 + 16 t + 15 t^2) t_1^2 - 3 (2 + 13 t + 7 t^2) t_1^3 + (16 + 53 t + 16 t^2) t_1^4 - 3 (7 + 13 t + 2 t^2) t_1^5 + (15 + 16 t + t^2) t_1^6 - 3 (2 + t) t_1^7 + t_1^8)$$

**Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3] // m[3, 6, 6] // m[6, 1, 1]**

$$\frac{1}{1 - t_1 + t_1^2} \mathbb{E} \left[ \frac{1}{1 - t_1 + t_1^2} \left( 3 b_1 c_1 - 3 b_1 c_1 t_1 + 3 b_1 c_1 t_1^2 + \frac{u_1 w_1}{t_1^2} + t_1 u_1 w_1 + t_1^{-2 - \frac{1}{1-t_1+t_1^2} + \frac{t_1}{1-t_1+t_1^2} - \frac{t_1^2}{1-t_1+t_1^2}} u_1 w_1 - t_1^{-1 - \frac{1}{1-t_1+t_1^2} + \frac{t_1}{1-t_1+t_1^2} - \frac{t_1^2}{1-t_1+t_1^2}} u_1 w_1 + t_1^{-\frac{1}{1-t_1+t_1^2} + \frac{t_1}{1-t_1+t_1^2} - \frac{t_1^2}{1-t_1+t_1^2}} u_1 w_1 \right) \right]$$

**Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3]**

$$\frac{1}{1 - t_3 + t_3^2} \mathbb{E} \left[ \frac{1}{1 - t_3 + t_3^2} \left( 2 b_3 c_3 + b_3 c_6 - 2 b_3 c_3 t_3 - b_3 c_6 t_3 + 2 b_3 c_3 t_3^2 + b_3 c_6 t_3^2 + \frac{u_3 w_3}{t_3^2} + t_3 u_3 w_3 + u_3 w_6 + \frac{u_3 w_6}{t_3^2} - \frac{u_3 w_6}{t_3} \right) \right]$$

**Rp[1, 2] Rp[3, 4] Rp[5, 6] // m[1, 4, 4] // m[4, 5, 5] // m[5, 2, 2] // m[2, 3, 3] // m[3, 6, 6]**

$$\frac{1}{1 - t_6 + t_6^2} \mathbb{E} \left[ \frac{1}{1 - t_6 + t_6^2} \left( 3 b_6 c_6 - 3 b_6 c_6 t_6 + 3 b_6 c_6 t_6^2 + \frac{u_6 w_6}{t_6^2} + t_6 u_6 w_6 + t_6^{-2 - \frac{1}{1-t_6+t_6^2} + \frac{t_6}{1-t_6+t_6^2} - \frac{t_6^2}{1-t_6+t_6^2}} u_6 w_6 - t_6^{-1 - \frac{1}{1-t_6+t_6^2} + \frac{t_6}{1-t_6+t_6^2} - \frac{t_6^2}{1-t_6+t_6^2}} u_6 w_6 + t_6^{-\frac{1}{1-t_6+t_6^2} + \frac{t_6}{1-t_6+t_6^2} - \frac{t_6^2}{1-t_6+t_6^2}} u_6 w_6 \right) \right]$$