Pensieve header: Normally Ordered Exponentials at 0-Co.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1606"];
```

"NO" for “Normal Order”.

```
Lg /: Lg[a_] + Lg[b_] := Lg[Simplify[a b]]
```

```
NO[u_i_, c_j_, k_][Q_] := Module[{α = ∂_{c_j} Q, β = ∂_{u_i} Q}, e^{-α} β u_k + (Q /. {u_i → 0, c_j → c_k})];
NO[w_i_, c_j_, k_][Q_] := Module[{α = ∂_{c_j} Q, β = ∂_{w_i} Q}, e^{α} β w_k + (Q /. {w_i → 0, c_j → c_k})];
NO[w_i_, u_j_, k_][Q_] := Module[{α = ∂_{w_i} Q /. u_j → 0, β = ∂_{u_j} Q /. w_i → 0, δ = ∂_{w_i,u_j} Q, ν},
    ν = (1 + b_k δ)^{-1};
    (Q /. w_i | u_j → 0) + (Lg[ν] - b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k /. {w_i → w_k, u_j → u_k})
  ];
```

```
NO[u1, c1, 1][b2 c1 + w2 u1 + u3 w4]
```

$$b_2\,c_1 + e^{-b_2}\,u_1\,w_2 + u_3\,w_4$$

```
NO[w2, u1, 0][b2 c1 + w2 u1 + u3 w4]
```

$$\text{Lg}\left[\frac{1}{1+b_0}\right] + b_2\,c_1 + \frac{u_0\,w_0}{1+b_0} + u_3\,w_4$$

```
m[i_, j_, k_][Q_] :=
 Module[{a}, (Q /. b_{i|j} → b_k // NO[w_i, c_j, a] // NO[u_i, c_a, a] // NO[w_a, u_j, a]) /.
    {c_i → c_k, w_j → w_k, y_a :> y_k}]
```

```
Q0 = Sum[β_i c_i, {i, 3}] + Sum[α_{i,j} u_i w_j, {i, 3}, {j, 3}]
```

$$c_1\,\beta_1 + c_2\,\beta_2 + c_3\,\beta_3 + u_1\,w_1\,\alpha_{1,1} + u_1\,w_2\,\alpha_{1,2} + u_1\,w_3\,\alpha_{1,3} + u_2\,w_1\,\alpha_{2,1} + u_2\,w_2\,\alpha_{2,2} + u_2\,w_3\,\alpha_{2,3} + u_3\,w_1\,\alpha_{3,1} + u_3\,w_2\,\alpha_{3,2} + u_3\,w_3\,\alpha_{3,3}$$

```
Q0 // m[1, 2, 1]
```

$$\text{Lg}\left[\frac{1}{1+e^{\beta_2}\,b_1\,\alpha_{2,1}}\right] + c_1\,\beta_1 + c_1\,\beta_2 + c_3\,\beta_3 + e^{-\beta_2}\,u_1\,(w_1\,\alpha_{1,2} + w_3\,\alpha_{1,3}) + \frac{e^{\beta_2}\,u_1\,w_1\,\alpha_{2,1}}{1+e^{\beta_2}\,b_1\,\alpha_{2,1}} + \frac{u_1\,(w_1\,\alpha_{2,2} + w_3\,\alpha_{2,3})}{1+e^{\beta_2}\,b_1\,\alpha_{2,1}} + \frac{w_1\,(u_1\,\alpha_{1,1} + e^{\beta_2}\,u_3\,\alpha_{3,1})}{1+e^{\beta_2}\,b_1\,\alpha_{2,1}} - \frac{b_1\,(w_1\,\alpha_{2,2} + w_3\,\alpha_{2,3})\,(u_1\,\alpha_{1,1} + e^{\beta_2}\,u_3\,\alpha_{3,1})}{1+e^{\beta_2}\,b_1\,\alpha_{2,1}} + u_3\,w_1\,\alpha_{3,2} + u_3\,w_3\,\alpha_{3,3}$$

```
Q0 // NO[w1, u2, 3]
```

$$\text{Lg}\left[\frac{1}{1+b_3\,\alpha_{2,1}}\right] + c_1\,\beta_1 + c_2\,\beta_2 + c_3\,\beta_3 + u_1\,w_2\,\alpha_{1,2} + u_1\,w_3\,\alpha_{1,3} + \frac{u_3\,w_3\,\alpha_{2,1}}{1+b_3\,\alpha_{2,1}} + \frac{u_3\,(w_2\,\alpha_{2,2} + w_3\,\alpha_{2,3})}{1+b_3\,\alpha_{2,1}} + \frac{w_3\,(u_1\,\alpha_{1,1} + u_3\,\alpha_{3,1})}{1+b_3\,\alpha_{2,1}} - \frac{b_3\,(w_2\,\alpha_{2,2} + w_3\,\alpha_{2,3})\,(u_1\,\alpha_{1,1} + u_3\,\alpha_{3,1})}{1+b_3\,\alpha_{2,1}} + u_3\,w_2\,\alpha_{3,2} + u_3\,w_3\,\alpha_{3,3}$$

```
t1 = Q0 // m[1, 2, 1] // m[1, 3, 1] // Simplify
```

$$\left(1 \Big/ \left(1 - e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) + b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right)\right)$$
$$e^{-\beta_2-\beta_3}\, \left(e^{\beta_2+\beta_3}\, \text{Lg}\left[1 \Big/ \left(1 + e^{\beta_3}\, b_1\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right) + e^{\beta_2}\, b_1\, \left(-e^{\beta_3}\, b_1\,\alpha_{2,2}\,\alpha_{3,1} + \alpha_{2,1}\, \left(1 + e^{\beta_3}\, b_1\,\alpha_{3,2}\right)\right)\right)\right]\right.$$
$$\left(1 - e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) + b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right) +$$
$$e^{\beta_2+\beta_3}\, c_1\, (\beta_1 + \beta_2 + \beta_3)\, \left(1 - e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) + b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right) +$$
$$u_1\, w_1\, \left(\alpha_{1,3} + e^{2\,\beta_2+\beta_3}\,\alpha_{2,1} + e^{\beta_2}\, b_1\,\alpha_{1,3}\,\alpha_{2,1} + e^{\beta_2+\beta_3}\,\alpha_{2,2} + e^{\beta_2}\,\alpha_{2,3} + e^{2\,(\beta_2+\beta_3)}\,\alpha_{3,1} + e^{\beta_2+\beta_3}\, b_1\,\alpha_{1,3}\right.$$
$$\alpha_{3,1} - e^{2\,(\beta_2+\beta_3)}\, b_1\,\alpha_{2,2}\,\alpha_{3,1} - e^{\beta_2+\beta_3}\, b_1^2\,\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1} + e^{\beta_2+2\,\beta_3}\,\alpha_{3,2} + e^{\beta_3}\, b_1\,\alpha_{1,3}\,\alpha_{3,2} +$$
$$e^{2\,(\beta_2+\beta_3)}\, b_1\,\alpha_{2,1}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\, b_1^2\,\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\, b_1\,\alpha_{2,3}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\,\alpha_{3,3} -$$
$$e^{\beta_2+\beta_3}\, b_1\,\alpha_{2,2}\,\alpha_{3,3} - e^{\beta_3}\,\alpha_{1,2}\, \left(-1 + b_1\, \left(-e^{\beta_2}\,\alpha_{2,1} + \alpha_{3,3}\right) - e^{\beta_2}\, b_1^2\, (\alpha_{2,3}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,3})\right) +$$
$$\left.\left.e^{\beta_2}\,\alpha_{1,1}\, \left(e^{\beta_3} - b_1\, \left(e^{\beta_3}\,\alpha_{2,2} + \alpha_{2,3} + e^{\beta_3}\,\alpha_{3,3}\right) - e^{\beta_3}\, b_1^2\, (\alpha_{2,3}\,\alpha_{3,2} - \alpha_{2,2}\,\alpha_{3,3})\right)\right)\right)$$

```
t2 = Q0 // m[2, 3, 2] // m[1, 2, 1] // Simplify
```

$$\left(1 \Big/ \left(-1 + e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) - b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right)\right)$$
$$e^{-\beta_2-\beta_3}\, \left(-e^{\beta_2+\beta_3}\, \text{Lg}\left[1 \Big/ \left(1 + e^{\beta_3}\, b_1\,\alpha_{3,2} + e^{\beta_2}\, b_1\, \left(-e^{\beta_3}\, (-1 + b_1\,\alpha_{2,2})\,\alpha_{3,1} + \alpha_{2,1}\, \left(1 + e^{\beta_3}\, b_1\,\alpha_{3,2}\right)\right)\right)\right]\right.$$
$$\left(1 - e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) + b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right) -$$
$$e^{\beta_2+\beta_3}\, c_1\, (\beta_1 + \beta_2 + \beta_3)\, \left(1 - e^{\beta_2+\beta_3}\, b_1^2\, (\alpha_{2,2}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,2}) + b_1\, \left(e^{\beta_2}\,\alpha_{2,1} + e^{\beta_3}\, \left(e^{\beta_2}\,\alpha_{3,1} + \alpha_{3,2}\right)\right)\right) -$$
$$u_1\, w_1\, \left(\alpha_{1,3} + e^{2\,\beta_2+\beta_3}\,\alpha_{2,1} + e^{\beta_2}\, b_1\,\alpha_{1,3}\,\alpha_{2,1} + e^{\beta_2+\beta_3}\,\alpha_{2,2} + e^{\beta_2}\,\alpha_{2,3} + e^{2\,(\beta_2+\beta_3)}\,\alpha_{3,1} + e^{\beta_2+\beta_3}\, b_1\,\alpha_{1,3}\right.$$
$$\alpha_{3,1} - e^{2\,(\beta_2+\beta_3)}\, b_1\,\alpha_{2,2}\,\alpha_{3,1} - e^{\beta_2+\beta_3}\, b_1^2\,\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1} + e^{\beta_2+2\,\beta_3}\,\alpha_{3,2} + e^{\beta_3}\, b_1\,\alpha_{1,3}\,\alpha_{3,2} +$$
$$e^{2\,(\beta_2+\beta_3)}\, b_1\,\alpha_{2,1}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\, b_1^2\,\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\, b_1\,\alpha_{2,3}\,\alpha_{3,2} + e^{\beta_2+\beta_3}\,\alpha_{3,3} -$$
$$e^{\beta_2+\beta_3}\, b_1\,\alpha_{2,2}\,\alpha_{3,3} - e^{\beta_3}\,\alpha_{1,2}\, \left(-1 + b_1\, \left(-e^{\beta_2}\,\alpha_{2,1} + \alpha_{3,3}\right) - e^{\beta_2}\, b_1^2\, (\alpha_{2,3}\,\alpha_{3,1} - \alpha_{2,1}\,\alpha_{3,3})\right) +$$
$$\left.\left.e^{\beta_2}\,\alpha_{1,1}\, \left(e^{\beta_3} - b_1\, \left(e^{\beta_3}\,\alpha_{2,2} + \alpha_{2,3} + e^{\beta_3}\,\alpha_{3,3}\right) - e^{\beta_3}\, b_1^2\, (\alpha_{2,3}\,\alpha_{3,2} - \alpha_{2,2}\,\alpha_{3,3})\right)\right)\right)$$

```
Simplify[t1 - t2]
```

0

```
Table[α_{i,j} → (Coefficient[t1 - t2, α_{i,j}] /. α__ → 0) // Simplify, {i, 3}, {j, 3}]
```

$$\{\{\alpha_{1,1} \to 0,\ \alpha_{1,2} \to 0,\ \alpha_{1,3} \to 0\},\ \{\alpha_{2,1} \to 0,\ \alpha_{2,2} \to 0,\ \alpha_{2,3} \to 0\},\ \{\alpha_{3,1} \to 0,\ \alpha_{3,2} \to 0,\ \alpha_{3,3} \to 0\}\}$$