

Pensieve header: One-Co computations in the abc presentation; continues pensieve://2015-11/.

The bracket

On the elements β , a , c , δa , ca , δaa .

Generalities

```

DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}]; (* tests for non-strict ordering;
also true when {is} is {i,i}. *)

Simp[expr_] := Simplify[expr];
S[ $\beta$ [f_]] :=  $\beta$ [Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[ $\delta a$ [f_, i_, j_]] :=  $\delta a$ [Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[ $\delta aa$ [f_, i_, j_, k_, l_]] :=  $\delta aa$ [Simp[f], i, j, k, l];
S[expr_] := expr /.
  ( $\lambda_\beta$  |  $\lambda_a$  |  $\lambda_{\delta a}$  |  $\lambda_c$  |  $\lambda_{ca}$  |  $\lambda_{\delta aa}$  |  $\lambda_{\beta 1}$  |  $\lambda_{a 1}$  |  $\lambda_{\delta a 1}$  |  $\lambda_{c 1}$  |  $\lambda_{ca 1}$  |  $\lambda_{\delta aa 1}$ )  $\rightarrow$  S[ $\lambda$ ];

 $\beta$ [0] := 0;
 $\beta$  /:  $\beta$ [f_] +  $\beta$ [g_] :=  $\beta$ [f+g] // S;
 $\beta$  /: g_* $\beta$ [f_] :=  $\beta$ [gf] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f+g, j, k] // S;
a /: g_*a[f_, j_, k_] := a[gf, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f+g, j] // S;
c /: g_*c[f_, j_] := c[gf, j] // S;
 $\delta a$ [0, _, _] := 0;
 $\delta a$  /:  $\delta a$ [f_, j_, k_] +  $\delta a$ [g_, j_, k_] :=  $\delta a$ [f+g, j, k] // S;
 $\delta a$  /: g_* $\delta a$ [f_, j_, k_] :=  $\delta a$ [gf, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f+g, j, k, l] // S;
ca /: g_*ca[f_, j_, k_, l_] := ca[gf, j, k, l] // S;
 $\delta aa$ [0, _, _, _, _] := 0;
 $\delta aa$  /:  $\delta aa$ [f_, i_, j_, k_, l_] +  $\delta aa$ [g_, i_, j_, k_, l_] :=
   $\delta aa$ [f+g, i, j, k, l] // S;
 $\delta aa$  /: g_* $\delta aa$ [f_, i_, j_, k_, l_] :=  $\delta aa$ [gf, i, j, k, l] // S;

```

```

β1[0] := 0;
β1 /: β1[f_] + β1[g_] := β1[f+g] // S;
β1 /: g_ * β1[f_] := β1[g f] // S;
a1[0, _, _] := 0;
a1 /: a1[f_, j_, k_] + a1[g_, j_, k_] := a1[f+g, j, k] // S;
a1 /: g_ * a1[f_, j_, k_] := a1[g f, j, k] // S;
c1[0, _] := 0;
c1 /: c1[f_, j_] + c1[g_, j_] := c1[f+g, j] // S;
c1 /: g_ * c1[f_, j_] := c1[g f, j] // S;
δa1[0, _, _] := 0;
δa1 /: δa1[f_, j_, k_] + δa1[g_, j_, k_] := δa1[f+g, j, k] // S;
δa1 /: g_ * δa1[f_, j_, k_] := δa1[g f, j, k] // S;
ca1[0, _, _, _] := 0;
ca1 /: ca1[f_, j_, k_, l_] + ca1[g_, j_, k_, l_] := ca1[f+g, j, k, l] // S;
ca1 /: g_ * ca1[f_, j_, k_, l_] := ca1[g f, j, k, l] // S;
δaa1[0, _, _, _, _] := 0;
δaa1 /: δaa1[f_, i_, j_, k_, l_] + δaa1[g_, i_, j_, k_, l_] :=
  δaa1[f+g, i, j, k, l] // S;
δaa1 /: g_ * δaa1[f_, i_, j_, k_, l_] := δaa1[g f, i, j, k, l] // S;

```

δaa relations

VS = False;

"First sort tails then sort heads"

Standard Swinging - sorts heads, if support is 4 strands:

```

S[δaa[f_, i_, j_, k_, l_]] /: DQ[i, j, k, l] ∧ OQ[i, k] ∧ !OQ[j, l] := (
  If[VS, Print["Standard swinging on ", δaa[f, i, j, k, l]]];
  S[δaa[f, i, l, k, j] + ca[bk f, l, i, j] -
    ca[bi f, l, k, j] - ca[bk f, j, i, l] + ca[bi f, j, k, l]]
);

```

Locality - sorts tails when supports are disjoint:

```

S[δaa[f_, i_, j_, k_, l_]] /: ({i, j} ∩ {k, l} == {}) ∧ !OQ[i, k] := (
  If[VS, Print["Locality on ", δaa[f, i, j, k, l]]];
  δaa[f, k, l, i, j] // S
);

```

Commute Heads - sorts tails when the heads are the same:

```

S[δaa[f_, i_, k_, j_, k_]] /: DQ[i, j, k] ∧ !OQ[i, j] := (
  If[VS, Print["Commute heads on ", δaa[f, i, k, j, k]]];
  S[δaa[f, j, k, i, k] + δa[-bi f, j, k] + δa[bj f, i, k]]
);

```

Commute Head/Tail - sorts tails:

```
S[ $\delta_{aa}[f_, i_, j_, k_, i_] /; DQ[i, j, k] \wedge !OQ[i, k] := ($ 
  If[VS, Print["Commute head/tail on ",  $\delta_{aa}[f, i, j, k, i]$ ]];
  S[
     $\delta_{aa}[f, k, i, i, j] + \delta_{aa}[f, k, j, i, j] - \delta_{aa}[f, i, j, k, j]$ 
  ]
);
```

Commute Head/Tail - sorts heads where heads & tails are both broken:

```
S[ $\delta_{aa}[f_, k_, j_, j_, i_] /; DQ[i, j, k] \wedge OQ[i, j, k] := ($ 
  If[VS, Print["Commute head/tail on ",  $\delta_{aa}[f, k, j, j, i]$ ]];
  S[
     $\delta_{aa}[f, j, i, k, j] + \delta_{aa}[f, j, i, k, i] - \delta_{aa}[f, k, i, j, i]$ 
  ]
);
```

2113 Swinging - sorts tails:

```
 $\delta_{aa}[f, j, i, ii, k] // S$ 
Locality on  $\delta_{aa}[f, j, i, ii, k]$ 
Standard swinging on  $\delta_{aa}[f, ii, k, j, i]$ 
 $ca[-fb_{ii}, i, j, k] + ca[fb_{ii}, k, j, i] +$ 
 $ca[-fb_j, k, ii, i] + ca[fb_j, i, ii, k] + \delta_{aa}[f, ii, i, j, k]$ 

S[ $\delta_{aa}[f_, j_, i_, i_, k_] /; DQ[i, j, k] \wedge OQ[i, j, k] := ($ 
  If[VS, Print["2113 swinging on ",  $\delta_{aa}[f, j, i, i, k]$ ]];
  S[ $ca[-fb_i, i, j, k] + ca[fb_i, k, j, i] +$ 
 $c[-fb_j, k] ** aop[1, i] + ca[fb_j, i, i, k] + aop[f, i] ** \delta_a[1, j, k]$ 
  ]
);
```

```
 $\delta_{aa}[f, j, i, i, k] // S$ 
 $c[fb_i b_j, k] + ca[-fb_i, i, j, k] + ca[fb_i, k, j, i] +$ 
 $ca[-fb_j, k, i, i] + ca[fb_j, i, i, k] + \delta_a[-fb_i, j, k] + \delta_{aa}[f, i, i, j, k]$ 
```

3112 Swinging - sorts tails:

```
 $\delta_{aa}[f, k, i, ii, j] // S$ 
Locality on  $\delta_{aa}[f, k, i, ii, j]$ 
Standard swinging on  $\delta_{aa}[f, ii, j, k, i]$ 
 $ca[-fb_{ii}, i, k, j] + ca[fb_{ii}, j, k, i] +$ 
 $ca[-fb_k, j, ii, i] + ca[fb_k, i, ii, j] + \delta_{aa}[f, ii, i, k, j]$ 
```

```
S[ $\delta_{aa}[f_, k_, i_, i_, j_]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["3112 swinging on ",  $\delta_{aa}[f, k, i, i, j]$ ]];
  S[ca[-fbi, i, k, j] + ca[fbi, j, k, i] +
    c[-fbk, j] ** aop[1, i] + ca[fbk, i, i, j] + aop[1, i] **  $\delta_a[f, k, j]$ ]
];
```

$\delta_{aa}[f, k, i, i, j]$ // S

```
c[fbi bk, j] + ca[-fbi, i, k, j] + ca[fbi, j, k, i] +
ca[-fbk, j, i, i] + ca[fbk, i, i, j] +  $\delta_a[-fb_i, k, j]$  +  $\delta_{aa}[f, i, i, k, j]$ 
```

Tails Commute - sorts heads when the tails are the same:

```
S[ $\delta_{aa}[f_, i_, j_, i_, l_]$ ] /; DQ[i, j, l]  $\wedge$  !OQ[j, l] := (
  If[VS, Print["Tails commute on ",  $\delta_{aa}[f, i, j, i, l]$ ]];
   $\delta_{aa}[f, i, l, i, j]$  // S
);
```

1321 Swinging - sorts heads:

$\delta_{aa}[f, i, k, j, ii]$ // S

Standard swinging on $\delta_{aa}[f, i, k, j, ii]$

```
ca[-fbi, ii, j, k] + ca[fbi, k, j, ii] +
ca[-fbj, k, i, ii] + ca[fbj, ii, i, k] +  $\delta_{aa}[f, i, ii, j, k]$ 
```

```
S[ $\delta_{aa}[f_, i_, k_, j_, i_]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["1321 swinging on ",  $\delta_{aa}[f, i, k, j, i]$ ]];
  S[ca[-fbi, i, j, k] + ca[fbi, k, j, i] +
    ca[-fbj, k, i, i] + ac[fbj, i, k, i] +  $\delta_{aa}[f, i, i, j, k]$ ]
);
```

1322 Swinging - sorts heads, but breaks tails:

$\delta_{aa}[f, i, k, j, jj]$ // S

Standard swinging on $\delta_{aa}[f, i, k, j, jj]$

```
ca[-fbi, jj, j, k] + ca[fbi, k, j, jj] +
ca[-fbj, k, i, jj] + ca[fbj, jj, i, k] +  $\delta_{aa}[f, i, jj, j, k]$ 
```

```
S[ $\delta_{aa}[f_, i_, k_, j_, j_]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["1322 swinging on ",  $\delta_{aa}[f, i, k, j, j]$ ]];
  S[ac[-fbi, j, k, j] + ca[fbi, k, j, j] +
    ca[-fbj, k, i, j] + ca[fbj, j, i, k] +  $\delta_{aa}[f, j, k, i, j]$ ]
);
```

1332 Swinging - sorts heads:

$\delta_{aa}[f, i, k, kk, j]$ // S

Standard swinging on $\delta_{aa}[f, i, k, kk, j]$

$ca[-fb_i, j, kk, k] + ca[fb_i, k, kk, j] +$
 $ca[-fb_{kk}, k, i, j] + ca[fb_{kk}, j, i, k] + \delta_{aa}[f, i, j, kk, k]$

$S[\delta_{aa}[f, i, k, k, j]]$ /; $DQ[i, j, k] \wedge OQ[i, j, k] :=$ (
 If[VS, Print["1332 swinging on ", $\delta_{aa}[f, i, k, k, j]$];
 $S[c[-fb_i, j] ** aop[1, k] + ca[fb_i, k, k, j] +$
 $ca[-fb_k, k, i, j] + ca[fb_k, j, i, k] + \delta_a[f, i, j] ** aop[1, k]]$
);

$\delta_{aa}[f, i, k, k, j]$ // S

$c[fb_i b_k, j] + ca[-fb_i, j, k, k] + ca[fb_i, k, k, j] +$
 $ca[-fb_k, k, i, j] + ca[fb_k, j, i, k] + \delta_a[-fb_k, i, j] + \delta_{aa}[f, i, j, k, k]$

1231 Swinging - sorts heads:

$\delta_{aa}[f, i, j, k, ii]$ // S

Standard swinging on $\delta_{aa}[f, i, j, k, ii]$

$ca[-fb_i, ii, k, j] + ca[fb_i, j, k, ii] +$
 $ca[-fb_k, j, i, ii] + ca[fb_k, ii, i, j] + \delta_{aa}[f, i, ii, k, j]$

$S[\delta_{aa}[f, i, j, k, i]]$ /; $DQ[i, j, k] \wedge OQ[i, j, k] :=$ (
 If[VS, Print["1231 swinging on ", $\delta_{aa}[f, i, j, k, i]$];
 $S[ac[-fb_i, k, j, i] + ca[fb_i, j, k, i] +$
 $ca[-fb_k, j, i, i] + ac[fb_k, i, j, i] + \delta_{aa}[f, i, i, k, j]]$
);

1211 sliding - sorts heads:

$S[\delta_{aa}[f, i, j, i, i]]$ /; $DQ[i, j] \wedge OQ[i, j] :=$ (
 If[VS, Print["1211 sliding on ", $\delta_{aa}[f, i, j, i, i]$];
 $S[\delta_{aa}[f, i, i, i, j]]$
);

2111 sliding - sorts tails:

$S[\delta_{aa}[f, j, i, i, i]]$ /; $DQ[i, j] \wedge OQ[i, j] :=$ (
 If[VS, Print["2111 sliding on ", $\delta_{aa}[f, j, i, i, i]$];
 $S[\delta_{aa}[f, i, i, j, i]]$
);

2212 sliding - sorts tails:

$S[\delta_{aa}[f, j, j, i, j]]$ /; $DQ[i, j] \wedge OQ[i, j] :=$ (
 If[VS, Print["2212 sliding on ", $\delta_{aa}[f, j, j, i, j]$];
 $S[\delta_{aa}[f, i, j, j, j]]$
);

2221 sliding - sorts heads:

```
S[ $\delta_{aa}[f_, j_, j_, j_, i_]$ ] /; DQ[i, j]  $\wedge$  OQ[i, j] := (
  If[VS, Print["2221 sliding on ",  $\delta_{aa}[f, j, j, j, i]$ ]];
  S[ $\delta_{aa}[f, j, i, j, j]$ ]
);
```

2231 Swinging - sorts heads:

```
 $\delta_{aa}[f, j, jj, k, i]$  // S
Standard swinging on  $\delta_{aa}[f, j, jj, k, i]$ 
ca[-fbj, i, k, jj] + ca[fbj, jj, k, i] +
ca[-fbk, jj, j, i] + ca[fbk, i, j, jj] +  $\delta_{aa}[f, j, i, k, jj]$ 

S[ $\delta_{aa}[f_, j_, j_, k_, i_]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["2231 swinging on ",  $\delta_{aa}[f, j, j, k, i]$ ]];
  S[ca[-fbj, i, k, j] + ca[fbj, j, k, i] +
    ac[-fbk, j, i, j] + ca[fbk, i, j, j] +  $\delta_{aa}[f, j, i, k, j]$ ]
);
```

2331 Swinging - sorts heads:

```
 $\delta_{aa}[f, j, k, kk, i]$  // S
ca[-fbj, i, kk, k] + ca[fbj, k, kk, i] +
ca[-fbkk, k, j, i] + ca[fbkk, i, j, k] +  $\delta_{aa}[f, j, i, kk, k]$ 

S[ $\delta_{aa}[f_, j_, k_, k_, i_]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["2331 swinging on ",  $\delta_{aa}[f, j, k, k, i]$ ]];
  S[c[-fbj, i] ** aop[1, k] + ca[fbj, k, k, i] +
    ca[-fbk, k, j, i] + ca[fbk, i, j, k] +  $\delta_a[f, j, i]$  ** aop[1, k]]
);
```

Backie jkkj Swinging - sorts tails or heads:

```
 $\delta_{aa}[f, j, k, kk, jj]$  // S
ca[-fbj, jj, kk, k] + ca[fbj, k, kk, jj] +
ca[-fbkk, k, j, jj] + ca[fbkk, jj, j, k] +  $\delta_{aa}[f, j, jj, kk, k]$ 

S[ $\delta_{aa}[f_, j_, k_, k_, j_]$ ] /; DQ[j, k] := (
  If[VS, Print["Backie swinging on ",  $\delta_{aa}[f, j, k, k, j]$ ]];
  S[c[-fbj, j] ** aop[1, k] + ca[fbj, k, k, j] +
    ca[-fbk, k, j, j] + ac[fbk, j, k, j] +  $\delta_a[f, j, j]$  ** aop[1, k]]
);
```

NonCommutativeMultiply

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_ , 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;

 $\beta[f_] ** a[g_, j_, k_] := a[fg, j, k];$ 
 $\beta[f_] ** c[g_, j_] := c[fg, j];$ 
 $c[g_, j_] ** \beta[f_] := c[fg, j];$ 
 $\beta[f_] ** \delta a[g_, j_, k_] := \delta a[fg, j, k];$ 
 $\beta[f_] ** ca[g_, i_, j_, k_] := ca[fg, i, j, k];$ 
 $ca[g_, i_, j_, k_] ** \beta[f_] := ca[fg, i, j, k];$ 
 $\beta[f_] ** \delta aa[g_, i_, j_, k_, l_] := \delta aa[fg, i, j, k, l];$ 
 $\delta a[g_, j_, k_] ** \beta[f_] := \delta a[fg, j, k];$ 
 $\delta ** a[f_, i_, j_] := \delta a[f, i, j];$ 
 $c[f_, i_] ** a[g_, j_, k_] := ca[fg, i, j, k];$ 
 $a[f_, i_, j_] ** \delta a[g_, k_, l_] := \delta aa[fg, i, j, k, l];$ 
 $\delta a[f_, i_, j_] ** a[g_, k_, l_] := \delta aa[fg, i, j, k, l];$ 

 $\delta ** _c = 0;$ 
 $\delta ** _\delta a = 0;$ 
 $\delta ** _ca = 0;$ 
 $\delta ** _\delta aa = 0;$ 
 $_c ** _c = 0;$ 
 $_c ** _\delta a = _\delta a ** _c = 0;$ 
 $_c ** _ca = _ca ** _c = 0;$ 
 $_c ** _\delta aa = _\delta aa ** _c = 0;$ 
 $_\delta a ** _\delta a = 0;$ 
 $_\delta a ** _\delta aa = _\delta aa ** _\delta a = 0;$ 
 $_\delta a ** _ca = _ca ** _\delta a = 0;$ 

NonCommutativeMultiply::ndef =
  "NonCommutativeMultiply is not defined on {`1`, `2`}."
NonCommutativeMultiply[x_, y_] :=
  (Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
NonCommutativeMultiply is not defined on {`1`, `2`}.
```

Bracket Generalities

```

B[0, _] = 0; B[_ , 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[# , y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;

```

The γ shortcuts

```

 $\gamma[f_, j_, k_] := \delta a[f, j, k] - c[b_j f, k] // S;$ 
 $\gamma[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;$ 
 $ac[f_, j_, k_, l_] := ca[f, l, j, k] + B[a[1, j, k], c[f, l]];$ 
 $aop[f_, j_] := a[f, j, j] + \beta[-f b_j] + c[-f, j];$ 

```

Fundamental Brackets

a- β , a-c, a-a, AS

```

B[a[j_, k_],  $\beta$ [g_]] :=  $\gamma[\partial_{b_j} g - \partial_{b_k} g, j, k];$ 
B[ $\beta$ [g_], a[j_, k_]] := -B[a[j, k],  $\beta$ [g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k}  $\cap$  {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] :=  $\gamma[1, j, k, l] // S;$ 
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] -  $\gamma[1, j, k, l] // S;$ 
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
(* backie *) B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[b_j, k, j] - a[b_k, j, k] + a[b_j, k, k] - a[b_k, j, j] + ca[1, k, k, j] -
  ca[1, j, j, k] + ca[1, k, j, j] - ca[1, j, k, k] +  $\gamma[1, j, k] - \gamma[1, k, j];$ 
(* [tail, selfie] *) B[a[j_, k_], a[j_, j_]] /; DQ[j, k] :=  $\gamma[1, j, k] // S;$ 
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
(* [head, selfie] *) B[a[j_, k_], a[k_, k_]] /; DQ[j, k] :=  $\gamma[-1, j, k] // S;$ 
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[k, k]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] :=  $\gamma[-f g, j, k];$ 
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] :=  $\gamma[f g, j, k];$ 
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k}  $\cap$  {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];

```

Vanishing brackets

```

B[_ $\beta$ , _ $\beta$  |  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa] = 0;
B[_ $\beta$  |  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa, _ $\beta$ ] = 0;
B[ $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa,  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa] = 0;

```


Composite Brackets

```

B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[l, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[l, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[l, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[l, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];

B::ndef = "B is not defined on {\`1`,\`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
B is not defined on {\`1`,\`2`}.

```

The MM Presentation

U2MM

On the elements β , a , c , δa , ca , δaa .

```

MM2U::ndef = "MM2U is not defined on `1`.";
U2MM[expr_] := expr /. {
  β[f_] => β1[f],
  a[f_, j_, k_] /; DQ[j, k] => a1[f, j, k],
  a[f_, j_, j_] => a1[f, j, j] + δa1[-∂bjf, j, j] + c1[f + bj ∂bjf, j],
  c[f_, j_] => c1[f, j],
  δa[f_, j_, k_] => δa1[f, j, k],
  ca[f_, i_, j_, k_] /; DQ[i, j] => ca1[f, i, j, k],
  ca[f_, j_, j_, k_] => ca1[f, j, j, k] + δa1[f, j, k] + c1[-bj f, k],
  δaa[f_, i_, j_, k_, l_] /; DQ[j, k] => δaa1[f, i, j, k, l],
  δaa[f_, i_, j_, j_, k_] =>
    δaa1[f, i, j, j, k] + δaa1[f, j, k, i, k] - δaa1[f, i, k, j, k],
  δaa[other___] => (Message[MM2U::ndef, δaa[other]]; Undefined)
}

```

MM2U

```

MM2U::ndef = "MM2U is not defined on `1`.";
MM2U[expr_] := expr /. {
   $\beta_1[f_] \Rightarrow \beta[f]$ ,
   $a_1[f_, j_, k_] /; DQ[j, k] \Rightarrow a[f, j, k]$ ,
   $a_1[f_, j_, j_] \Rightarrow a[f, j, j] + \delta a[\partial_b, f, j, j] + c[-f - b_j \partial_b, f, j]$ ,
   $c_1[f_, j_] \Rightarrow c[f, j]$ ,
   $\delta a_1[f_, j_, k_] \Rightarrow \delta a[f, j, k]$ ,
   $ca_1[f_, i_, j_, k_] /; DQ[i, j] \Rightarrow ca[f, i, j, k]$ ,
   $ca_1[f_, j_, j_, k_] \Rightarrow ca[f, j, j, k] + \delta a[-f, j, k] + c[b_j f, k]$ ,
   $\delta a a_1[f_, i_, j_, k_, l_] /; DQ[j, k] \Rightarrow \delta a a[f, i, j, k, l]$ ,
   $\delta a a_1[f_, i_, i_, i_, j_] \Rightarrow \delta a a[f, i, j, i, i]$ ,
   $\delta a a_1[f_, i_, j_, j_, k_] /; DQ[i, j, k] \Rightarrow$ 
     $\delta a a[f, i, j, j, k] - \delta a a[f, j, k, i, k] + \delta a a[f, i, k, j, k]$ ,
   $\delta a a_1[f_, i_, j_, j_, j_] /; DQ[i, j] \Rightarrow$ 
     $\delta a a[f, j, j, i, j] + \delta a[b_j f, i, j] - \delta a[b_i f, j, j]$ ,
   $\delta a a_1[f_, i_, j_, j_, i_] /; DQ[i, j] \Rightarrow \delta a a[f, i, j, j, i] +$ 
     $\delta a a[f, i, i, j, i] - \delta a a[f, i, i, j, i] - \delta a[b_i f, j, i] + \delta a[b_j f, i, i]$ ,
   $\delta a a_1[other\_ ] \Rightarrow (Message[MM2U::ndef, \delta a a_1[other]]); Undefined$ 
}

```

S for MM

```

VS1 = False;
S[β1[f_]] := β1[Simp[f]];
S[a1[i_, j_]] := a1[i, j];
S[a1[f_, i_, j_]] := a1[Simp[f], i, j];
S[c1[f_, k_]] := c1[Simp[f], k];
S[δa1[f_, i_, j_]] := δa1[Simp[f], i, j];
S[ca1[f_, j_, k_, l_]] := ca1[Simp[f], j, k, l];
S[δaa1[f_, i_, j_, k_, l_]] := δaa1[Simp[f], i, j, k, l];
(* Bad heads - interchange arrows no matter what: *)
S[δaa1[f_, i_, j_, k_, l_]] /; !OQ[j, l] := (
  If[VS1, Print["Interchange arrows on ", δaa1[f, i, j, k, l]]];
  S[δaa1[f, k, l, i, j]]
);
(* Bad tails and good heads - swing: *)
S[δaa1[f_, i_, j_, k_, l_]] /; !OQ[i, k] ^ DQ[j, l] ^ OQ[j, l] := (
  If[VS1, Print["Swinging on ", δaa1[f, i, j, k, l]]];
  S[δaa1[f, i, l, k, j] + ca1[b_k f, l, i, j] -
    ca1[b_i f, l, k, j] - ca1[b_k f, j, i, l] + ca1[b_i f, j, k, l]]
);
(* Bad tails and equal heads - commute heads: *)
S[δaa1[f_, i_, k_, j_, k_]] /; !OQ[i, j] := (
  If[VS1, Print["Commute heads on ", δaa1[f, i, k, j, k]]];
  S[δaa1[f, j, k, i, k] + δa1[-b_i f, j, k] + δa1[b_j f, i, k]]
);

```

Bases

```

FormalBasis[S_List, f_] := Module[{ff, n = Length@S, i, j, k, l},
  ff = f@@Table[bS[[i]], {i, n}];
  Flatten@{
    β[ff],
    Table[a[ff, S[[i]], S[[j]]], {i, n}, {j, n}],
    Table[c[ff, S[[i]]], {i, n}],
    Table[δa[ff, S[[i]], S[[j]]], {i, n}, {j, n}],
    Table[ca[ff, S[[i]], S[[j]], S[[k]]], {i, n}, {j, n}, {k, n}],
    Table[δaa[ff, S[[i]], S[[j]], S[[k]], S[[l]]], {i, n}, {j, n}, {k, i, n}, {l, j, n}]
  } /. 1[___] → 1
];

FormalBasis[n_Integer, f_] := FormalBasis[Range[n], f];
FormalPlusBasis[n_, f_] := Module[{ff},
  ff = f@@Table[bi, {i, n}];
  Flatten@{
    β[ff],
    Table[a[ff, i, j], {i, n-1}, {j, i+1, n}],
    Table[c[ff, i], {i, n}],
    Table[δa[ff, i, j], {i, n-1}, {j, i+1, n}],
    Table[ca[ff, i, j, k], {i, n}, {j, n-1}, {k, j+1, n}],
    Table[δaa[ff, i, j, k, l], {i, n-1}, {j, i+1, n}, {k, n-1}, {l, k+1, n}]
  } /. 1[___] → 1
];

Jacobi[x1_, x2_, x3_] := Module[{Jac},
  Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
  If[Jac === 0, Jac, {x1, x2, x3} → Jac]
];

```

The Adjoint action

AutoAd

```

AutoAd[x_][y_] :=
Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
  pows = NestList[B[x, #] &, y, 20];
  Print["pows computed for ", {x, y}, "..."];
  states = Union[
    Cases[pows, s_β | s_a | s_c | s_δa | s_ca | s_δaa > ReplacePart[s, 1 → _], ∞]];
  Sum[
    seq = Cases[{-#}, states[[i]], ∞] & /@ pows;
    seq = Replace[seq, {_{f_, ___}} > f, {} > 0}, {1}];
    Print["seq computed... ", states[[i]], " is ", i, "/", Length@states];
    dseq = Drop[seq, sh];
    If[Union[Length[MonomialList[{-#}] & /@ dseq] === {1} &
      Union[Length[FactorTermsList[{-#}] & /@ dseq] === {2},
      sf1 = FindSequenceFunction[FactorTermsList[{-#][[1]] & /@ dseq];
      sf2 = FindSequenceFunction[FactorTermsList[{-#][[2]] & /@ dseq];
      Print["sf1: ", sf1, " sf2: ", sf2];
      sf = (sf1[{-#] sf2[{-#] &),
      (*Else*) sf = FindSequenceFunction[dseq,
        FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
          "Polynomial", "RationalFunction", "HypergeometricTerm"}];
      Print["sf: ", sf];
    ];
    ReplacePart[states[[i]], 1 → Simplify[
      Sum[seq[[n + 1]] / n!, {n, 0, sh - 1}] + Sum[sf[[n + 1 - sh]] / n!, {n, sh, ∞}]],
    {i, Length@states}];
  ];
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)

```

Ad

```

Ad[a[t_, j_, k_]][β[f_]] /; FreeQ[t, b_] :=
  β[f] + c[(1 - e^{-tb_j}) (∂_{b_k} f - ∂_{b_j} f), k] + δa[
    (e^{-tb_j} - 1) (∂_{b_k} f - ∂_{b_j} f) / b_j, j, k];
Ad[a[t_, j_, k_]][a[1, j_, k_]] /; FreeQ[t, b_] := a[1, j, k];
Ad[a[t_, j_, k_]][a[1, n_, i_]] /;
  FreeQ[t, b_] & {j, k} ∩ {n, i} === {} := a[1, n, i];
Ad[a[t_, j_, k_]][a[1, j_, j_]] /; DQ[j, k] & FreeQ[t, b_] :=

```

$$\begin{aligned}
& a[1, j, j] + c[-1 + e^{-tb_j}, k] + \delta a\left[\frac{1 - e^{-tb_j}}{b_j}, j, k\right]; \\
\text{Ad}[a[t_-, j_-, k_-]][a[1, k_-, k_-]] /; \text{DQ}[j, k] \wedge \text{FreeQ}[t, b_-] := \\
& a[1, k, k] + c[1 - e^{-tb_j}, k] + \delta a\left[\frac{-1 + e^{-tb_j}}{b_j}, j, k\right]; \\
\text{Ad}[a[t_-, j_-, k_-]][a[1, i_-, j_-]] /; \text{DQ}[i, j, k] \wedge \text{FreeQ}[t, b_-] := \\
& a[1, i, j] + a[1 - e^{-tb_j}, i, k] + a\left[\frac{(e^{-tb_j} - 1) b_i}{b_j}, j, k\right] + ca\left[\frac{1 - e^{-tb_j}}{b_j}, k, i, j\right] + \\
& ca\left[\frac{e^{-tb_j} - 1}{b_j}, j, i, k\right] + ca\left[\frac{b_i (1 - e^{-tb_j} - tb_j)}{b_j^2}, j, j, k\right] + \\
& ca\left[\frac{e^{-2tb_j} b_i (1 - e^{tb_j} - e^{-tb_j} (e^{tb_j} - 2) tb_j)}{b_j^2}, k, j, k\right] + ca\left[\frac{e^{-2tb_j} (e^{tb_j} (1 - tb_j) - 1)}{b_j}, \right. \\
& \left. k, i, k\right] + \delta a\left[\frac{b_i (1 - e^{-tb_j} - tb_j)}{b_j^2} + \frac{-b_i (1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j)}{b_j^2}, j, k\right] + \\
& \delta a\left[\frac{(-1 + e^{-tb_j} + tb_j)}{b_j} + \frac{(1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j)}{b_j}, i, k\right] + \\
& \delta aa\left[\frac{2e^{-tb_j} b_i (\text{Sinh}[tb_j] - tb_j)}{b_j^3}, j, k, j, k\right] + \delta aa\left[\frac{-1 + e^{-tb_j} + tb_j}{b_j^2}, i, j, j, k\right] + \\
& \delta aa\left[-\frac{1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j}{b_j^2}, i, k, j, k\right]; \\
\text{Ad}[a[t_-, j_-, k_-]][a[1, i_-, k_-]] /; \text{DQ}[i, j, k] \wedge \text{FreeQ}[t, b_-] := \\
& a[e^{-tb_j}, i, k] + a\left[\frac{(1 - e^{-tb_j}) b_i}{b_j}, j, k\right] + ca\left[\frac{2e^{-tb_j} b_i (\text{Sinh}[tb_j] - tb_j)}{b_j^2}, k, j, k\right] + \\
& ca\left[\frac{e^{-2tb_j} (1 + e^{tb_j} (-1 + tb_j))}{b_j}, k, i, k\right] + \delta a\left[\frac{e^{-2tb_j} b_i (-1 + e^{tb_j} (1 - tb_j))}{b_j^2}, j, k\right] + \\
& \delta a\left[\frac{e^{-2tb_j} (1 - e^{tb_j} (1 - tb_j))}{b_j}, i, k\right] + \delta aa\left[\frac{2e^{-tb_j} b_i (-\text{Sinh}[tb_j] + tb_j)}{b_j^3}, j, k, j, k\right] + \\
& \delta aa\left[\frac{e^{-2tb_j} (-1 + e^{tb_j} (1 - tb_j))}{b_j^2}, i, k, j, k\right]; \\
\text{Ad}[a[t_-, j_-, k_-]][a[1, j_-, l_-]] /; \text{DQ}[j, k, l] \wedge \text{FreeQ}[t, b_-] := \\
& a[1, j, l] + ca[t, l, j, k] + ca\left[\frac{e^{-tb_j} - 1}{b_j}, k, j, l\right] + \delta aa\left[\frac{1 - e^{-tb_j} - tb_j}{b_j^2}, j, k, j, l\right]; \\
\text{Ad}[a[t_-, j_-, k_-]][a[1, k_-, l_-]] /; \text{DQ}[j, k, l] \wedge \text{FreeQ}[t, b_-] := \\
& a[e^{tb_j}, k, l] + a\left[-\frac{(-1 + e^{tb_j}) b_k}{b_j}, j, l\right] + ca\left[\frac{-1 + e^{tb_j} (1 - tb_j)}{b_j}, k, k, l\right] + \\
& ca\left[\frac{b_j - e^{-tb_j} b_j + b_k + e^{tb_j} (-1 + tb_j) b_k}{b_j^2}, k, j, l\right] + \\
& ca\left[\frac{b_j + b_k + tb_j b_k - e^{tb_j} (b_j + b_k)}{b_j^2}, l, j, k\right] + \delta aa\left[\frac{1 + e^{tb_j} (-1 + tb_j)}{b_j^2}, j, k, k, l\right] +
\end{aligned}$$

$$\delta aa \left[\frac{1}{b_j^3} e^{-tb_j} (b_j + e^{2tb_j} (b_j + (2 - tb_j) b_k)) - e^{tb_j} (2b_k + b_j (2 + tb_k)) \right], j, k, j, 1];$$

$$\text{Ad}[a[t_, j_, k_]] [a[1, k_, j_]] /; \text{DQ}[j, k] \wedge \text{FreeQ}[t, b_] :=$$

$$a[e^{tb_j}, k, j] + a[-1 + e^{tb_j}, k, k] + a\left[-\frac{(-1 + e^{tb_j}) b_k}{b_j}, j, j\right] + a\left[-\frac{(-1 + e^{tb_j}) b_k}{b_j}, j, k\right] +$$

$$c\left[\frac{(1 - e^{tb_j} + 2e^{tb_j} tb_j) b_k}{b_j}, j\right] + c\left[-1 + e^{-tb_j} + \frac{e^{-tb_j} (1 - e^{2tb_j} + 2e^{2tb_j} tb_j) b_k}{b_j}, k\right] +$$

$$ca[e^{tb_j} t, k, k, j] + ca\left[\frac{-1 + e^{tb_j} - 2e^{tb_j} tb_j}{b_j}, j, k, k\right] + ca\left[\frac{-1 + e^{tb_j} - e^{tb_j} tb_j}{b_j}, k, k, k\right] +$$

$$ca\left[\frac{-2(-1 + e^{tb_j}) b_k + b_j (1 - e^{tb_j} + 2e^{tb_j} tb_k)}{b_j^2}, j, j, k\right] +$$

$$ca\left[\frac{e^{-tb_j} ((-1 + e^{tb_j}) b_k + b_j (-1 + e^{tb_j} - e^{2tb_j} tb_k))}{b_j^2}, k, j, j\right] +$$

$$ca\left[\frac{1}{b_j^2} e^{-tb_j} (- (1 - 3e^{tb_j} + 2e^{2tb_j}) b_k + b_j (- (-1 + e^{tb_j})^2 + e^{2tb_j} tb_k)) \right], k, j, k] +$$

$$\delta a[-e^{tb_j} t, k, j] + \delta a\left[\frac{-1 + e^{tb_j} - e^{tb_j} tb_j}{b_j}, k, k\right] + \delta a\left[-\frac{(1 - e^{tb_j} + e^{tb_j} tb_j) b_k}{b_j^2}, j, j\right] +$$

$$\delta a\left[\frac{e^{-tb_j} ((-1 + e^{tb_j}) b_k + b_j (-1 + e^{tb_j} - e^{2tb_j} tb_k))}{b_j^2}, j, k\right] +$$

$$\delta aa\left[\frac{2 - 2e^{tb_j} + 2e^{tb_j} tb_j}{b_j^2}, j, j, k, k\right] + \delta aa\left[\frac{2 - 2e^{tb_j} + 2e^{tb_j} tb_j}{b_j^2}, j, k, k, k\right] +$$

$$\delta aa\left[\frac{1}{b_j^3} e^{-tb_j} ((1 - 4e^{tb_j} + 3e^{2tb_j}) b_k + b_j ((-1 + e^{tb_j})^2 - 2e^{2tb_j} tb_k)) \right], j, j, j, k] +$$

$$\delta aa\left[\frac{1}{b_j^3} e^{-tb_j} ((1 - 4e^{tb_j} + 3e^{2tb_j}) b_k + b_j ((-1 + e^{tb_j})^2 - 2e^{2tb_j} tb_k)) \right], j, k, j, k];$$

$$\text{Ad}[a[t_, j_, k_]] [c[1, i_]] /; \text{FreeQ}[t, b_] \wedge (\{j, k\} \cap \{i\} == \{\}) := c[1, i];$$

$$\text{Ad}[a[t_, j_, k_]] [c[1, j_]] /; \text{DQ}[j, k] \wedge \text{FreeQ}[t, b_] :=$$

$$c[1, j] + c[1 - e^{-tb_j}, k] + \delta a\left[\frac{e^{-tb_j} - 1}{b_j}, j, k\right];$$

$$\text{Ad}[a[t_, j_, k_]] [c[1, k_]] /; \text{DQ}[j, k] \wedge \text{FreeQ}[t, b_] :=$$

$$c[e^{-tb_j}, k] + \delta a\left[\frac{1 - e^{-tb_j}}{b_j}, j, k\right];$$

$$\text{Ad}[x_\beta | x_c | x_{\delta a} | x_{ca} | x_{\delta aa}] [y_] := y + B[x, y];$$

$$\text{Ad}[x_] [a[f_, i_, j_]] /; f != 1 := \text{Ad}[x] [\beta[f]] ** \text{Ad}[x] [a[1, i, j]];$$

$$\text{Ad}[x_] [c[f_, i_]] /; f != 1 := \text{Ad}[x] [\beta[f]] ** \text{Ad}[x] [c[1, i]];$$

$$\text{Ad}[x_] [\delta a[f_, j_, k_]] := \delta ** (\beta[f] ** \text{Ad}[x] [a[1, j, k]]);$$

$$\text{Ad}[x_] [ca[f_, i_, j_, k_]] := \text{Ad}[x] [c[f, i]] ** \text{Ad}[x] [a[1, j, k]];$$

$$\text{Ad}[x_] [\delta aa[f_, i_, j_, k_, l_]] := \text{Ad}[x] [\delta a[f, i, j]] ** \text{Ad}[x] [a[1, k, l]];$$

$$\text{Ad}[x_] [y_Plus] := \text{Ad}[x] /@ y;$$

```
Ad::ndef = "Ad[`1` is not defined on `2`.";
Ad[x_][y_] := (Message[Ad::ndef, x, y]; Undefined);
```

Verifying R3

```
VerifyR3[t_, expr_] := Module[{lhs, rhs},
  lhs = expr // R[t, 1, 2] // R[t, 1, 3] // R[t, 2, 3] // S;
  rhs = expr // R[t, 2, 3] // R[t, 1, 3] // R[t, 1, 2] // S;
  expr -> S[lhs - rhs] == 0
];
VerifyR3[expr_] := VerifyR3[1, expr];

Total[MapIndexed[ (#1 /. f -> f#[[1]] ) &, DeleteCases[FormalBasis[{j, k}, f], _β | _a]]]
c[f1[bj, bk], j] + c[f2[bj, bk], k] + ca[f7[bj, bk], j, j, j] + ca[f8[bj, bk], j, j, k] +
ca[f9[bj, bk], j, k, j] + ca[f10[bj, bk], j, k, k] + ca[f11[bj, bk], k, j, j] +
ca[f12[bj, bk], k, j, k] + ca[f13[bj, bk], k, k, j] + ca[f14[bj, bk], k, k, k] +
δa[f3[bj, bk], j, j] + δa[f4[bj, bk], j, k] + δa[f5[bj, bk], k, j] + δa[f6[bj, bk], k, k] +
δaa[f15[bj, bk], j, j, j, j] + δaa[f16[bj, bk], j, j, j, k] + δaa[f17[bj, bk], j, j, k, j] +
δaa[f18[bj, bk], j, j, k, k] + δaa[f19[bj, bk], j, k, j, k] + δaa[f20[bj, bk], j, k, k, k] +
δaa[f21[bj, bk], k, j, k, j] + δaa[f22[bj, bk], k, j, k, k] + δaa[f23[bj, bk], k, k, k, k]
```



```
f22[___] = 0; f21[___] = 0; f9[___] = 0; f5[___] = 0; f13[___] = 0; f17[___] = 0; f7[___] = 0;
f8[___] = 0;
```

```
f10[bj_, bk_] := g2[bk];
```

```
f1[bj_, bk_] := g3[bk];
```

```
f12[bj_, bk_] := -bj f19[bj, bk];
```

```
f19[bj_, bk_] := - $\frac{e^{-bj} (2 - 2 e^{bj} + bj + e^{bj} bj)}{2 bj^3}$ ;
```

```
f16[bj_, bk_] := g1[bj];
```

```
f11[bj_, bk_] := g4[bj];
```

```
f14[bj_, bk_] := -bj f20[bj, bk];
```

```
f20[bj_, bk_] :=  $\epsilon \frac{-1 + e^{bj}}{e^{bj} bj} \frac{-2 + 2 e^{bk} - bk - e^{bk} bk + 2 bj^2 g2[bk] + 2 e^{bk} bj^2 g4[bk]}{2 (-1 + e^{bk}) bk^2}$ ;
```

```
f2[bj_, bk_] := g5[bj] - bj f4[bj, bk];
```

```
(* Non-forced choice: *) f4[___] = 0;
```

```
g5[bj_] := - $\frac{e^{-bj} (2 - 2 e^{bj} + bj + e^{bj} bj + 2 bj g3[bj])}{2 bj}$ ;
```

```
(* Non-forced choices: *) g1[_] = 0; g2[_] = 0; g3[_] = 0;
```

```
g4[_] = 0; f3[___] = 0; f6[___] = 0; f15[___] = 0; f18[___] = 0; f23[___] = 0;
```

```
Total[
```

```
MapIndexed[(#1 /. f -> f#2[[1]]) &, DeleteCases[FormalBasis[{j, k}, f], _\beta | _a]] // S
```

```
c[- $\frac{e^{-bj} (2 - 2 e^{bj} + (1 + e^{bj}) bj)}{2 bj}$ , k] + ca[ $\frac{e^{-bj} (2 - 2 e^{bj} + (1 + e^{bj}) bj)}{2 bj^2}$ , k, j, k] +
```

```
ca[ $\frac{e^{-bj} (-1 + e^{bj}) \in (2 - 2 e^{bk} + (1 + e^{bk}) bk)}{2 (-1 + e^{bk}) bk^2}$ , k, k, k] +
```

```
deltaa[- $\frac{e^{-bj} (2 - 2 e^{bj} + (1 + e^{bj}) bj)}{2 bj^3}$ , j, k, j, k] +
```

```
deltaa[- $\frac{e^{-bj} (-1 + e^{bj}) \in (2 - 2 e^{bk} + (1 + e^{bk}) bk)}{2 (-1 + e^{bk}) bj bk^2}$ , j, k, k, k]
```

```
R[t_, j_, k_][x_] := Expand[
```

```
x // Ad[a[t, j, k]] //
```

```
(# + t B[Total[MapIndexed[(#1 /. f -> f#2[[1]]) &, DeleteCases[
FormalBasis[{j, k}, f], _\beta | _a]]], #)] &
```

```
];
```

```
R[j_, k_][x_] := R[1, j, k][x];
```

```
Print[VerifyR3[#]] & /@
```

```
{a[f[b1, b2, b3, b4], 1, 4], a[f[b1, b2, b3, b4], 2, 4], a[f[b1, b2, b3, b4], 3, 4],
a[f[b1, b2, b3, b4], 4, 1], a[f[b1, b2, b3, b4], 4, 2], a[f[b1, b2, b3, b4], 4, 3]};
```

$$a[f[b_1, b_2, b_3, b_4], 1, 4] \rightarrow \text{True}$$

$$a[f[b_1, b_2, b_3, b_4], 2, 4] \rightarrow \text{True}$$

$$a[f[b_1, b_2, b_3, b_4], 3, 4] \rightarrow c\left[-\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_2}, 4\right] +$$

$$ca\left[-\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_2^2}, 2, 2, 4\right] +$$

$$ca\left[-\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_1 b_2}, 4, 1, 2\right] +$$

$$ca\left[\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_1 b_2}, 2, 1, 4\right] +$$

$$\delta a\left[\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_1 b_2}, 1, 4\right] +$$

$$\delta aa\left[\frac{(-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_3}{2 b_1 b_2^2}, 1, 2, 2, 4\right] = 0$$

$$a[f[b_1, b_2, b_3, b_4], 4, 1] \rightarrow$$

$$c\left[-\frac{1}{2 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 3\right] +$$

$$ca\left[-\frac{1}{2 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 2, 2, 3\right] +$$

$$ca\left[\frac{1}{2 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 3, 2, 2\right] +$$

$$ca\left[-\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 3, 1, 2\right] +$$

$$ca\left[\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 2, 1, 3\right] +$$

$$\delta a\left[\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 1, 3\right] +$$

$$\delta aa\left[-\frac{1}{2 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 2, 2, 4, 3\right] +$$

$$\delta aa\left[\frac{1}{2 b_1 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 1, 2, 2, 3\right] = 0$$

$$a[f[b_1, b_2, b_3, b_4], 4, 2] \rightarrow \text{True}$$

$$a[f[b_1, b_2, b_3, b_4], 4, 3] \rightarrow$$

$$c\left[\frac{1}{2 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 3\right] +$$

$$ca\left[\frac{1}{2 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 2, 2, 3\right] +$$

$$ca\left[-\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 2, 1, 3\right] +$$

$$ca\left[\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 3, 1, 2\right] +$$

$$\delta a\left[-\frac{1}{2 b_1 b_2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 1, 3\right] +$$

$$\delta aa\left[-\frac{1}{2 b_1 b_2^2} e^{-b_1 - b_2} (-1 + e^{b_1}) (-1 + \epsilon) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2) b_4, 1, 2, 2, 3\right] = 0$$

```
Print[VerifyR3[#] & /@
```

```
{a[f[b1, b2, b3, b4], 1, 2], a[f[b1, b2, b3, b4], 1, 3], a[f[b1, b2, b3, b4], 2, 3],
a[f[b1, b2, b3, b4], 2, 1], a[f[b1, b2, b3, b4], 3, 1], a[f[b1, b2, b3, b4], 3, 2]};
```

```
a[f[b1, b2, b3, b4], 1, 2] → True
```

```
a[f[b1, b2, b3, b4], 1, 3] → True
```

```
a[f[b1, b2, b3, b4], 2, 3] → True
```

```
a[f[b1, b2, b3, b4], 2, 1] → True
```

```
a[f[b1, b2, b3, b4], 3, 1] → True
```

```
a[f[b1, b2, b3, b4], 3, 2] → True
```

```
Print[VerifyR3[t, #] & /@
```

```
{a[f[b1, b2, b3, b4], 1, 4], a[f[b1, b2, b3, b4], 2, 4], a[f[b1, b2, b3, b4], 3, 4],
a[f[b1, b2, b3, b4], 4, 1], a[f[b1, b2, b3, b4], 4, 2], a[f[b1, b2, b3, b4], 4, 3]};
```

```
a[f[b1, b2, b3, b4], 1, 4] → True
```

```
a[f[b1, b2, b3, b4], 2, 4] →
```

$$ca \left[-\frac{1}{b_1} e^{-b_1-t b_2} (-1 + e^{t b_2}) f[b_1, b_2, b_3, b_4] (2 (e^{b_1} + e^{(1+t) b_1} (-1+t) - e^{t b_1} t) + (e^{b_1} - e^{t b_1}) t b_1), \right.$$

$$\left. 3, 2, 4 \right] + ca \left[\frac{1}{b_1^2} e^{-b_1-t b_2} (-1 + e^{t b_2}) f[b_1, b_2, b_3, b_4] \right.$$

$$\left. (2 (e^{b_1} + e^{(1+t) b_1} (-1+t) - e^{t b_1} t) + (e^{b_1} - e^{t b_1}) t b_1) b_2, 3, 1, 4 \right] +$$

$$\delta aa \left[\frac{1}{b_1^2} e^{-b_1-t b_2} (-1 + e^{t b_2}) f[b_1, b_2, b_3, b_4] (2 (e^{b_1} + e^{(1+t) b_1} (-1+t) - e^{t b_1} t) + (e^{b_1} - e^{t b_1}) t b_1), \right.$$

$$\left. 1, 3, 2, 4 \right] + \delta aa \left[-\frac{1}{b_1^3} e^{-b_1-t b_2} (-1 + e^{t b_2}) f[b_1, b_2, b_3, b_4] \right.$$

$$\left. (2 (e^{b_1} + e^{(1+t) b_1} (-1+t) - e^{t b_1} t) + (e^{b_1} - e^{t b_1}) t b_1) b_2, 1, 3, 1, 4 \right] = 0$$

```
$Aborted
```

```
 $\rho_0 =$ 
```

```
Total[MapIndexed[(#1 /. f → f[#2[[1]])] &, DeleteCases[FormalBasis[{j, k}, f], _β | _a]]]
```

$$c \left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + b_j + e^{b_j} b_j)}{2 b_j}, k \right] + ca \left[\frac{e^{-b_j} (2 - 2 e^{b_j} + b_j + e^{b_j} b_j)}{2 b_j^2}, k, j, k \right] +$$

$$ca \left[-\frac{e^{-b_j} (-1 + e^{b_j}) \in (-2 + 2 e^{b_k} - b_k - e^{b_k} b_k)}{2 (-1 + e^{b_k}) b_k^2}, k, k, k \right] +$$

$$\delta aa \left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + b_j + e^{b_j} b_j)}{2 b_j^3}, j, k, j, k \right] +$$

$$\delta aa \left[\frac{e^{-b_j} (-1 + e^{b_j}) \in (-2 + 2 e^{b_k} - b_k - e^{b_k} b_k)}{2 (-1 + e^{b_k}) b_j b_k^2}, j, k, k, k \right]$$

$$\phi_1[x_-] := e^{-x} - 1; \quad \phi_2[x_-] := \frac{(x+2)e^{-x} - 2 + x}{2x};$$

$$\rho = c[-\phi_2[b_j], k] + ca\left[\frac{\phi_2[b_j]}{b_j}, k, j, k\right] + ca\left[\frac{\phi_1[b_j]}{b_k \phi_1[b_k]} \phi_2[b_k], k, k, k\right] +$$

$$\delta aa\left[\frac{-\phi_2[b_j]}{b_j^2}, j, k, j, k\right] + \delta aa\left[\frac{-\phi_1[b_j]}{b_j b_k \phi_1[b_k]} \phi_2[b_k], j, k, k, k\right];$$

$\rho_0 - \rho$

$$ca\left[\frac{e^{-b_j}(-1 + e^{b_j})(-1 + \epsilon)(2 - 2e^{b_k} + (1 + e^{b_k})b_k)}{2(-1 + e^{b_k})b_k^2}, k, k, k\right] +$$

$$\delta aa\left[-\frac{e^{-b_j}(-1 + e^{b_j})(-1 + \epsilon)(2 - 2e^{b_k} + (1 + e^{b_k})b_k)}{2(-1 + e^{b_k})b_j b_k^2}, j, k, k, k\right]$$

{Series[$\phi_1[x]$, {x, 0, 5}], Series[$\phi_2[x]$, {x, 0, 5}]}

$$\left\{-x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O[x]^6, \frac{x^2}{12} - \frac{x^3}{24} + \frac{x^4}{80} - \frac{x^5}{360} + O[x]^6\right\}$$