Cheat Sheet OneCo-1604

http://drorbn.net/AcademicPensieve/Projects/OneCo-1604/
Continues 2016-04; restarted Projects/OneCo-1606; modified 22/6/16

Deriving Gassner. $\mathcal{L}^{2Dw}$ is $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo locality, $[a_{ij},a_{ik}]=0$, $[a_{ik},a_{jk}]=-[a_{ij},a_{jk}]=b_ja_{ik}-b_ia_{jk}$, and $[a_{ij},a_{ji}]=b_ia_{ji}-b_ja_{ij}+b_ia_{jj}-b_ja_{ii}$. Acts on $V=\mathbb{Q}[\![b_i]\!]\langle x_i=a_{i\infty}\rangle$ by $[a_{ij},x_i]=0$, $[a_{ij},x_j]=b_ix_j-b_jx_i$. Hence $e^{\operatorname{ad}a_{ij}}x_i=x_i$, $e^{\operatorname{ad}a_{ij}}x_j=e^{b_i}x_j+\frac{b_j}{b_i}(1-e^{b_i})x_i$. Renaming $\bar{x}_i=x_i/b_i$, $t_i=e^{b_i}$, get $[e^{\operatorname{ad}a_{ij}}]_{\bar{x}_i,\bar{x}_j}=\begin{pmatrix}1 & 1-t_i\\ 0 & t_i\end{pmatrix}$. The radical contains 1 and a_{ii} (if included) and $\left\{\sum_i\frac{\alpha_i}{b_i}\sum_j a_{ij}\colon\ \sum\alpha_i=0\right\}$.

The $\mathcal{L}^{2Dw}$ Adjoint representation. $e^{\operatorname{ad}a_{ij}}$ acts by

$$a_{kl}\mapsto a_{kl},\qquad a_{ik}\mapsto a_{ik},\qquad a_{kj}\mapsto e^{-b_i}a_{kj}+\frac{b_k}{b_i}(1-e^{-b_i})a_{ij},$$

$$a_{ki}\mapsto a_{ki}+(1-e^{-b_i})a_{kj}+b_k\frac{e^{-b_i}-1}{b_i}a_{ij},$$

$$a_{jk}\mapsto e^{b_i}a_{jk}+\frac{b_j}{b_i}(1-e^{b_i})a_{ik},\quad a_{ji}\mapsto e^{b_i}a_{ji}+\frac{b_j}{b_i}(1-e^{b_i})a_{ij}.$$

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf, pensieve://2016-04/nb/BurauAndAd.pdf.

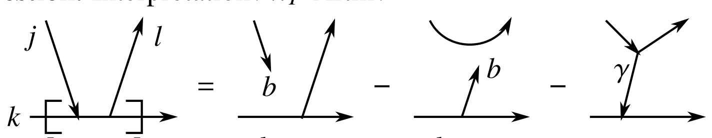
Adjoint Gassner. Renaming $\bar{a}_{ij}=a_{ij}/b_i$ and $t_i=e^{b_i}$, get $[\bar{a}_{ij},\bar{a}_{ik}]=0$, $[\bar{a}_{ik},\bar{a}_{jk}]=-[\bar{a}_{ij},\bar{a}_{jk}]=\bar{a}_{ik}-\bar{a}_{jk}$, and (mod $\langle\bar{a}_{ii}\rangle$) $[\bar{a}_{ij},\bar{a}_{ji}]=\bar{a}_{ji}-\bar{a}_{ij}$, so $e^{\operatorname{ad}a_{ij}}$ acts by

$$\bar{a}_{kj}\mapsto t_i^{-1}\bar{a}_{kj}+(1-t_i^{-1})\bar{a}_{ij},$$

$$\bar{a}_{ki}\mapsto\bar{a}_{ki}+(1-t_i^{-1})\bar{a}_{kj}+(t_i^{-1}-1)\bar{a}_{ij},$$

$$\bar{a}_{jk}\mapsto t_i\bar{a}_{jk}+(1-t_i)\bar{a}_{ik},\quad\bar{a}_{ji}\mapsto t_i\bar{a}_{ji}+(1-t_i)\bar{a}_{ij}.$$

Question. Interpretation? π_T-Artin?



$$[a_{jk},a_{kl}]\ =\ b_ja_{kl}\ -\ b_ka_{jl}\ -\ \gamma_{jkl}$$

2Dv. b: bracket trace; c: cobracket trace; $\langle b,c\rangle=\delta\in\{0,1\}$; $\deg b_i=\deg c_j=\deg a_{ij}=\deg\delta=1$. Implementation/verification: pensieve://2015-08/nb/abc.pdf.

$\mathcal{A}^{2Dv}$ is $\mathbb{Q}[\![\delta]\!]FA(b_i,c_j,a_{ij})$ (so $\mathcal{L}^v=\{f+f^{ij}a_{ij}\}$) modulo locality,

tt. $[a_{jk},a_{jl}]=c_la_{jk}-c_ka_{jl}$,

hh. $[a_{jk},a_{ik}]=b_ia_{jk}-b_ja_{ik}$,

Swinging. $\delta a_{ij}a_{kl}-\delta a_{il}a_{kj}=b_kc_la_{ij}-b_ic_la_{kj}-b_kc_ja_{il}+b_ic_ja_{kl}$

ht. $[a_{jk},a_{kl}]=b_ja_{kl}-b_ka_{jl}-c_la_{jk}+c_ka_{jl}$,

ab,ac. $\operatorname{ad}a_{jk}\colon b_j,-b_k,-c_j,c_k\mapsto\gamma_{jk}:=\delta a_{jk}-b_jc_k$,

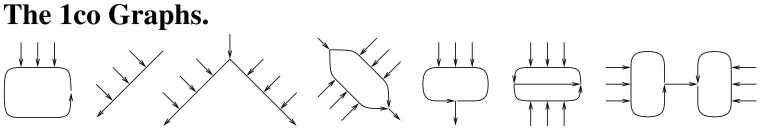
Backie. $[a_{jk},a_{kj}]=(b_j+c_k)a_{kj}-(b_k+c_j)a_{jk}+(b_j-c_j)a_{kk}-(b_k-c_k)a_{jj}+\gamma_{jk}-\gamma_{kj}$, with $\gamma_{jk}:=\delta a_{jk}-b_jc_k$,

bc. $[b_i,c_j]=0$.

The Ascending Algebra $\mathcal{A}_+^{2Dv}$. Same but with only a_{ij}, $i<j$.

The OneCo Sub-Quotient is $\langle a_{ij}\rangle$ modulo $\delta^2=\delta c_i=c_jc_k=0$, so $\mathcal{L}^{1co}$ is (coefficient functions non-central, in $\mathbb{Q}[\![b_i]\!]$)

The 1co Graphs.



In abc.nb: $R^{jk}=e^{a_{jk}}\rho$ with $\rho:=$

$$\psi(b_j)\left(-c_k+\frac{c_ka_{jk}}{b_j}-\frac{\delta a_{jk}a_{jk}}{b_j^2}\right)+\frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_ka_{kk}-\frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

and with $\phi(x):=e^{-x}-1=-x+x^2/2-\dots$, and $\psi(x):=((x+2)e^{-x}-2+x)/(2x)=x^2/12-x^3/24+\dots$.

OneCo Monoblog.

(161621) 1-co low algebra ($\epsilon^2=0$): $a_{12}=I=b_1c_2+u_1w_2\in\mathfrak{b}_\epsilon^*\otimes\mathfrak{b}_\epsilon$

$[w,c]=w$ $\quad[b,u]=-\epsilon u$ $\quad\delta c=0$ $\quad\delta w=\epsilon(c\wedge u)$

$[b,c]=0$ $\quad[b,w]=\epsilon w$ $\quad[c,u]=u$ $\quad[u,w]=b-\epsilon c$

(verification in pensieve://2016-06)

Also, $ad(-a_{12})=\{u_1\mapsto\epsilon u_1c_2,\ u_2\mapsto-b_1u_2+b_2u_1-\epsilon u_1c_2,\ b_1\mapsto-\epsilon u_1w_2,\ b_2\mapsto\epsilon u_1w_2,\ w_1\mapsto-b_1w_2-\epsilon w_1c_2+\epsilon c_1w_2,\ w_2\mapsto b_1w_2,\ c_1\mapsto u_1w_2,\ c_2\mapsto-u_1w_2\}$.

(161620) I need concise descriptions of $\mathcal{P}^{2,2}$ in $\mathcal{A}^v$ terms & in $\mathfrak{g}$ terms.

(160618b) A “defining rep” for the Euler ext. of Alexander-Gassner?

(160618a) The “Euler extension” of a graded Lie algebra L over $R:=\mathbb{Q}[\![b_i]\!]$, with $\deg b_i=1$, assuming as graded v.s. $L=R\otimes_{\mathbb{Q}}L_0$ with a graded f.d. L_0: Set $R_\epsilon:=R\otimes\mathbb{Q}[\epsilon]/(\epsilon^2=0)$ with $\deg\epsilon=1$ and $\iota\colon R\to R_\epsilon$ via $b_i\mapsto b_i+\epsilon$ (not R-linear!), set $L_E:=(R_\epsilon\otimes L_0)\rtimes R\langle E\rangle$ with $[E,b_i]=0$, $[E,\epsilon]=\epsilon$ and $[E,x]=(\deg x)x$ for $x\in L_0$, and with $\iota\colon L\to L_E$ in the obvious way. Then L_E is finite rank over R with a faithful adjoint rep, and it contains a $\mathbb{Q}$-linear copy of L.

(160616) Consider “degrons” and their “duals”, and non-linear embeddings of the Gassner Lie algebra.

(160612) Let $\mathfrak{b}_0:=\langle c,w\rangle$ with $[c,w]=w$, let $\mathfrak{b}_0^*=\langle b,u\rangle$ with $c(b)=u(w)=1$ and $c(w)=u(b)=0$ be Abelian, let $\mathfrak{g}_0=I\mathfrak{b}_0=\mathfrak{b}_0^*\rtimes\mathfrak{b}_0$ so $[b,c]=[b,w]=[b,u]=0$ while $[c,u]=-u$ and $[w,u]=b$. Let $r=Id=b_1c_2+u_1w_2\in\mathfrak{b}_0^*\otimes\mathfrak{b}_0\subset\mathfrak{g}_0\otimes\mathfrak{g}_0$. Let $\mathcal{U}=\mathcal{U}(\mathfrak{g}_0)$, degree-completed with respect to $\deg b,u=1$ and $\deg c,w=0$. Then $R=\exp(r)\in\mathcal{U}\otimes\mathcal{U}$ satisfies Yang-Baxter, $bc+uw$, $cb+wu$, and b are central, and $(cb+wu)-(bc+uw)=b$. Also, $ad(-r_{ij})=\{b_k\mapsto 0,u_i\mapsto 0,u_j\mapsto b_iu_j-b_ju_i,c_i\mapsto-u_iw_j,c_j\mapsto u_iw_j,w_i\mapsto b_iw_j,w_j\mapsto-b_iw_j\}$.

$a_{12}=I\in\mathfrak{b}_0^*\otimes\mathfrak{b}_0$:

$b_1c_2+u_1w_2$

$[c,w]=w\quad[b,\cdot]=0$

$[u,c]=u\quad[w,u]=b$

(160611) I should re-write `Local.nb` using $\{1,\delta,a_{ij},\eta_k,\eta_ka_{ij},u_i,w_j\}$, and aim to split the a_{ij}’s.

(160608) Likely, $G^*\cong\Delta t\infty$.

(160607) How is $G\otimes G^*$ related to “Adjoint Gassner” above?

(160529) In 2016-06/TurboGassner.nb: (make presentable?)

```
GP_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j :> (1 - t_i) u_i + t_i u_j,
    f_. v_j :> f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i + f t_i u_i }]
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
FTG_{i_,j_}[ℰ_] := Expand[ℰ /. {
     f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i - η[i]) v_i + (t_i + η[i]) v_j,
       (t_i Coefficient[f, η[i]] - t_j Coefficient[f, η[j]])
        (1 - t_i^-1) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
       Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
     u_j → (1 - t_i) u_i + t_i u_j,
     w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
TG_{i_,j_}[ℰ_] := Expand[ℰ /. {
     f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
       (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f)
        (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
       Kδ_{k,i} f (u_j - u_i) u_i w_j],
     u_j → (1 - t_i) u_i + t_i u_j,
     w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
```

(160529) Is there a use for $\delta^{-1}c_ic_j$ terms?
(160527) Try rewriting R as an exponential. Explains the ϕ_1's?
(160505) A faithful representation for $\mathcal{A}^{2,2}$? Ado suggests existence.
(1504) If $S_n := \sum_{k=0}^{n-1} A^k C B^{n-1-k}$ then $AS_n - S_n B = A^n C - CB^n$ so $S_n = (L_A - R_B)^{-1}(A^n C - CB^n)$.
(151019a) To do: Find and implement the group-like condition.

Recycling.

(160510) The next few steps: • $h\infty$ scattering in 1-co. • Solve again for R. • Find a manifestly polynomial formula for R. • Revisit stitching in 0-co. • Stitching for $\delta h\infty$ scattering. • Stitching for $h\infty$ scattering. • Full adjoint scattering in 1-co. • Stitching for full adjoint. • Glow. • Caps and cups.

(151019c) Make the braid representation presentable?

(160508) How would I present the TS stitching formula?

(160317) To do: For 0-co a and b, compute the 1-co part of $e^{-a}be^{a}$.

(151019b) Switch to an EK basis?

So $a_{ij}f = f^{\delta}a_{ij} - \dfrac{b_ic_j}{\delta}(f^{\delta} - f)$, $[a_{ij}, f] = (f^{\delta} - f)\left(a_{ij} - \dfrac{b_ic_j}{\delta}\right)$,

with $f^{\delta} := f /\!/ \begin{pmatrix} b_i\to b_i+\delta & b_j\to b_j-\delta \\ c_i\to c_i-\delta & c_j\to c_j+\delta \end{pmatrix}$.

The Abstract Context. (From LesDiablerets-1508)

Definition. A meta-monoid is a functor M: (finite sets, injections)$\to$(sets) along with natural operations $*\colon M(S_1)\times M(S_2)\to M(S_1\sqcup S_2)$ whenever $S_1\cap S_2=\emptyset$ and $m^{ab}_c\colon M(S)\to M((S\setminus\{a,b\})\sqcup\{c\})$ whenever $a\neq b\in S$ and $c\notin S\setminus\{a,b\}$, such that

$$\text{meta-associativity:}\quad m^{ab}_y /\!/ m^{yc}_x = m^{bc}_y /\!/ m^{ay}_x$$

$$\text{meta-locality:}\quad m^{ab}_x /\!/ m^{de}_y = m^{de}_y /\!/ m^{ab}_x$$

and, with $\epsilon_b = M(S\hookrightarrow S\sqcup\{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Theorem. $S\mapsto\Gamma_0(S)$ is a meta-monoid and $z_0\colon P\!v\!T\to\Gamma_0$ is a morphism of meta-monoids.

Theorem. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S)=\Gamma_0(S)\times\Gamma_1(S)$, along with an extension of z_0 to $z_{01}\colon P\!v\!T\to\Gamma_{01}$, with

$$\Gamma_1(S) = R_S\oplus V\oplus V^{\otimes 2}\oplus V^{\otimes 3}\oplus \mathcal{S}^2(V)^{\otimes 2}\qquad(\text{with } V := R_S\langle S\rangle).$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c\to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c}\colon\Gamma_1(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$, a meta-right-action $\alpha^{ab}\colon\Gamma_1(S)\times\Gamma_0(S)\to\Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c\colon\Gamma_0(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$ such that

$$(\zeta_0,\zeta_1)/\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c},\ (\zeta_1,\zeta_0)/\!/\alpha^{ab}/\!/ m^{ab}_{1c} + \zeta_0/\!/\rho^{ab}_c\right)$$

In MostGeneralR.nb:

```
rule2 = {gg3|4|5|7|8[_] → 0, gg2[x_] :→ e^x/x, ff_[__] → 0};
ρ0[j, k] /. rule2 // S
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$$c\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j}, k\right] + ca\left[\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^2}, k, j, k\right] + ca\left[\frac{e^{b_k}}{b_k}, j, k, k\right] + ca\left[-\frac{e^{-b_j}\left(-1+e^{b_j}\right)(2+b_k)}{2b_k^2}, k, k, k\right] + \delta a\left[\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^2}, j, k\right] + \delta aa\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^3}, j, k, j, k\right] + \delta aa\left[\frac{e^{-b_j}\left(-1+e^{b_j}\right)(2+b_k)}{2b_jb_k^2}, j, k, k, k\right]$$

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R[1, 2]@a[1, 1, ∞] /. rule2 // S
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$$a[1,1,\infty] + c\left[-e^{b_2}b_1, \infty\right] + ca\left[1-e^{b_2}, \infty, 1, 2\right] + ca\left[e^{b_2}+\frac{-1+e^{-b_1}}{b_1}, 2, 1, \infty\right] + ca\left[-\frac{e^{b_2}b_1}{b_2}, 2, 2, \infty\right] + \delta a\left[e^{b_2}, 1, \infty\right] + \delta aa\left[-\frac{-1+e^{-b_1}+b_1}{b_1^2}, 1, 2, 1, \infty\right] + \delta aa\left[\frac{e^{b_2}}{b_2}, 1, 2, 2, \infty\right]$$

```
R[1, 2]@a[1, 2, ∞] /. rule2 // S
```

$$a\left[e^{b_1}, 2, \infty\right] + a\left[-\frac{\left(-1+e^{b_1}\right)b_2}{b_1}, 1, \infty\right] + c\left[\frac{-\left(-1+e^{b_1}\right)b_2+b_1\left(-1+e^{b_1}+\left(e^{b_1}-e^{b_2}+e^{b_1+b_2}\right)b_2\right)}{b_1}, \infty\right] + ca\left[\frac{e^{b_2}\left(-1+e^{b_1}\right)b_2}{b_1}, \infty, 1, 2\right] + ca\left[\frac{\left(-1+e^{b_1}\right)\left(1+e^{b_2}b_2\right)}{b_2}, 2, 2, \infty\right] + ca\left[-\frac{e^{-b_1}\left(-1+e^{b_1}\right)\left(-1+e^{b_1}+e^{b_1+b_2}b_2\right)}{b_1}, 2, 1, \infty\right] + \delta a\left[\frac{\left(-1+e^{b_1}\right)b_2-b_1\left(-1+e^{b_1}+\left(e^{b_1}-e^{b_2}+e^{b_1+b_2}\right)b_2\right)}{b_1^2}, 1, \infty\right] + \delta aa\left[\frac{e^{-b_1}\left(-1+e^{b_1}\right)^2}{b_1^2}, 1, 2, 1, \infty\right] + \delta aa\left[-\frac{\left(-1+e^{b_1}\right)\left(1+e^{b_2}b_2\right)}{b_1b_2}, 1, 2, 2, \infty\right]$$

(151019d) Perhaps I should switch to a circuit algebra perspective, plus meta-monoid ops.

(160612) Let $\mathfrak{g} = FL(x^1,x^2)/[x^1,x^2] = x^2$, let $\mathfrak{g}^* = \langle\phi_1,\phi_2\rangle$ with $\phi_i(x^j)=\delta^j_i$, let $I\mathfrak{g}=\mathfrak{g}^*\rtimes\mathfrak{g}$ so $[\phi_i,\phi_j]=[\phi_1,x^i]=0$ while $[x^1,\phi_2]=-\phi_2$ and $[x^2,\phi_2]=\phi_1$. Let $r = Id = \phi_1\otimes x^1+\phi_2\otimes x^2\in\mathfrak{g}^*\otimes\mathfrak{g}\subset I\mathfrak{g}\otimes I\mathfrak{g}$. Let $\mathcal{U}=\mathcal{U}(I\mathfrak{g})$, degree-completed with respect to $\deg\phi_i=1$ and $\deg x^i=0$ (so $\mathcal{U}\equiv$ (power series is 4 variables)). Then $R=\exp(r)\in\mathcal{U}\otimes\mathcal{U}$ satisfies CYBE, ϕ_ix^i, $x^i\phi_i$, ϕ_1 are central, $x^i\phi_i-\phi_ix^i=\phi_1$, and $[x^j,\phi_i]=\delta^j_i\phi_1-\delta^j_1\phi_i$. ϕ_1 is "bracket-trace".