

In  $\mathcal{U}(T) \otimes \mathcal{U}(H)$  conventions.

Generalities.

```
Simp[ε_] := Simplify[ε];
CF[ε_] :=
  ε /. {λβ | λa | λδβ | λc | λao | λca => MapAt[Simp, λ, 1],
  aao[f_, i_, j_, k_, l_] =>
  aao[Simp@f, Sort[{i, k}][[1]], Sort[{j, l}][[1]],
  Sort[{i, k}][[2]], Sort[{j, l}][[2]]];
AutoCollecting[λ_] := (λ /: λ[0, ___] = 0;
  λ /: λ[f_, r___] + λ[g_, r___] := λ[Simp[f+g], r];
  λ /: g_*λ[f_, r___] := λ[Simp[gf], r]);
AutoCollecting /@ {β, a, δβ, c, ao, ca, aao};
UU /: UU[x_] + UU[y_] := UU[x+y];
UU /: a_*UU[x_] := UU[Expand[a x]];
UU /: D[u_UU, vs_] :=
  CF[u /. β | λa | λδβ | λc | λao | λca | λaao =>
  MapAt[D[#, vs] &, λ, 1]];
hb[f_, i_, j_, k_] := a[bj f, i, k] - a[bi f, j, k]
  (* = f · [aik, ajk] *));
UU /: Coefficient[u_UU, λ[js___]] :=
  Total[Cases[u, λ[f_, js] => f, ∞]];
Kδ /: Kδis := KroneckerDelta[1, Length[Union[{is}]]];
```

```
UU /: UU[L_] ** UU[R_] :=
  CF@UU[Expand[Distribute[pp[L, R]] /. {
  pp[0, _] → 0, pp[_ , 0] → 0,
  pp[β[f_], λβ | λa | λδβ | λc | λao | λca |
  λaao] => MapAt[f # &, λ, 1],
  pp[_δβ | _c | _ao | _ca | _aao,
  _δβ | _c | _ao | ca | _aao] → 0,
  pp[ao[f_, i_, j_], a[g_, k_, l_]] =>
  (* Not done! *) aao[f g, i, j, k, l] -
  Kδjk δhb[f g, i, j, l],
  pp[δβ[f_], β[g_]] => δβ[f g],
  pp[δβ[f_], a[g_, i_, j_]] => ao[f g, i, j]
  }]];
**
```

Definition of *tm*.

```
UU[ε_] // tm[x_, y_, z_] := (rr = Replace[x | y → z];
  CF[UU[Expand[ε /. {
  a[f_, x, j_] => a[f, z, j] + ao[∂by f, z, j],
  a[f_, y, j_] => a[f, z, j],
  ao[f_, x | y, j_] => ao[f, z, j],
  ca[f_, i_, x | y, k_] => ca[f, i, z, k],
  aao[f_, i_, j_, k_, l_] =>
  aao[f, rr@i, j, rr@k, l]
  } /. bx|y → bz]]]);
```

Definition of *hm*.

```
UU[ε_] // hm[x_, y_, z_] := (rr = Replace[x | y → z];
  CF[UU[Expand[ε /. {
  a[f_, i_, x | y] => a[f, i, z],
  c[f_, x | y] => c[f, z],
  ao[f_, i_, x | y] => ao[f, i, z],
  ca[f_, y, j_, x] => ca[f, z, j, z] + ao[f, j, z],
  ca[f_, i_, j_, k_] => ca[f, rr@i, j, rr@k],
  aao[f_, i_, y, k_, x] => aao[f, k, z, i, z],
  aao[f_, i_, j_, k_, l_] => aao[f, i, rr@j, k, rr@l]
  }]]]);
```

Definition of *hts*.

```
UU[ε_] // hts[y_, x_] := CF[UU[Expand[ε /. {
  a[f_, i_, j_] => a[f, i, j] - Kδyy ao[∂bx f, i, y] +
  Kδix Kδyy (β[-bx f] + δβ[bx ∂bx f] + c[f, y]),
  ao[f_, x, y] => ao[f, x, y] - δβ[f bx],
  ca[f_, i_, j_, k_] =>
  ca[f, i, j, k] + Kδiy Kδjx ao[f, x, k] +
  Kδjx Kδky c[-f bx, i],
  aao[f_, i_, j_, k_, l_] =>
  aao[f, i, j, k, l] + Kδix Kδyy ao[-bx f, k, l] +
  Kδix Kδly ao[-bx f, k, j] + Kδkx Kδyy ao[-bx f, i, l] +
  Kδkx Kδly ao[-bx f, i, j] + 2 Kδxik Kδyjl δβ[bx bx f]
  }]]];
```

Definition of *dm*.

```
dm[x_, y_, z_][ε_] :=
  ε // hts[x, y] // tm[x, y, z] // hm[x, y, z]
```

Definition of *tb*.

```
tb[x_] [UU[L_], UU[R_]] :=
  CF[UU[Expand[Distribute[pp[L, R]] /. {
  pp[0, _] → 0, pp[_ , 0] → 0,
  pp[_β | _δβ | _c | _ao | _ca | _aao,
  _β | _δβ | _c | _ao | _ca | _aao] → 0,
  pp[u_β | u_δβ | u_c | u_ao | u_ca | u_aao, v_a] =>
  -pp[v, u]
  } /. {
  pp[a[f_, x, j_], u_] => (u /. {
  β[g_] => ao[f ∂bx g, x, j],
  a[g_, k_, l_] => aao[f ∂bx g, x, j, k, l] +
  ca[bx f ∂bx g, l, x, j] +
  Kδkx (ca[-f g - bx g ∂bx f, j, x, l] +
  ca[f g, l, x, j] + aao[-g ∂bx f, x, j,
  x, l]),
  _ → 0
  }),
  pp[a[f_, j_, k_], a[g_, x, l_]] (*/; j!=x*) =>
  -aao[g ∂bx f, x, l, j, k] - ca[bj g ∂bx f, k, x, l],
  pp[_ , _] → 0
  }]]];
```

<pre> thb[x_, y_][UU[L_], UU[R_]] := CF[UU[Expand[Distribute[pp[L, R]] /. {   pp[0, _] → 0, pp[_ , 0] → 0,   pp[_β   _δβ   _c   _ao   _ca   _aao,     _β   _δβ   _c   _ao   _ca   _aao] → 0,   pp[_a, _β   _δβ] → 0,   pp[β[f_], a[g_, i_, j_]] ⇒ Kδ<sub>yj</sub> ao[g ∂<sub>b<sub>x</sub></sub> f, i, y],   pp[a[f_, i_, j_], a[g_, k_, l_]] ⇒ Kδ<sub>yl</sub> (     aao[g ∂<sub>b<sub>x</sub></sub> f, k, l, i, j] + ca[g b<sub>i</sub> ∂<sub>b<sub>x</sub></sub> f, j, k, y] +     Kδ<sub>xi</sub> (a[-g f b<sub>k</sub>, x, j] + a[g f b<sub>x</sub>, k, j] +     aao[-g ∂<sub>b<sub>x</sub></sub> f, x, j, k, y] +     aao[g ∂<sub>b<sub>x</sub></sub> f, k, j, x, y] + ao[-g b<sub>x</sub> ∂<sub>b<sub>x</sub></sub> f, k, j] +     ca[-g f, y, k, j] + ca[g f, j, k, y]))),   pp[a[f_, i_, j_], c[g_, k_]] ⇒     Kδ<sub>ix</sub> Kδ<sub>ky</sub> ao[-f g, x, j],   pp[a[f_, i_, j_], ao[g_, k_, l_]] ⇒     Kδ<sub>xi</sub> Kδ<sub>yl</sub> ao[f g b<sub>x</sub>, k, j],   pp[a[f_, i_, j_], ca[g_, k_, l_, m_]] ⇒ Kδ<sub>xi</sub> (     Kδ<sub>yk</sub> (aao[-f g, l, j, x, m] + ca[-f g b<sub>l</sub>, m, x, j]) +     Kδ<sub>ym</sub> (ca[-f g b<sub>l</sub>, k, x, j] + ca[f g b<sub>x</sub>, k, l, j]) -     Kδ<sub>y,k,m</sub> ao[f g b<sub>l</sub>, x, j]),   pp[a[f_, i_, j_], aao[g_, k_, l_, m_, n_]] ⇒ Kδ<sub>xi</sub> (     Kδ<sub>yl</sub> (aao[f g b<sub>x</sub>, k, j, m, n] +     ca[f g b<sub>k</sub> b<sub>m</sub>, n, x, j]) +     Kδ<sub>yn</sub> (aao[f g b<sub>x</sub>, k, j, m, l] +     ca[f g b<sub>k</sub> b<sub>m</sub>, l, x, j]) +     Kδ<sub>y,l,n</sub> ao[f g b<sub>k</sub> b<sub>m</sub>, x, j]),   pp[_δβ   _c, _a] → 0,   pp[ao[f_, i_, j_], a[g_, k_, l_]] ⇒     Kδ<sub>xi</sub> Kδ<sub>yl</sub> (-ao[b<sub>k</sub> f g, i, j] + ao[b<sub>i</sub> f g, k, j]),   pp[ca[f_, m_, i_, j_], a[g_, k_, l_]] ⇒     Kδ<sub>xi</sub> Kδ<sub>yl</sub> (ca[-f g b<sub>k</sub>, m, x, j] + ca[f g b<sub>x</sub>, m, k, j]),   pp[aao[f_, x, j_, x, n_], a[g_, k_, y]] ⇒     aao[-2 f g b<sub>k</sub>, x, j, x, n] + aao[2 f g b<sub>x</sub>, k, j, x, n],   pp[aao[f_, i_, j_, m_, n_], a[g_, k_, l_]] ⇒     Kδ<sub>xi</sub> Kδ<sub>yl</sub> (aao[-f g b<sub>k</sub>, m, j, x, n] +     aao[f g b<sub>x</sub>, k, j, m, n]) +     Kδ<sub>xm</sub> Kδ<sub>yl</sub> (aao[-f g b<sub>k</sub>, i, j, x, n] +     aao[f g b<sub>x</sub>, i, j, k, n])}]];  htb[x_, y_][L_UU, R_UU] := -thb[y, x][R, L]; </pre>	<p style="text-align: center; color: red; margin: 0;"><b>Definition of thb.</b></p> <pre> hb[y_][UU[L_], UU[R_]] := CF[UU[Expand[Distribute[pp[L, R]] /. {   pp[0, _] → 0, pp[_ , 0] → 0,   pp[_β   _δβ, _] → 0,   pp[_ , _β   _δβ] → 0,   pp[_c   _ao   _ca   _aao, _c   _ao   _ca   _aao] → 0,   pp[u_c   u_ao   u_ca   u_aao, v_a] ⇒ -pp[v, u] } /. {   pp[a[f_, i_, y], u_] ⇒ (u /. {     a[g_, j_, k_] ⇒ Kδ<sub>yk</sub> hb[f g, i, j, k],     c[g_, j_] ⇒ Kδ<sub>yj</sub> ao[f g, i, y],     ao[g_, j_, k_] ⇒ Kδ<sub>yk</sub> ao[-f g b<sub>i</sub>, j, y],     ca[g_, j_, k_, l_] ⇒       Kδ<sub>yj</sub> (ca[f g b<sub>k</sub>, l, i, y] + aao[f g, i, l, k, y]) +       Kδ<sub>yl</sub> (ca[-f g b<sub>i</sub>, j, k, y] + ca[f g b<sub>k</sub>, j, i, y]) +       Kδ<sub>yjl</sub> ao[f g b<sub>k</sub>, i, y],     aao[g_, j_, k_, l_, m_] ⇒       Kδ<sub>yk</sub> (ca[-f g b<sub>j</sub> b<sub>l</sub>, m, i, y] +       aao[-f g b<sub>i</sub>, j, m, l, y]) +       Kδ<sub>ym</sub> (ca[-f g b<sub>j</sub> b<sub>l</sub>, k, i, y] +       aao[-f g b<sub>i</sub>, j, k, l, y]) +       Kδ<sub>ykm</sub> ao[-f g b<sub>j</sub> b<sub>l</sub>, i, y]     )),     _pp → 0   }]]];  h<sub>1</sub>h<sub>2</sub>t<sub>1</sub>t<sub>2</sub> → h<sub>1</sub>h<sub>2</sub>t<sub>1</sub>t<sub>2</sub> → h<sub>1</sub>h<sub>2</sub>t<sub>2</sub>t<sub>1</sub> → h<sub>2</sub>h<sub>1</sub>t<sub>2</sub>t<sub>1</sub> → h<sub>2</sub>h<sub>1</sub>t<sub>2</sub>t<sub>1</sub>: db. db[x_][u_UU, v_UU] := Module[{t, h}, Plus[   htb[x, x][u // τ[x, t], v // hσ[x, h]] // tm[t, x, x] //   hm[x, h, x],   tb[x][u, v // hσ[x, h]] // hm[x, h, x],   hb[x][u, v // τ[x, t]] // tm[t, x, x],   thb[x, x][u // hσ[x, h], v // τ[x, t]] //   tm[t, x, x] // hm[x, h, x]];  bb[S_List] := Module[{w, bar, t, n = 0, i, k}, The bracket.   w = #2 // dσ[S, bar /@ S];   Sum[t = db[S[[k]]][#1, w // dσ[bar[S[[k]]], S[[k]]]];   Do[t = t // dm[bar[S[[i]]], S[[i]], S[[i]], {i, 1, k - 1}];   Do[t = t // dm[S[[i]], bar[S[[i]]], S[[i]],     {i, k + 1, Length@S}];   t, {k, Length@S}] &amp; bb[S_++] := bb[{S}] </pre>
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ct[s_] := ct[s, s]; ct[] = ct[0, 0];
ct[h_, t_][UU[L_], UU[R_]] := CF[UU[Distribute[pp[L, R]] /. {
  pp[_β | _δβ, _] → 0,
  pp[a[f_, i_, h], β[g_]] ⇒ β[f bi ((∂btg) /. bt → 0)],
  pp[a[f_, i_, h], a[g_, t, j_]] ⇒ a[f (g /. bt → 0), i, j],
  pp[a[f_, i_, h], a[g_, j_, k_]] ⇒ a[f bi ((∂btg) /. bt → 0), j, k],
  pp[a[f_, i_, h], c[g_, j_]] ⇒ c[f bi ((∂btg) /. bt → 0), j],
  pp[a[f_, i_, h], ao[g_, t, j_]] ⇒ ao[f (g /. bt → 0), i, j],
  pp[a[f_, i_, h], ao[g_, j_, k_]] ⇒ ao[f bi ((∂btg) /. bt → 0), j, k],
  pp[a[f_, i_, h], ca[g_, k_, t, j_]] ⇒ ca[f (g /. bt → 0), k, i, j],
  pp[a[f_, i_, h], ca[g_, l_, j_, k_]] ⇒ ca[f bi ((∂btg) /. bt → 0), l, j, k],
  pp[a[f_, i_, h], aao[g_, t, j_, t, k_]] → 0,
  pp[a[f_, i_, h], aao[g_, t, j_, k_, l_]] ⇒ aao[f (g /. bt → 0), i, j, k, l],
  pp[a[f_, i_, h], aao[g_, j_, k_, t, l_]] ⇒ aao[f (g /. bt → 0), j, k, i, l],
  pp[a[f_, i_, h], aao[g_, j_, k_, l_, m_]] ⇒ aao[f bi ((∂btg) /. bt → 0), j, k, l, m],
  pp[a[_], _] → 0,
  pp[c[f_, h], β[g_]] ⇒ δβ[f ((∂btg) /. bt → 0)],
  pp[_c, _β] → 0,
  pp[c[f_, h], a[g_, t, j_]] ⇒ c[f (g /. bt → 0), j],
  pp[c[f_, h], a[g_, j_, k_]] ⇒ ao[f ((∂btg) /. bt → 0), j, k] + c[f bj ((∂btg) /. bt → 0), k],
  pp[_c, _a] → 0,
  pp[_c | _ao | _ca | _aao, _c | _ao | _ca | _aao] → 0,
  pp[ao[f_, i_, h], β[g_]] ⇒ 0,
  pp[ao[f_, i_, h], a[g_, t, j_]] ⇒ ao[f (g /. bt → 0), i, j],
  pp[ao[f_, i_, h], a[g_, j_, k_]] ⇒ 0,
  pp[_ao, _] → 0,
  pp[ca[_ , h, _ , h], _] → 0,
  pp[ca[f_, h, i_, j_], β[g_]] ⇒ ao[f ((∂btg) /. bt → 0), i, j] + c[f bi ((∂btg) /. bt → 0), j],
  pp[ca[f_, i_, j_, h], β[g_]] ⇒ c[f bj ((∂btg) /. bt → 0), i],
  pp[ca[f_, h, i_, j_], a[g_, t, k_]] ⇒ ca[f (g /. bt → 0), k, i, j],
  pp[ca[f_, h, i_, j_], a[g_, k_, l_]] ⇒ aao[f ((∂btg) /. bt → 0), i, j, k, l] + ca[f bi ((∂btg) /. bt → 0), j, k, l] +
    ca[f bk ((∂btg) /. bt → 0), l, i, j],
  pp[ca[f_, i_, j_, h], a[g_, t, k_]] ⇒ ca[f (g /. bt → 0), i, j, k],
  pp[ca[f_, i_, j_, h], a[g_, k_, l_]] ⇒ ca[f bj ((∂btg) /. bt → 0), i, k, l],
  pp[_ca, _] → 0,
  pp[aao[_ , _ , h, _ , h], _] → 0,
  pp[aao[f_, i_, h, j_, k_], β[g_]] ⇒ c[-f bi bj ((∂btg) /. bt → 0), k],
  pp[aao[f_, i_, h, j_, k_], a[g_, t, l_]] ⇒ aao[f (g /. bt → 0), i, l, j, k],
  pp[aao[f_, i_, h, j_, k_], a[g_, l_, m_]] ⇒ ca[-f bi bj ((∂btg) /. bt → 0), k, l, m],
  pp[aao[f_, i_, j_, k_, h], β[g_]] ⇒ c[-f bi bk ((∂btg) /. bt → 0), j],
  pp[aao[f_, i_, j_, k_, h], a[g_, t, l_]] ⇒ aao[f (g /. bt → 0), i, j, k, l],
  pp[aao[f_, i_, j_, k_, h], a[g_, l_, m_]] ⇒ ca[-f bi bk ((∂btg) /. bt → 0), j, l, m],
  pp[_aao, _] → 0 }]}];

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