# **Informal assessment:** *Over then under tangles* by D. Bar-Natan, Z. Dancso and R. Van Der Veen

This manuscript is an investigation into *over-then-under* tangle diagrams. Specifically, an over-then-under (OU) tangle diagram is a tangle whose strands complete all of their over crossings before any of their under crossings, and an OU tangle is an oriented tangle that can be represented by an OU tangle diagram. This is equivalent to considering a bridge presentation for $n$-strand tangles that have only $2n$ extrema points ($n$ max and $n$ min) which occur at strands' ends. The authors develop tangle projection moves—*glide moves*—and establish that when projections satisfy an *acyclic* condition a sequence of glide moves exist for going from an arbitrary projection to an OU projection/diagram. From there the authors investigate the OU diagrams for the collection of acyclic $n$-tangles that correspond to $n$-braids. The tools used in this investigation are algebraic and categorical. The prominent feature of this manuscript is its unorthodox choice of expository style. Section 1 is a presentation of false theorems and corollaries. The authors tag these false statements with a deliberate misspelling of the statements' signifiers—Fheorem and Forollary. The moral lesson the authors are trying to illustrate is that the bad ideas of the "theorems" in section 1 lead to good ideas of theorems in section 2.

As far as the reviewer knows a study of $n$-strand tangles in $n$-bridge positions is not present in the literature. However, knot and link diagrams that minimize the number of over segments are well know since a diagram having $n$ over-crossing segments correspond to an $n$-braid presentation. In deed, the seminal paper by Hatcher and Thurston on incompressible surfaces in the 2-bridge knots exteriors [Inventiones Mathematicae, 1985] features a knot projection in its first figure where there are just two over-crossing segments and the figure is suggesting that, except for these to segments, the knot is transversely contained in the fibre neighborhood of a train-track in the plane.

Even though this is an informal assessment, the reviewer is undecided on whether the results in this manuscript are of sufficient caliber to warrant publication in a journal of AG&T's stature. The is largely due to the previously mentioned expository style. To put it bluntly it is confusing and distracting. It's aim is to illustrate a lesson—bad ideas can lead to good ideas—that is well known to all established scholars. (If the authors goal is to illustrate this lesson, an undergraduate level journal may be a more appropriate publication venue.) More importantly, the authors choice of expository style cuts short a literature review—which is scattered and placed towards the end of the work—that would place their investigation into the larger mathematical landscape. As such the authors miss an opportunity to expand their audience beyond that of the quantum topology community. OU $n$-strand tangle diagrams correspond to closed $n$-braids when viewed as $n$-bridge presentations and as such should pertain to the Nielsen-Thurston classification of mapping class groups from the punctured disc to itself. (The sparse literature review the authors provide for this connection they relegate to the footnote on

page 7.) Here I would think the work of Murasugi on 3-braids. Murasugi's 3-braid classification has the Nielsen-Thurston classification implicit in its normal forms and, as such, the OU diagram for all pseudo-Anovos 3-braids will have all the under crossing segments transversely racing around the fibre neighborhood of the unique train-track associated with mappings of the 3-punctured disc. How does the 3-braid calculations in section 3 relate to Murasugi's normal form? More generally, since Corollary 2.9 asserts the equivalence of OU tangles with braids and Corollary 2.8 asserts the $\bar{\Gamma} \circ \bar{\iota}$ as a complete braid invariant, the absence of any reference to the large existing body of work on braid word algorithms is a glaring omission.

With all that said, with the manuscript's current expository structure this reviewer cannot recommend that AG&T's editorial review process be advanced.