Pensieve header: OU matters around the Diamond Lemma.

Old program, from Gamma.nb.

```
In[•]:=   SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OU"];
          << KnotTheory`
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
In[•]:=   SetAttributes[VD, Orderless]
```

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```
In[•]:=   bp[i_, j_] := σ_{i,j}; bm[i_, j_] := σ̄_{i,j};
```

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```
In[•]:=   Tidy[vd_VD] := Module[{ps = Union @@ (List @@@ vd)},
            Replace[vd, Thread[ps → Range@Length@ps], {2}]]
```

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```
In[•]:=   R12Reduce1[vd_VD] := Tidy@Module[{R2s, R2}, Which[
            Length[R2s = Cases[vd, X_s_[i_, j_] :> X_{-s}[i + 1, j + 1]] ∩ (List @@ vd)] > 0,
            Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :> X_{-s}[i - 1, j - 1]]],
            Length[R2s = Cases[vd, X_s_[i_, j_] :> X_{-s}[i + 1, j - 1]] ∩ (List @@ vd)] > 0,
            Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] :> X_{-s}[i - 1, j + 1]]],
            True, DeleteCases[vd, X_[i_, j_] /; Abs[i - j] == 1] ]]
```

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```
In[•]:=   R12Reduce[vd_VD] := FixedPoint[R12Reduce1, vd]
```

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```
In[•]:=   γ[vd_VD] := Module[{js, s1, i1, j1, s2, i2, j2},
            js = Cases[vd, X_[_, j_] :> j] ∩ Cases[vd, X_[i_, _] :> i - 1];
            If[Length[js] == 0, vd,
              j1 = RandomChoice[js]; i2 = j1 + 1;
              Cases[vd, X_s_[i_, j1] :> (s1 = s; i1 = i)];
              Cases[vd, X_s_[i2, j_] :> (s2 = s; j2 = j)];
              Tidy@Join[Complement[vd, VD[X_{s1}[i1, j1], X_{s2}[i2, j2]]],
                VD[X_{s2}[j1, j2], X_{s1}[i1, i2], X_{s1 s2}[i1 - s1/3, j2 + s2/3], X_{-s1 s2}[i1 + s1/3, j2 - s2/3]]
              ] ]]
```

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```
In[•]:=   Γ[vd_VD] := FixedPoint[γ, vd, 2^16]
```

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```
In[•]:=   Γ[T_] /; Head[T] =!= VD := Γ[VD[T]]
```

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In[ ]:=
```
$\overline{\Gamma}$[vd_VD] := FixedPoint[$\gamma$@*R12Reduce, vd, $2^{16}$]
```

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In[ ]:=
```
$\overline{\Gamma}$[T_] /; Head[T] =!= VD := $\overline{\Gamma}$[VD[T]]
```

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In[ ]:=
```
VPB[n_, {$\sigma$s___}] := VPB[n, $\sigma$s];
```

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In[ ]:=
```
VD /: vd1_VD ** vd2_VD := Module[{es1, es2, m2},
   es1 = Cases[vd1, EOS[i_] :> i];
   m2 = Max[es2 = Cases[vd2, EOS[i_] :> i]];
   Tidy[vd1 ⋃ Replace[DeleteCases[vd2, _EOS],
     i_ :> i / m2 - 1 + es1[[1 + Count[es2, e_ /; i > e]]], {2}]]
  ]
```

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In[ ]:=
```
VD[VPB[n_]] := VD @@ (EOS /@ Range[n]);
VD[VPB[n_, $\sigma_{i\_,j\_}$]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X$_{+1}$[i - 0.5, j - 0.5]];
VD[VPB[n_, $\overline{\sigma}_{i\_,j\_}$]] := Tidy@Append[VD @@ (EOS /@ Range[n]), X$_{-1}$[i - 0.5, j - 0.5]];
VD[VPB[n_, $\sigma$_, $\sigma$s__]] := VD[VPB[n, $\sigma$]] ** VD[VPB[n, $\sigma$s]]
```

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In[ ]:=
```
VPBGenerators[n_] :=
  VPBGenerators[n] = Flatten@Table[{$\sigma_{i,j}$, $\overline{\sigma}_{i,j}$}, {i, n}, {j, DeleteCases[Range@n, i]}];
```

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In[ ]:=
```
ProudFollowers[n_, $\sigma_{i\_,j\_}$] := ProudFollowers[n, $\sigma_{i,j}$] = Module[{p, q, s},
    Flatten@{$\sigma_{i,j}$, $\sigma_{j,i}$, $\overline{\sigma}_{j,i}$,
      Table[{$\sigma_{p,q}$, $\sigma_{q,p}$, $\overline{\sigma}_{p,q}$, $\overline{\sigma}_{q,p}$}, {p, {i, j}}, {q, Complement[Range[n], {i, j}]}],
      Table[{$\sigma_{p,q}$, $\overline{\sigma}_{p,q}$},
       {p, Complement[Range[i + 1, n], {j}]}, {q, Complement[Range[n], {i, j, p}]}]
     }];
ProudFollowers[n_, $\overline{\sigma}_{i\_,j\_}$] := ProudFollowers[n, $\overline{\sigma}_{i,j}$] = ProudFollowers[n, $\sigma_{i,j}$] /. $\sigma_{i,j}$ → $\overline{\sigma}_{i,j}$
```

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In[ ]:=
```
ProudVPBs[n_, 0] := {VPB[n]};
ProudVPBs[n_, 1] := VPB[n, #] & /@ VPBGenerators[n];
ProudVPBs[n_, m_] /; m > 1 := Flatten[
  ProudVPBs[n, m - 1] /. VPB[n, $\sigma$s___, $\sigma$_] :> (VPB[n, $\sigma$s, $\sigma$, #] & /@ ProudFollowers[n, $\sigma$])]
```

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In[ ]:=
```
CountOUForms[n_, m_] := Module[{k},
  Length@Union@Flatten@Table[$\overline{\Gamma}$@vpb, {k, 0, m}, {vpb, ProudVPBs[n, k]}]]
```

New Code

```mathematica
In[ ]:=  AllOUs[n_, 0] := {VD @@ Flatten@Table[{BT[2 i - 1], EOS[2 i]}, {i, n}]};
         AllOUs[n_, m_] /; m > 0 :=
          Sort@Flatten[AllOUs[n, m - 1] /. vd_VD :> Module[{BTs, EOSs, k, maxO, s},
                 BTs = Sort@Cases[vd, BT[i_] :> i];
                 EOSs = Sort@Cases[vd, EOS[i_] :> i];
                 maxO = Max[1, Max[Cases[vd, X_[i_, _] :> i]]];
                 Table[
                  Tidy[Append[vd, X_s[p - 0.5, q + 0.5]]],
                  {s, {-1, 1}}, {k, Length[BTs]},
                  {q, BTs[[k]], EOSs[[k]] - 1}, {p, Select[BTs, (# ≥ maxO) &]}
                 ]]]
```

```mathematica
In[ ]:=  AllOUs[3, 0]
```

```mathematica
Out[ ]=  {VD[BT[1], BT[3], BT[5], EOS[2], EOS[4], EOS[6]]}
```

```mathematica
In[ ]:=  AllOUs[3, 4] // Length
```

```mathematica
Out[ ]=  86400
```

```mathematica
In[ ]:=  4 × 3 × 4 Binomial[4, 2]
```

```mathematica
Out[ ]=  288
```

```mathematica
In[ ]:=  2^3 Binomial[5, 2] 5! / 2
```

```mathematica
Out[ ]=  4800
```

```mathematica
In[ ]:=  2^4 Binomial[6, 2] 6! / 2
```

```mathematica
Out[ ]=  86400
```

```mathematica
In[ ]:=  2^4 Binomial[7, 3] 7! / 3!
```

```mathematica
Out[ ]=  470400
```

```mathematica
In[ ]:=  AllROUs[n_, m_] :=
          Select[AllOUs[n, m] /. vd_VD :> Tidy@DeleteCases[vd, _BT], (# === R12Reduce[#]) &]
```

```mathematica
In[ ]:=  AllROUs[3, 4] // Length
```

```mathematica
Out[ ]=  41682
```

```mathematica
In[ ]:=  ξ[vd_VD] := Count[Γ̄[vd], X_[_, _]]
```

In[ ]:= ξ /@ **AllROUs[3, 2]**

Out[ ]= {2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
     2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
     2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
     2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
     2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}

In[ ]:=
```
VD /: (σ_{i_,j_} | vd_VD) := Switch[Order[ξ[vd], ξ[VD[VPB[Count[vd, _EOS], σ̄_{i,j}]] ** vd]],
    0, Print["OMG, Trouble!"],
    1, False, -1, True];
VD /: (σ̄_{i_,j_} | vd_VD) := Switch[Order[ξ[vd], ξ[VD[VPB[Count[vd, _EOS], σ_{i,j}]] ** vd]],
    0, Print["OMG, Trouble!"],
    1, False, -1, True];
```

In[ ]:= $\left(\# \mid \text{VD}[\text{VPB}[3, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}]]\right)$ & /@ {$\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{2,3}$}

Out[ ]= {True, False, True}

In[ ]:=
```
VD /: Divisors[vd_VD] := Select[VPBGenerators[Count[vd, _EOS]], (# | vd) &];
VD /: Quotients[vd_VD] :=
  Γ̄[VD[VPB[Count[vd, _EOS], # /. {σ → σ̄, σ̄ → σ}]] ** vd] & /@ Divisors[vd];
```

In[ ]:= {**Divisors**[$\overline{\Gamma}$[VPB[3, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{2,3}$]]], **Quotients**[$\overline{\Gamma}$[VPB[3, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{2,3}$]]]}

Out[ ]= {{$\sigma_{1,2}$, $\sigma_{2,3}$}, {VD[EOS[2], EOS[4], EOS[7], X$_1$[1, 5], X$_1$[3, 6]],
     VD[EOS[3], EOS[5], EOS[7], X$_1$[1, 6], X$_1$[2, 4]]}}

In[ ]:= **Divisors** /@ **AllROUs[3, 2]**

Out[ ]= {{}, {}, {}, {}, {}, {}, {$\overline{\sigma}_{3,2}$}, {}, {$\sigma_{3,2}$}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {},
     {}, {}, {}, {}, {}, {$\overline{\sigma}_{3,1}$}, {$\overline{\sigma}_{3,1}$}, {$\overline{\sigma}_{3,2}$}, {$\overline{\sigma}_{3,2}$}, {$\sigma_{3,2}$}, {$\sigma_{3,1}$}, {$\sigma_{3,1}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{2,3}$},
     {}, {$\sigma_{2,3}$}, {}, {}, {}, {}, {}, {$\overline{\sigma}_{3,1}$}, {$\sigma_{3,1}$}, {$\overline{\sigma}_{3,1}$}, {$\sigma_{3,1}$}, {}, {}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{2,1}$},
     {$\sigma_{2,1}$}, {$\sigma_{2,1}$}, {$\overline{\sigma}_{3,1}$}, {}, {$\sigma_{3,1}$}, {}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{2,3}$}, {$\overline{\sigma}_{2,3}$}, {$\sigma_{2,3}$}, {$\sigma_{2,1}$},
     {$\sigma_{2,1}$}, {$\sigma_{2,3}$}, {}, {}, {}, {}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{3,1}$}, {$\sigma_{3,1}$}, {$\overline{\sigma}_{3,1}$}, {$\sigma_{2,1}$}, {$\sigma_{2,1}$},
     {$\sigma_{3,1}$}, {$\overline{\sigma}_{2,1}$}, {}, {$\sigma_{2,1}$}, {}, {}, {}, {}, {}, {$\overline{\sigma}_{3,2}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{3,2}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{1,2}$},
     {$\overline{\sigma}_{1,2}$}, {$\overline{\sigma}_{3,2}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{3,2}$}, {$\sigma_{1,2}$}, {$\sigma_{1,2}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{1,3}$}, {$\overline{\sigma}_{1,3}$}, {$\overline{\sigma}_{2,3}$}, {$\sigma_{2,3}$},
     {$\overline{\sigma}_{2,3}$}, {$\sigma_{1,3}$}, {$\sigma_{1,3}$}, {$\sigma_{2,3}$}, {$\overline{\sigma}_{2,3}$}, {$\sigma_{2,3}$}, {$\overline{\sigma}_{2,3}$}, {$\sigma_{2,3}$}, {}, {}, {}, {}, {}, {}, {},
     {}, {$\overline{\sigma}_{1,2}$}, {$\overline{\sigma}_{1,2}$}, {$\sigma_{1,2}$}, {$\sigma_{1,2}$}, {$\overline{\sigma}_{1,3}$}, {$\overline{\sigma}_{1,3}$}, {$\sigma_{1,3}$}, {$\sigma_{1,3}$}, {}, {}, {}, {}, {}, {},
     {}, {}, {}, {}, {}, {}, {$\overline{\sigma}_{1,3}$}, {}, {$\sigma_{1,3}$}, {}, {$\overline{\sigma}_{1,2}$}, {$\overline{\sigma}_{1,2}$}, {$\overline{\sigma}_{1,3}$}, {$\overline{\sigma}_{1,3}$}, {$\sigma_{1,3}$},
     {$\sigma_{1,2}$}, {$\sigma_{1,2}$}, {$\sigma_{1,3}$}, {$\overline{\sigma}_{1,2}$}, {}, {$\sigma_{1,2}$}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}, {}}

In[ ]:= **Union[Divisors** /@ **AllROUs[3, 3]]**

Out[ ]= {{}, {$\sigma_{1,2}$}, {$\sigma_{1,3}$}, {$\sigma_{2,1}$}, {$\sigma_{2,3}$}, {$\sigma_{3,1}$}, {$\sigma_{3,2}$}, {$\overline{\sigma}_{1,2}$}, {$\overline{\sigma}_{1,3}$}, {$\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{2,3}$}, {$\overline{\sigma}_{3,1}$},
     {$\overline{\sigma}_{3,2}$}, {$\sigma_{1,2}$, $\sigma_{2,3}$}, {$\sigma_{1,2}$, $\sigma_{3,1}$}, {$\sigma_{1,2}$, $\overline{\sigma}_{1,3}$}, {$\sigma_{1,3}$, $\sigma_{2,1}$}, {$\sigma_{1,3}$, $\sigma_{3,2}$}, {$\sigma_{2,1}$, $\sigma_{3,2}$},
     {$\sigma_{2,1}$, $\overline{\sigma}_{2,3}$}, {$\sigma_{2,3}$, $\sigma_{3,1}$}, {$\sigma_{3,1}$, $\overline{\sigma}_{3,2}$}, {$\overline{\sigma}_{1,2}$, $\sigma_{1,3}$}, {$\overline{\sigma}_{1,2}$, $\overline{\sigma}_{2,3}$}, {$\overline{\sigma}_{1,2}$, $\overline{\sigma}_{3,1}$},
     {$\overline{\sigma}_{1,3}$, $\overline{\sigma}_{2,1}$}, {$\overline{\sigma}_{1,3}$, $\overline{\sigma}_{3,2}$}, {$\overline{\sigma}_{2,1}$, $\sigma_{2,3}$}, {$\overline{\sigma}_{2,1}$, $\overline{\sigma}_{3,2}$}, {$\overline{\sigma}_{2,3}$, $\overline{\sigma}_{3,1}$}, {$\overline{\sigma}_{3,1}$, $\sigma_{3,2}$}}

$In[\circ]:=$ **Union[Divisors /@ AllROUs[3, 5]]**

$Out[\circ]=$ $\big\{\{\}, \{\sigma_{1,2}\}, \{\sigma_{1,3}\}, \{\sigma_{2,1}\}, \{\sigma_{2,3}\}, \{\sigma_{3,1}\}, \{\sigma_{3,2}\}, \{\overline{\sigma}_{1,2}\}, \{\overline{\sigma}_{1,3}\}, \{\overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{3,1}\},$
$\{\overline{\sigma}_{3,2}\}, \{\sigma_{1,2}, \sigma_{2,3}\}, \{\sigma_{1,2}, \sigma_{3,1}\}, \{\sigma_{1,2}, \overline{\sigma}_{1,3}\}, \{\sigma_{1,3}, \sigma_{2,1}\}, \{\sigma_{1,3}, \sigma_{3,2}\}, \{\sigma_{2,1}, \sigma_{3,2}\},$
$\{\sigma_{2,1}, \overline{\sigma}_{2,3}\}, \{\sigma_{2,3}, \sigma_{3,1}\}, \{\sigma_{3,1}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{1,2}, \sigma_{1,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,1}\},$
$\{\overline{\sigma}_{1,3}, \overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,1}, \sigma_{2,3}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{3,1}, \sigma_{3,2}\}\big\}$

$In[\circ]:=$ **Union[Divisors /@ AllROUs[3, 6]]**

$In[\circ]:=$ **Union[Divisors /@ AllROUs[4, 4]]**

$Out[\circ]=$ $\big\{\{\}, \{\sigma_{1,2}\}, \{\sigma_{1,3}\}, \{\sigma_{1,4}\}, \{\sigma_{2,1}\}, \{\sigma_{2,3}\}, \{\sigma_{2,4}\}, \{\sigma_{3,1}\}, \{\sigma_{3,2}\}, \{\sigma_{3,4}\}, \{\sigma_{4,1}\}, \{\sigma_{4,2}\},$
$\{\sigma_{4,3}\}, \{\overline{\sigma}_{1,2}\}, \{\overline{\sigma}_{1,3}\}, \{\overline{\sigma}_{1,4}\}, \{\overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{2,4}\}, \{\overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{4,1}\},$
$\{\overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{4,3}\}, \{\sigma_{1,2}, \sigma_{2,3}\}, \{\sigma_{1,2}, \sigma_{2,4}\}, \{\sigma_{1,2}, \sigma_{3,1}\}, \{\sigma_{1,2}, \sigma_{3,4}\}, \{\sigma_{1,2}, \sigma_{4,1}\},$
$\{\sigma_{1,2}, \sigma_{4,3}\}, \{\sigma_{1,2}, \overline{\sigma}_{1,3}\}, \{\sigma_{1,2}, \overline{\sigma}_{1,4}\}, \{\sigma_{1,2}, \overline{\sigma}_{3,4}\}, \{\sigma_{1,2}, \overline{\sigma}_{4,3}\}, \{\sigma_{1,3}, \sigma_{2,1}\}, \{\sigma_{1,3}, \sigma_{2,4}\},$
$\{\sigma_{1,3}, \sigma_{3,2}\}, \{\sigma_{1,3}, \sigma_{3,4}\}, \{\sigma_{1,3}, \sigma_{4,1}\}, \{\sigma_{1,3}, \sigma_{4,2}\}, \{\sigma_{1,3}, \overline{\sigma}_{1,4}\}, \{\sigma_{1,3}, \overline{\sigma}_{2,4}\}, \{\sigma_{1,3}, \overline{\sigma}_{4,2}\},$
$\{\sigma_{1,4}, \sigma_{2,1}\}, \{\sigma_{1,4}, \sigma_{2,3}\}, \{\sigma_{1,4}, \sigma_{3,1}\}, \{\sigma_{1,4}, \sigma_{3,2}\}, \{\sigma_{1,4}, \sigma_{4,2}\}, \{\sigma_{1,4}, \sigma_{4,3}\}, \{\sigma_{1,4}, \overline{\sigma}_{2,3}\},$
$\{\sigma_{1,4}, \overline{\sigma}_{3,2}\}, \{\sigma_{2,1}, \sigma_{3,2}\}, \{\sigma_{2,1}, \sigma_{3,4}\}, \{\sigma_{2,1}, \sigma_{4,2}\}, \{\sigma_{2,1}, \sigma_{4,3}\}, \{\sigma_{2,1}, \overline{\sigma}_{2,3}\}, \{\sigma_{2,1}, \overline{\sigma}_{2,4}\},$
$\{\sigma_{2,1}, \overline{\sigma}_{3,4}\}, \{\sigma_{2,1}, \overline{\sigma}_{4,3}\}, \{\sigma_{2,3}, \sigma_{3,1}\}, \{\sigma_{2,3}, \sigma_{3,4}\}, \{\sigma_{2,3}, \sigma_{4,1}\}, \{\sigma_{2,3}, \sigma_{4,2}\}, \{\sigma_{2,3}, \overline{\sigma}_{2,4}\},$
$\{\sigma_{2,3}, \overline{\sigma}_{4,1}\}, \{\sigma_{2,4}, \sigma_{3,1}\}, \{\sigma_{2,4}, \sigma_{3,2}\}, \{\sigma_{2,4}, \sigma_{4,1}\}, \{\sigma_{2,4}, \sigma_{4,3}\}, \{\sigma_{2,4}, \overline{\sigma}_{3,1}\}, \{\sigma_{3,1}, \sigma_{4,2}\},$
$\{\sigma_{3,1}, \sigma_{4,3}\}, \{\sigma_{3,1}, \overline{\sigma}_{3,2}\}, \{\sigma_{3,1}, \overline{\sigma}_{3,4}\}, \{\sigma_{3,1}, \overline{\sigma}_{4,2}\}, \{\sigma_{3,2}, \sigma_{4,1}\}, \{\sigma_{3,2}, \sigma_{4,3}\}, \{\sigma_{3,2}, \overline{\sigma}_{3,4}\},$
$\{\sigma_{3,2}, \overline{\sigma}_{4,1}\}, \{\sigma_{3,4}, \sigma_{4,1}\}, \{\sigma_{3,4}, \sigma_{4,2}\}, \{\sigma_{4,1}, \overline{\sigma}_{4,2}\}, \{\sigma_{4,1}, \overline{\sigma}_{4,3}\}, \{\sigma_{4,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{1,2}, \sigma_{1,3}\},$
$\{\overline{\sigma}_{1,2}, \sigma_{1,4}\}, \{\overline{\sigma}_{1,2}, \sigma_{3,4}\}, \{\overline{\sigma}_{1,2}, \sigma_{4,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,4}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,4}\},$
$\{\overline{\sigma}_{1,2}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{1,3}, \sigma_{1,4}\}, \{\overline{\sigma}_{1,3}, \sigma_{2,4}\}, \{\overline{\sigma}_{1,3}, \sigma_{4,2}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{2,4}\},$
$\{\overline{\sigma}_{1,3}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{1,4}, \sigma_{2,3}\}, \{\overline{\sigma}_{1,4}, \sigma_{3,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{2,1}\},$
$\{\overline{\sigma}_{1,4}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{2,1}, \sigma_{2,3}\}, \{\overline{\sigma}_{2,1}, \sigma_{2,4}\},$
$\{\overline{\sigma}_{2,1}, \sigma_{3,4}\}, \{\overline{\sigma}_{2,1}, \sigma_{4,3}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{2,3}, \sigma_{2,4}\},$
$\{\overline{\sigma}_{2,3}, \sigma_{4,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{2,4}, \sigma_{3,1}\},$
$\{\overline{\sigma}_{2,4}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,1}, \sigma_{3,2}\}, \{\overline{\sigma}_{3,1}, \sigma_{3,4}\},$
$\{\overline{\sigma}_{3,1}, \sigma_{4,2}\}, \{\overline{\sigma}_{3,1}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{3,1}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,2}, \sigma_{3,4}\}, \{\overline{\sigma}_{3,2}, \sigma_{4,1}\}, \{\overline{\sigma}_{3,2}, \overline{\sigma}_{4,1}\},$
$\{\overline{\sigma}_{3,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,4}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{3,4}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{4,1}, \sigma_{4,2}\}, \{\overline{\sigma}_{4,1}, \sigma_{4,3}\}, \{\overline{\sigma}_{4,2}, \sigma_{4,3}\}\big\}$

$In[\circ]:=$ **MinQ[*divs*_] :=**
    Sort[Sort[*divs* /. Thread[Range@4 → #]] & /@ Permutations[Range@4]]⟦1⟧ === *divs*

In[•]:= **Select**$\Big[\{\{\}, \{\sigma_{1,2}\}, \{\sigma_{1,3}\}, \{\sigma_{1,4}\}, \{\sigma_{2,1}\}, \{\sigma_{2,3}\}, \{\sigma_{2,4}\}, \{\sigma_{3,1}\}, \{\sigma_{3,2}\}, \{\sigma_{3,4}\}, \{\sigma_{4,1}\},$
$\{\sigma_{4,2}\}, \{\sigma_{4,3}\}, \{\overline{\sigma}_{1,2}\}, \{\overline{\sigma}_{1,3}\}, \{\overline{\sigma}_{1,4}\}, \{\overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{2,4}\}, \{\overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{3,4}\},$
$\{\overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{4,3}\}, \{\sigma_{1,2}, \sigma_{2,3}\}, \{\sigma_{1,2}, \sigma_{2,4}\}, \{\sigma_{1,2}, \sigma_{3,1}\}, \{\sigma_{1,2}, \sigma_{3,4}\}, \{\sigma_{1,2}, \sigma_{4,1}\},$
$\{\sigma_{1,2}, \sigma_{4,3}\}, \{\sigma_{1,2}, \overline{\sigma}_{1,3}\}, \{\sigma_{1,2}, \overline{\sigma}_{1,4}\}, \{\sigma_{1,2}, \overline{\sigma}_{3,4}\}, \{\sigma_{1,2}, \overline{\sigma}_{4,3}\}, \{\sigma_{1,3}, \sigma_{2,1}\}, \{\sigma_{1,3}, \sigma_{2,4}\},$
$\{\sigma_{1,3}, \sigma_{3,2}\}, \{\sigma_{1,3}, \sigma_{3,4}\}, \{\sigma_{1,3}, \sigma_{4,1}\}, \{\sigma_{1,3}, \sigma_{4,2}\}, \{\sigma_{1,3}, \overline{\sigma}_{1,4}\}, \{\sigma_{1,3}, \overline{\sigma}_{2,4}\}, \{\sigma_{1,3}, \overline{\sigma}_{4,2}\},$
$\{\sigma_{1,4}, \sigma_{2,1}\}, \{\sigma_{1,4}, \sigma_{2,3}\}, \{\sigma_{1,4}, \sigma_{3,1}\}, \{\sigma_{1,4}, \sigma_{3,2}\}, \{\sigma_{1,4}, \sigma_{4,2}\}, \{\sigma_{1,4}, \sigma_{4,3}\}, \{\sigma_{1,4}, \overline{\sigma}_{2,3}\},$
$\{\sigma_{1,4}, \overline{\sigma}_{3,2}\}, \{\sigma_{2,1}, \sigma_{3,2}\}, \{\sigma_{2,1}, \sigma_{3,4}\}, \{\sigma_{2,1}, \sigma_{4,2}\}, \{\sigma_{2,1}, \sigma_{4,3}\}, \{\sigma_{2,1}, \overline{\sigma}_{2,3}\}, \{\sigma_{2,1}, \overline{\sigma}_{2,4}\},$
$\{\sigma_{2,1}, \overline{\sigma}_{3,4}\}, \{\sigma_{2,1}, \overline{\sigma}_{4,3}\}, \{\sigma_{2,3}, \sigma_{3,1}\}, \{\sigma_{2,3}, \sigma_{3,4}\}, \{\sigma_{2,3}, \sigma_{4,1}\}, \{\sigma_{2,3}, \sigma_{4,2}\}, \{\sigma_{2,3}, \overline{\sigma}_{2,4}\},$
$\{\sigma_{2,3}, \overline{\sigma}_{4,1}\}, \{\sigma_{2,4}, \sigma_{3,1}\}, \{\sigma_{2,4}, \sigma_{3,2}\}, \{\sigma_{2,4}, \sigma_{4,1}\}, \{\sigma_{2,4}, \sigma_{4,3}\}, \{\sigma_{2,4}, \overline{\sigma}_{3,1}\}, \{\sigma_{3,1}, \sigma_{4,2}\},$
$\{\sigma_{3,1}, \sigma_{4,3}\}, \{\sigma_{3,1}, \overline{\sigma}_{3,2}\}, \{\sigma_{3,1}, \overline{\sigma}_{3,4}\}, \{\sigma_{3,1}, \overline{\sigma}_{4,2}\}, \{\sigma_{3,2}, \sigma_{4,1}\}, \{\sigma_{3,2}, \sigma_{4,3}\}, \{\sigma_{3,2}, \overline{\sigma}_{3,4}\},$
$\{\sigma_{3,2}, \overline{\sigma}_{4,1}\}, \{\sigma_{3,4}, \sigma_{4,1}\}, \{\sigma_{3,4}, \sigma_{4,2}\}, \{\sigma_{4,1}, \overline{\sigma}_{4,2}\}, \{\sigma_{4,1}, \overline{\sigma}_{4,3}\}, \{\sigma_{4,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{1,2}, \sigma_{1,3}\},$
$\{\overline{\sigma}_{1,2}, \sigma_{1,4}\}, \{\overline{\sigma}_{1,2}, \sigma_{3,4}\}, \{\overline{\sigma}_{1,2}, \sigma_{4,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,4}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,4}\},$
$\{\overline{\sigma}_{1,2}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{1,3}, \sigma_{1,4}\}, \{\overline{\sigma}_{1,3}, \sigma_{2,4}\}, \{\overline{\sigma}_{1,3}, \sigma_{4,2}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{2,1}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{2,4}\},$
$\{\overline{\sigma}_{1,3}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{1,3}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{1,4}, \sigma_{2,3}\}, \{\overline{\sigma}_{1,4}, \sigma_{3,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{2,1}\},$
$\{\overline{\sigma}_{1,4}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{1,4}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{2,1}, \sigma_{2,3}\}, \{\overline{\sigma}_{2,1}, \sigma_{2,4}\},$
$\{\overline{\sigma}_{2,1}, \sigma_{3,4}\}, \{\overline{\sigma}_{2,1}, \sigma_{4,3}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{2,1}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{2,3}, \sigma_{2,4}\},$
$\{\overline{\sigma}_{2,3}, \sigma_{4,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{2,3}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{2,4}, \sigma_{3,1}\},$
$\{\overline{\sigma}_{2,4}, \overline{\sigma}_{3,1}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{3,2}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{2,4}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,1}, \sigma_{3,2}\}, \{\overline{\sigma}_{3,1}, \sigma_{3,4}\},$
$\{\overline{\sigma}_{3,1}, \sigma_{4,2}\}, \{\overline{\sigma}_{3,1}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{3,1}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,2}, \sigma_{3,4}\}, \{\overline{\sigma}_{3,2}, \sigma_{4,1}\}, \{\overline{\sigma}_{3,2}, \overline{\sigma}_{4,1}\},$
$\{\overline{\sigma}_{3,2}, \overline{\sigma}_{4,3}\}, \{\overline{\sigma}_{3,4}, \overline{\sigma}_{4,1}\}, \{\overline{\sigma}_{3,4}, \overline{\sigma}_{4,2}\}, \{\overline{\sigma}_{4,1}, \sigma_{4,2}\}, \{\overline{\sigma}_{4,1}, \sigma_{4,3}\}, \{\overline{\sigma}_{4,2}, \sigma_{4,3}\}\}, \text{MinQ}\Big]$

Out[•]= $\Big\{\{\}, \{\sigma_{1,2}\}, \{\overline{\sigma}_{1,2}\}, \{\sigma_{1,2}, \sigma_{2,3}\}, \{\sigma_{1,2}, \sigma_{3,4}\},$
$\{\sigma_{1,2}, \overline{\sigma}_{1,3}\}, \{\sigma_{1,2}, \overline{\sigma}_{3,4}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{2,3}\}, \{\overline{\sigma}_{1,2}, \overline{\sigma}_{3,4}\}\Big\}$

In[•]:=
```
OUGraph[n_, m_] := Module[{gens, OUs, k, d, g, q, m1, m2},
   gens = VPBGenerators[n];
   OUs = Flatten@Table[AllROUs[n, k], {k, 0, m}];
   OURule = Dispatch@Thread[OUs → Range@Length@OUs];
   Graph[
    Range@Length@OUs,
    Union@Flatten@Table[
       m1 = Count[d, X_[_, _]];
       m2 = Count[q = Γ̅[VD[VPB[n, g]] ** d], X_[_, _]];
       If[m2 < m1, Labeled[(d ↦ q) /. OURule, g], Nothing],
       {d, OUs}, {g, gens}
       ]
    ]
   ]
```

In[•]:= $\overline{\Gamma}[\text{VPB}[3, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}]]$

Out[•]= $\text{VD}[\text{EOS}[7], \text{EOS}[14], \text{EOS}[17], X_1[1, 16],$
$X_1[2, 13], X_1[3, 6], X_1[4, 11], X_1[8, 15], X_1[9, 12], X_1[10, 5]]$

In[•]:= $\overline{\Gamma}\big[\text{VPB}\big[3, \overline{\sigma}_{2,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}\big]\big]$

Out[•]= $\text{VD}[\text{EOS}[7], \text{EOS}[13], \text{EOS}[15], X_1[1, 14], X_1[2, 12], X_1[3, 6], X_1[4, 10], X_1[8, 11], X_1[9, 5]]$

$In[\bullet]:=$ $\overline{\Gamma}[$VPB$[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}]]$

$Out[\bullet]=$ VD[EOS[7], EOS[14], EOS[17], X$_1$[1, 16],
    X$_1$[2, 13], X$_1$[3, 6], X$_1$[4, 11], X$_1$[8, 15], X$_1$[9, 12], X$_1$[10, 5]]

$In[\bullet]:=$ $\overline{\Gamma}[$VPB$[3, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]]$

$Out[\bullet]=$ VD[EOS[9], EOS[18], EOS[25], X$_{-1}$[4, 20], X$_{-1}$[10, 19], X$_{-1}$[14, 21], X$_1$[1, 17],
    X$_1$[2, 24], X$_1$[3, 8], X$_1$[5, 15], X$_1$[6, 22], X$_1$[11, 16], X$_1$[12, 23], X$_1$[13, 7]]

$In[\bullet]:=$ $\overline{\Gamma}[$VPB$[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]]$

$Out[\bullet]=$ VD[EOS[9], EOS[17], EOS[23], X$_{-1}$[4, 18], X$_{-1}$[13, 19], X$_1$[1, 16],
    X$_1$[2, 22], X$_1$[3, 8], X$_1$[5, 14], X$_1$[6, 20], X$_1$[10, 15], X$_1$[11, 21], X$_1$[12, 7]]

$In[\bullet]:=$ $\overline{\Gamma}[$VPB$[3, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}]]$

$Out[\bullet]=$ VD[EOS[9], EOS[17], EOS[23], X$_{-1}$[4, 18], X$_{-1}$[13, 19], X$_1$[1, 16],
    X$_1$[2, 22], X$_1$[3, 8], X$_1$[5, 14], X$_1$[6, 20], X$_1$[10, 15], X$_1$[11, 21], X$_1$[12, 7]]

$In[\bullet]:=$ **chb1 = VPB$[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]$; chb2 = VPB$[3, \overline{\sigma}_{2,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}]$;**
    {chb3, chb4} = {chb1, chb2} /. $\{\sigma \to \overline{\sigma}, \overline{\sigma} \to \sigma\}$

$Out[\bullet]=$ $\{$VPB$[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \sigma_{2,3}]$, VPB$[3, \sigma_{2,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,3}]\}$

$In[\bullet]:=$ $\overline{\Gamma}[$**chb1**$]$

$Out[\bullet]=$ VD[EOS[9], EOS[17], EOS[23], X$_{-1}$[4, 18], X$_{-1}$[13, 19], X$_1$[1, 16],
    X$_1$[2, 22], X$_1$[3, 8], X$_1$[5, 14], X$_1$[6, 20], X$_1$[10, 15], X$_1$[11, 21], X$_1$[12, 7]]

$In[\bullet]:=$ **Divisors$[\overline{\Gamma}[$chb1$]]$**

$Out[\bullet]=$ $\{\sigma_{1,2}\}$

$In[\bullet]:=$ **Quotients$[\overline{\Gamma}[$chb1$]]$**

$Out[\bullet]=$ {VD[EOS[4], EOS[11], EOS[15], X$_{-1}$[8, 12], X$_1$[1, 9], X$_1$[2, 13], X$_1$[5, 10], X$_1$[6, 14], X$_1$[7, 3]]}

$In[\bullet]:=$ **Divisors /@ Quotients$[\overline{\Gamma}[$chb1$]]$**

$Out[\bullet]=$ $\{\{\sigma_{2,1}\}\}$

$In[\bullet]:=$ **Flatten$[$Quotients /@ Quotients$[\overline{\Gamma}[$chb1$]]]$**

$Out[\bullet]=$ {VD[EOS[3], EOS[5], EOS[7], X$_1$[1, 4], X$_1$[2, 6]]}

$In[\bullet]:=$ **Divisors /@ Flatten$[$Quotients /@ Quotients$[\overline{\Gamma}[$chb1$]]]$**

$Out[\bullet]=$ $\{\{\sigma_{1,2}\}\}$

$In[\bullet]:=$ **Quotients /@ Flatten$[$Quotients /@ Quotients$[\overline{\Gamma}[$chb1$]]]$**

$Out[\bullet]=$ {{VD[EOS[2], EOS[3], EOS[5], X$_1$[1, 4]]}}

$In[\bullet]:=$ $\overline{\Gamma}[$**chb2**$]$

$Out[\bullet]=$ VD[EOS[7], EOS[13], EOS[15], X$_1$[1, 14], X$_1$[2, 12], X$_1$[3, 6], X$_1$[4, 10], X$_1$[8, 11], X$_1$[9, 5]]

$In[\bullet]:=$ **Divisors$\big[\overline{\Gamma}[\text{chb2}]\big]$**

$Out[\bullet]=$ $\{\sigma_{1,3}\}$

$In[\bullet]:=$ **$\overline{\Gamma}[\text{chb3}]$**

$Out[\bullet]=$ VD[EOS[9], EOS[17], EOS[23], X$_{-1}$[1, 16], X$_{-1}$[2, 22], X$_{-1}$[3, 8], X$_{-1}$[5, 14],
　　　X$_{-1}$[6, 20], X$_{-1}$[10, 15], X$_{-1}$[11, 21], X$_{-1}$[12, 7], X$_{1}$[4, 18], X$_{1}$[13, 19]]

$In[\bullet]:=$ **Divisors$\big[\overline{\Gamma}[\text{chb3}]\big]$**

$Out[\bullet]=$ $\{\overline{\sigma}_{1,2}\}$

$In[\bullet]:=$ **Divisors /@ Quotients$\big[\overline{\Gamma}[\text{chb3}]\big]$**

$Out[\bullet]=$ $\big\{\{\overline{\sigma}_{2,1}\}\big\}$

$In[\bullet]:=$ **Divisors /@ Flatten$\big[$Quotients /@ Quotients$\big[\overline{\Gamma}[\text{chb3}]\big]\big]$**

$Out[\bullet]=$ $\big\{\{\overline{\sigma}_{1,2}\}\big\}$

$In[\bullet]:=$ **Quotients /@ Flatten$\big[$Quotients /@ Quotients$\big[\overline{\Gamma}[\text{chb3}]\big]\big]$**

$Out[\bullet]=$ {{VD[EOS[2], EOS[3], EOS[5], X$_{-1}$[1, 4]]}}

$In[\bullet]:=$ **$\overline{\Gamma}[\text{chb4}]$**

$Out[\bullet]=$ VD[EOS[7], EOS[13], EOS[15], X$_{-1}$[1, 14],
　　　X$_{-1}$[2, 12], X$_{-1}$[3, 6], X$_{-1}$[4, 10], X$_{-1}$[8, 11], X$_{-1}$[9, 5]]

$In[\bullet]:=$ **Divisors$\big[\overline{\Gamma}[\text{chb4}]\big]$**

$Out[\bullet]=$ $\{\overline{\sigma}_{1,3}\}$

$In[\bullet]:=$
```
ExtractVPB[vd_VD] := Module[{n, ds, d},
    n = Count[vd, _EOS];
    If[Length[ds = Divisors[vd]] == 0, VPB[n],
      d = First@Sort[ds];
      q = Γ̄[VD[VPB[n, d /. {σ → σ̄, σ̄ → σ}]] ** vd];
      Insert[ExtractVPB[q], d, 2]
    ]];
 CF[vpb_VPB] := ExtractVPB[Γ̄[vpb]];
```

$In[\bullet]:=$ **{chb1, CF[chb1]}**

$Out[\bullet]=$ $\Big\{$VPB$\big[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}\big]$, VPB$\big[3, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}\big]\Big\}$

$In[\bullet]:=$ **{chb2, CF[chb2]}**

$Out[\bullet]=$ $\Big\{$VPB$\big[3, \overline{\sigma}_{2,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}\big]$, VPB$\big[3, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}\big]\Big\}$

$In[\bullet]:=$ **{chb3, CF[chb3]}**

$Out[\bullet]=$ $\Big\{$VPB$\big[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \sigma_{2,3}\big]$, VPB$\big[3, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}\big]\Big\}$

$In[\bullet]:=$ **{chb4, CF[chb4]}**

$Out[\bullet]=$ $\Big\{$VPB$\big[3, \sigma_{2,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,3}\big]$, VPB$\big[3, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}\big]\Big\}$

```
In[ ]:= Table[{vpb = Module[{n = 4, m = 12, i},
          VPB[n, Sequence @@
            Table[RandomChoice[{σ, σ̄}]_{i=RandomChoice@Range@n,RandomChoice[Complement[Range@n,{i}]]}, {m}]]],
        cf = CF[vpb], Length[cf] - 1}, {30}] // MatrixForm
```
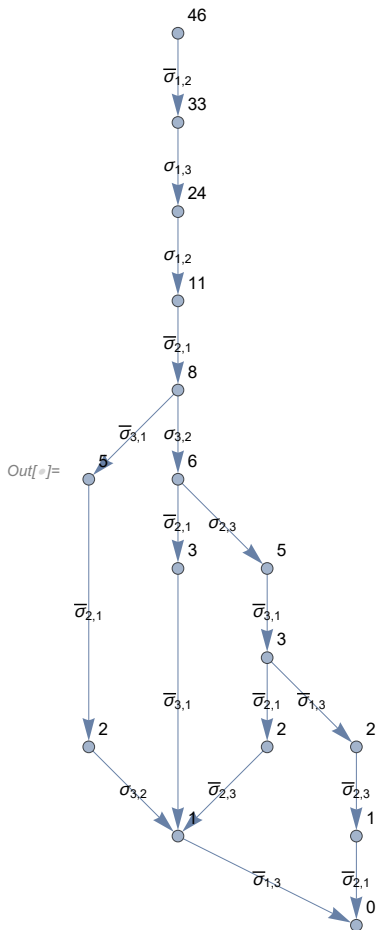
Out[ ]//MatrixForm=

$$
\begin{array}{ll}
\text{VPB}\left[4,\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{1,2},\ \sigma_{1,2},\ \sigma_{1,2},\ \sigma_{2,4},\ \sigma_{1,4},\ \sigma_{3,2},\ \overline{\sigma}_{3,4}\right] & \text{VPB}\left[4,\ \overline{\sigma}_3\right. \\
\text{VPB}\left[4,\ \sigma_{4,3},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{2,1},\ \sigma_{2,3},\ \sigma_{2,1},\ \sigma_{3,1},\ \sigma_{1,2},\ \sigma_{3,4},\ \sigma_{4,3},\ \sigma_{2,1},\ \overline{\sigma}_{4,3}\right] & \text{VPB}\left[\right. \\
\text{VPB}\left[4,\ \sigma_{1,2},\ \sigma_{3,1},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{3,2},\ \sigma_{4,2},\ \sigma_{4,3},\ \sigma_{3,1},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{4,2},\ \sigma_{2,3},\ \overline{\sigma}_{2,1}\right] & \text{VPB}\left[4,\ \sigma_{1,2},\ \sigma_3\right. \\
\text{VPB}\left[4,\ \sigma_{3,2},\ \overline{\sigma}_{2,3},\ \sigma_{4,3},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{3,1},\ \sigma_{2,4},\ \sigma_{3,1},\ \sigma_{4,3},\ \sigma_{1,3},\ \sigma_{2,1},\ \overline{\sigma}_{4,2},\ \sigma_{3,2}\right] & \text{VPB}\left[\right. \\
\text{VPB}\left[4,\ \sigma_{1,4},\ \sigma_{1,3},\ \sigma_{4,3},\ \overline{\sigma}_{4,3},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{3,1},\ \sigma_{1,3},\ \overline{\sigma}_{4,2},\ \sigma_{3,2},\ \overline{\sigma}_{4,3},\ \sigma_{4,1}\right] & \text{VPB}\left[\right. \\
\text{VPB}\left[4,\ \sigma_{4,1},\ \sigma_{2,4},\ \sigma_{2,3},\ \overline{\sigma}_{4,2},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{4,2},\ \sigma_{2,3},\ \overline{\sigma}_{1,4},\ \sigma_{1,3},\ \overline{\sigma}_{4,1},\ \sigma_{2,1}\right] & \text{VPB}\left[4,\ \sigma_{4,1},\ \sigma_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{2,3},\ \sigma_{2,3},\ \sigma_{4,1},\ \overline{\sigma}_{1,3},\ \sigma_{4,1},\ \overline{\sigma}_{2,4},\ \sigma_{1,4},\ \sigma_{1,2},\ \overline{\sigma}_{4,3},\ \sigma_{2,3},\ \sigma_{3,2}\right] & \text{VPB}\left[4,\ \sigma_{3,1},\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{2,3},\ \sigma_{2,1},\ \overline{\sigma}_{3,1},\ \sigma_{3,1},\ \overline{\sigma}_{3,1},\ \sigma_{3,2},\ \overline{\sigma}_{4,1},\ \sigma_{3,1},\ \sigma_{1,3},\ \overline{\sigma}_{2,1},\ \sigma_{3,1}\right] & \text{VPB}\left[\right. \\
\text{VPB}\left[4,\ \sigma_{2,4},\ \overline{\sigma}_{4,2},\ \sigma_{3,2},\ \overline{\sigma}_{4,2},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{3,4},\ \sigma_{4,1},\ \sigma_{1,2},\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{3,1},\ \sigma_{3,2}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{4,1},\ \sigma_{3,1},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{2,3},\ \sigma_{3,2},\ \sigma_{2,4},\ \sigma_{3,2},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{4,3},\ \overline{\sigma}_{1,4},\ \sigma_{1,4},\ \overline{\sigma}_{2,1}\right] & \text{VPB}\left[4,\ \overline{\sigma}_3\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{3,4},\ \sigma_{4,1},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{1,3},\ \sigma_{2,1},\ \overline{\sigma}_{3,1},\ \sigma_{4,1},\ \overline{\sigma}_{4,3},\ \sigma_{1,3},\ \sigma_{1,3},\ \sigma_{4,1},\ \sigma_{3,2}\right] & \text{VPB}\left[4,\ \sigma_{3,1},\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{2,3},\ \sigma_{3,1},\ \overline{\sigma}_{2,4},\ \sigma_{1,3},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{4,3},\ \sigma_{4,2},\ \overline{\sigma}_{3,4},\ \sigma_{3,4},\ \sigma_{2,4},\ \overline{\sigma}_{1,3}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \sigma_{1,4},\ \overline{\sigma}_{3,2},\ \sigma_{4,3},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{1,4},\ \sigma_{4,3},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{4,2},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{3,4}\right] & \text{VPB}\left[4,\ \sigma_{1,4},\ \overline{\sigma}_3\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{1,2},\ \sigma_{4,3},\ \sigma_{4,2},\ \sigma_{4,1},\ \sigma_{4,2},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{1,3},\ \sigma_{2,4},\ \sigma_{4,3}\right] & \text{VPB}\left[4,\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{3,4},\ \sigma_{3,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{1,3},\ \sigma_{3,2},\ \sigma_{3,4},\ \sigma_{2,1},\ \overline{\sigma}_{3,2},\ \sigma_{2,4},\ \overline{\sigma}_{2,3}\right] & \text{VPB}\left[4,\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_3\right. \\
\text{VPB}\left[4,\ \sigma_{3,2},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{4,2},\ \sigma_{1,2},\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{2,3},\ \sigma_{2,4},\ \sigma_{2,3},\ \sigma_{4,1}\right] & \text{VPB}\left[4,\ \sigma_3\right. \\
\text{VPB}\left[4,\ \sigma_{2,3},\ \sigma_{4,1},\ \overline{\sigma}_{4,2},\ \overline{\sigma}_{4,2},\ \overline{\sigma}_{4,2},\ \sigma_{2,4},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{1,4},\ \sigma_{1,3},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{2,4},\ \sigma_{3,4}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{2,1},\ \sigma_{2,4},\ \sigma_{4,2},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{3,4},\ \sigma_{3,1},\ \overline{\sigma}_{1,4},\ \sigma_{3,4},\ \overline{\sigma}_{4,3}\right] & \text{VPB}\left[4,\ \overline{\sigma}_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{1,4},\ \sigma_{3,4},\ \overline{\sigma}_{2,3},\ \sigma_{4,2},\ \overline{\sigma}_{2,3},\ \sigma_{3,4},\ \sigma_{3,2},\ \sigma_{3,1},\ \overline{\sigma}_{1,2},\ \sigma_{1,4},\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{3,2}\right] & \text{VPB}\left[4,\ \overline{\sigma}_{1,4},\ \sigma_3\right. \\
\text{VPB}\left[4,\ \sigma_{3,1},\ \sigma_{3,1},\ \sigma_{2,4},\ \sigma_{3,2},\ \overline{\sigma}_{3,1},\ \sigma_{2,3},\ \overline{\sigma}_{4,3},\ \sigma_{4,1},\ \sigma_{4,3},\ \sigma_{2,4},\ \sigma_{4,1},\ \sigma_{1,4}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \sigma_{2,3},\ \sigma_{2,4},\ \sigma_{3,1},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{1,3},\ \sigma_{3,4},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{2,1},\ \sigma_{1,2},\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{2,3}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{3,2},\ \sigma_{3,1},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{4,3},\ \sigma_{1,2},\ \overline{\sigma}_{2,1},\ \sigma_{4,3},\ \overline{\sigma}_{2,3}\right] & \\
\text{VPB}\left[4,\ \sigma_{1,4},\ \sigma_{4,1},\ \overline{\sigma}_{4,3},\ \sigma_{1,4},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{2,1},\ \sigma_{4,3},\ \sigma_{2,3},\ \sigma_{3,2},\ \sigma_{2,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{3,2}\right] & \text{VPB}\left[4,\ \sigma_1\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{3,1},\ \sigma_{1,2},\ \sigma_{4,3},\ \sigma_{4,1},\ \sigma_{2,4},\ \overline{\sigma}_{2,3},\ \sigma_{3,1},\ \overline{\sigma}_{3,4},\ \sigma_{2,3},\ \sigma_{4,2},\ \overline{\sigma}_{2,1}\right] & \text{VPB}\left[4,\ \sigma_{3,2},\ \sigma_{4,2},\ \sigma_{1,2},\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{4,3},\ \sigma_{2,4},\ \sigma_{3,2},\ \sigma_{1,2},\ \sigma_{1,2},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{3,1},\ \sigma_{4,2},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{4,2},\ \sigma_{2,3}\right] & \text{VPB}\left[4,\ \overline{\sigma}_{4,3},\ \sigma_2\right. \\
\text{VPB}\left[4,\ \sigma_{3,1},\ \overline{\sigma}_{4,3},\ \sigma_{4,2},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{3,2},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{4,3},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{1,2},\ \sigma_{2,3}\right] & \text{VPB}\left[4,\ \sigma_{3,1},\ \overline{\sigma}_{4,3},\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{3,4},\ \sigma_{1,3},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{4,1},\ \overline{\sigma}_{1,3},\ \sigma_{1,3},\ \overline{\sigma}_{1,4},\ \sigma_{2,1},\ \overline{\sigma}_{4,3},\ \sigma_{3,2},\ \sigma_{1,4}\right] & \text{VPB}\left[\right. \\
\text{VPB}\left[4,\ \sigma_{4,3},\ \sigma_{1,4},\ \overline{\sigma}_{1,3},\ \sigma_{2,3},\ \overline{\sigma}_{1,3},\ \sigma_{4,1},\ \sigma_{2,1},\ \sigma_{1,4},\ \sigma_{2,1},\ \overline{\sigma}_{1,3},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{1,4}\right] & \text{VPB}\left[4,\ \sigma_4\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{3,1},\ \sigma_{2,4},\ \sigma_{2,3},\ \sigma_{4,2},\ \overline{\sigma}_{4,1},\ \sigma_{4,3},\ \sigma_{4,3},\ \sigma_{1,3},\ \overline{\sigma}_{2,4},\ \overline{\sigma}_{1,2},\ \sigma_{4,1}\right] & \text{VPB}\left[4,\ \sigma_2\right. \\
\text{VPB}\left[4,\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{4,1},\ \sigma_{2,4},\ \sigma_{1,4},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{2,3},\ \sigma_{1,3},\ \overline{\sigma}_{1,4},\ \overline{\sigma}_{4,3}\right] & \text{VPB}\left[4,\ \sigma_{3,1},\ \overline{\sigma}_1\right.
\end{array}
$$

In[•]:= `Table[{vpb = Module[{n = 3, m = 10, i},`
     `VPB[n, Sequence @@`
       `Table[RandomChoice[{σ, σ̄}]`$_{i=\text{RandomChoice@Range@n,RandomChoice[Complement[Range@n,\{i\}]]}}$`, {m}]]],`
   `cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm`

Out[•]//MatrixForm=

$$\begin{pmatrix}
\text{VPB}[3, \sigma_{1,2}, \sigma_{2,1}, \overline{\sigma}_{2,1}, \sigma_{3,1}, \overline{\sigma}_{2,3}, \overline{\sigma}_{2,1}, \sigma_{2,1}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,3}, \sigma_{3,1}] & \text{VPB}[3, \sigma_{1,2}, \sigma_{3,1}, \overline{\sigma}_2] \\
\text{VPB}[3, \sigma_{3,1}, \sigma_{1,3}, \overline{\sigma}_{3,1}, \sigma_{3,1}, \overline{\sigma}_{1,3}, \overline{\sigma}_{3,1}, \overline{\sigma}_{3,1}, \sigma_{2,1}, \overline{\sigma}_{2,1}, \sigma_{3,2}] & \text{VPB}[3, \overline{\sigma}_3] \\
\text{VPB}[3, \overline{\sigma}_{3,1}, \overline{\sigma}_{1,3}, \sigma_{3,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,2}, \sigma_{2,3}, \sigma_{2,3}, \sigma_{1,3}, \overline{\sigma}_{2,1}] & \text{VPB}[3, \overline{\sigma}_{3,1}, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,1}, \overline{\sigma}_3] \\
\text{VPB}[3, \overline{\sigma}_{1,3}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{1,3}, \overline{\sigma}_{2,3}, \overline{\sigma}_{2,1}, \overline{\sigma}_{2,3}, \sigma_{2,3}, \sigma_{2,3}] & \text{VPB}[3, \overline{\sigma}_{1,3}, \sigma_{1,2}, \sigma_{1,3}, \sigma_1] \\
\text{VPB}[3, \overline{\sigma}_{2,3}, \sigma_{2,3}, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \sigma_{3,1}, \overline{\sigma}_{3,1}, \sigma_{1,2}, \overline{\sigma}_{3,1}, \sigma_{2,1}, \sigma_{2,1}] & \text{VPB}[3, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \sigma_1] \\
\text{VPB}[3, \sigma_{3,2}, \overline{\sigma}_{3,1}, \overline{\sigma}_{3,2}, \sigma_{3,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,1}, \overline{\sigma}_{3,1}, \sigma_{2,1}, \overline{\sigma}_{2,3}] & \text{VPB}[3, \sigma_{3,2}, \overline{\sigma}_{3,1}, \overline{\sigma}_{3,2}, \overline{\sigma}_2] \\
\text{VPB}[3, \sigma_{1,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{3,1}, \sigma_{3,1}, \sigma_{3,2}, \sigma_{2,1}, \overline{\sigma}_{1,2}, \sigma_{2,1}, \sigma_{2,1}, \overline{\sigma}_{2,1}] & \text{VPB}[3, \sigma_{1,2}, \sigma_{1,2}, \sigma_{3,2}, \overline{\sigma}_1] \\
\text{VPB}[3, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \sigma_{2,1}, \sigma_{3,1}, \sigma_{2,1}, \sigma_{2,1}, \overline{\sigma}_{3,1}, \overline{\sigma}_{2,1}] & \text{VPB}[3, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}, \sigma_{3,1}, \sigma_{2,1}, \sigma_3] \\
\text{VPB}[3, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,2}, \sigma_{2,1}, \overline{\sigma}_{2,3}, \sigma_{3,2}, \overline{\sigma}_{3,1}, \sigma_{2,3}, \sigma_{1,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,2}] & \text{VPB}[3, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,2}, \sigma_{2,1}, \overline{\sigma}_{2,3}, \sigma_3] \\
\text{VPB}[3, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,1}, \sigma_{3,2}, \overline{\sigma}_{1,2}, \sigma_{1,3}, \overline{\sigma}_{2,3}, \sigma_{1,2}, \overline{\sigma}_{1,2}, \sigma_{1,3}, \sigma_{1,2}] & \text{VPB}[3, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,1}, \sigma_{3,2}, \overline{\sigma}_1] \\
\text{VPB}[3, \sigma_{3,2}, \overline{\sigma}_{3,2}, \sigma_{2,1}, \overline{\sigma}_{3,1}, \sigma_{2,3}, \sigma_{1,2}, \overline{\sigma}_{1,2}, \sigma_{2,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{3,1}] & \text{VPB}[3, \sigma_{2,1}, \overline{\sigma}_{3,1}, \sigma_2] \\
\text{VPB}[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,2}, \sigma_{3,1}, \sigma_{2,1}, \overline{\sigma}_{2,3}, \sigma_{3,2}, \overline{\sigma}_{1,3}, \sigma_{3,1}, \overline{\sigma}_{2,3}] & \text{VPB}[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,2}, \sigma_{3,1}, \sigma_2] \\
\text{VPB}[3, \sigma_{2,1}, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,3}, \sigma_{2,1}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{3,1}] & \text{VPB}[3, \sigma_{2,1}, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,2}, \overline{\sigma}_2] \\
\text{VPB}[3, \sigma_{3,2}, \sigma_{3,2}, \overline{\sigma}_{3,1}, \sigma_{2,1}, \overline{\sigma}_{3,2}, \sigma_{1,2}, \sigma_{1,3}, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \sigma_{3,2}] & \text{VPB}[3, \sigma_{3,2}, \sigma_{3,2}, \overline{\sigma}_{3,1}, \sigma_{2,1}, \overline{\sigma}_{3,2}, \sigma_1] \\
\text{VPB}[3, \sigma_{3,2}, \overline{\sigma}_{2,1}, \sigma_{3,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{3,2}, \sigma_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \sigma_{1,3}] & \text{VPB}[3, \sigma_{3,2}, \overline{\sigma}_{2,1}, \sigma_{3,1}, \overline{\sigma}_1] \\
\text{VPB}[3, \overline{\sigma}_{2,3}, \sigma_{2,3}, \overline{\sigma}_{2,3}, \sigma_{3,1}, \overline{\sigma}_{2,3}, \overline{\sigma}_{2,1}, \sigma_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{3,2}, \overline{\sigma}_{3,2}] & \text{VPB}[3, \sigma_{2,1}, \sigma_{3,1}, \sigma_{1,3}, \overline{\sigma}_{2,1}, \overline{\sigma}_2] \\
\text{VPB}[3, \overline{\sigma}_{1,2}, \sigma_{3,1}, \overline{\sigma}_{3,2}, \overline{\sigma}_{2,3}, \overline{\sigma}_{3,1}, \overline{\sigma}_{1,2}, \sigma_{3,1}, \sigma_{3,2}, \sigma_{3,2}, \overline{\sigma}_{1,3}] & \text{VPB}[3, \overline{\sigma}_{1,2}, \sigma_{3,1}, \overline{\sigma}_{3,2}, \overline{\sigma}_{2,3}, \overline{\sigma}_3] \\
\text{VPB}[3, \sigma_{1,2}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,3}, \overline{\sigma}_{1,2}, \sigma_{1,3}, \sigma_{3,2}, \sigma_{3,1}, \sigma_{1,2}, \overline{\sigma}_{3,2}] & \text{VPB}[3, \sigma_{1,2}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,3}, \sigma_1] \\
\text{VPB}[3, \sigma_{2,3}, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,3}, \sigma_{3,1}, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,1}, \sigma_{1,2}, \overline{\sigma}_{2,1}, \sigma_{1,2}] & \text{VPB}[3, \sigma_{2,3}, \overline{\sigma}_{3,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,3}, \sigma_{3,1}, \overline{\sigma}_2] \\
\text{VPB}[3, \overline{\sigma}_{1,2}, \sigma_{1,3}, \sigma_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{3,1}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \sigma_{3,2}, \sigma_{1,2}] & \text{VPB}[3, \overline{\sigma}_{1,2}, \sigma_{1,3}, \sigma_{1,2}, \overline{\sigma}_{2,1}, \sigma_3]
\end{pmatrix}$$

In[•]:= `Table[{vpb = Module[{n = 3, m = 12, i},`
     `VPB[n, Sequence @@`
       `Table[RandomChoice[{σ, σ̄}]`$_{i=\text{RandomChoice@Range@n,RandomChoice[Complement[Range@n,\{i\}]]}}$`, {m}]]],`
   `cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm`

Out[•]= `$Aborted`

`Table[{vpb = Module[{n = 3, m = 16, i},`
   `VPB[n, Sequence @@`
     `Table[RandomChoice[{σ, σ̄}]`$_{i=\text{RandomChoice@Range@n,RandomChoice[Complement[Range@n,\{i\}]]}}$`, {m}]]],`
   `cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm`

$In[\circ]:=$
```
ExtractionGraph[obj_] := Module[{vd, n, gs, vs, es, p, m1, m2, g, q, k},
   gs = VPBGenerators[n = Count[vd = Γ̄[obj], _EOS]];
   vs = {vd}; es = {}; p = 0;
   While[p < Length[vs],
     m1 = Count[vd = vs[[++p]], X_[_, _]];
     Do[
       m2 = Count[q = Γ̄[VD[VPB[n, g /. {σ → σ̄, σ̄ → σ}]] ** vd], X_[_, _]];
       If[m2 < m1,
         If[! MemberQ[vs, q], AppendTo[vs, q]];
         k = Position[vs, q][[1, 1]];
         AppendTo[es, Labeled[p ↦ k, g]]
       ],
       {g, gs}
     ]
   ];
   Graph[Table[Labeled[k, k_Length[vs[[k]]]-n], {k, p}], es]
  ]
```

$In[\circ]:=$ $\mathbf{ExtractionGraph\big[VPB\big[3,\ \overline{\sigma}_{1,2},\ \sigma_{1,3},\ \sigma_{1,2},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{1,2},\ \overline{\sigma}_{1,3},\ \sigma_{3,2},\ \sigma_{1,2}\big]\big]}$

$Out[\circ]=$

In[ ]:= **ExtractionGraph$\left[\text{VPB}\left[3,\ \sigma_{2,3},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{1,3}\right]\right]$**

Out[ ]=
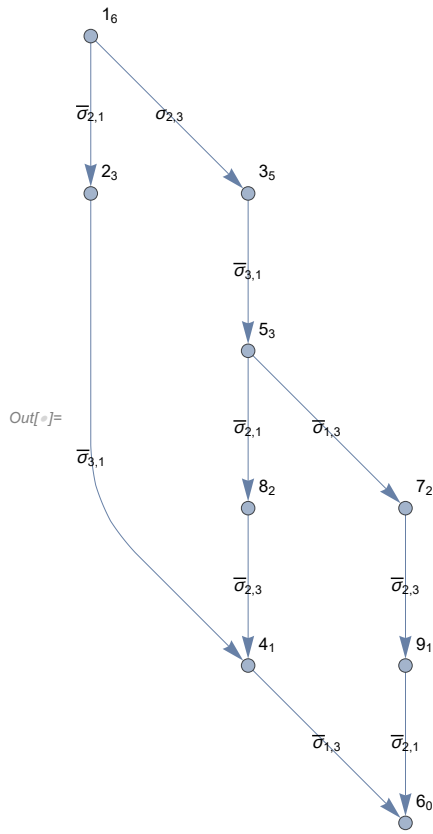
In[ ]:= **ExtractionGraph[VPB[3, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{2,3}$]]**

Out[ ]=
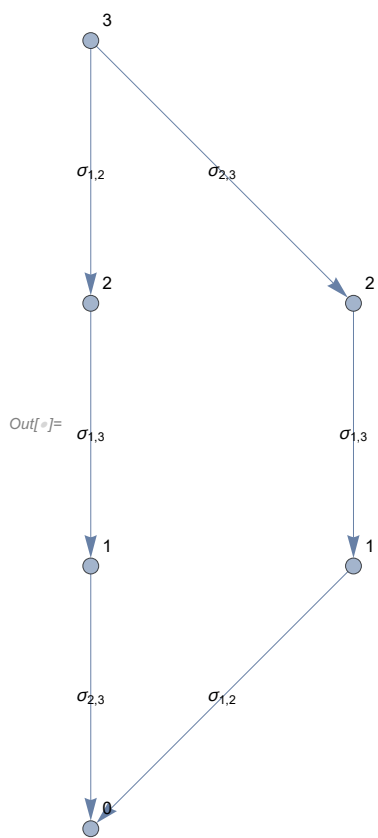
$In[\bullet]:=$ **ExtractionGraph[VPB[4, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{1,4}$, $\sigma_{2,3}$, $\sigma_{2,4}$, $\sigma_{3,4}$]]**

$Out[\bullet]=$
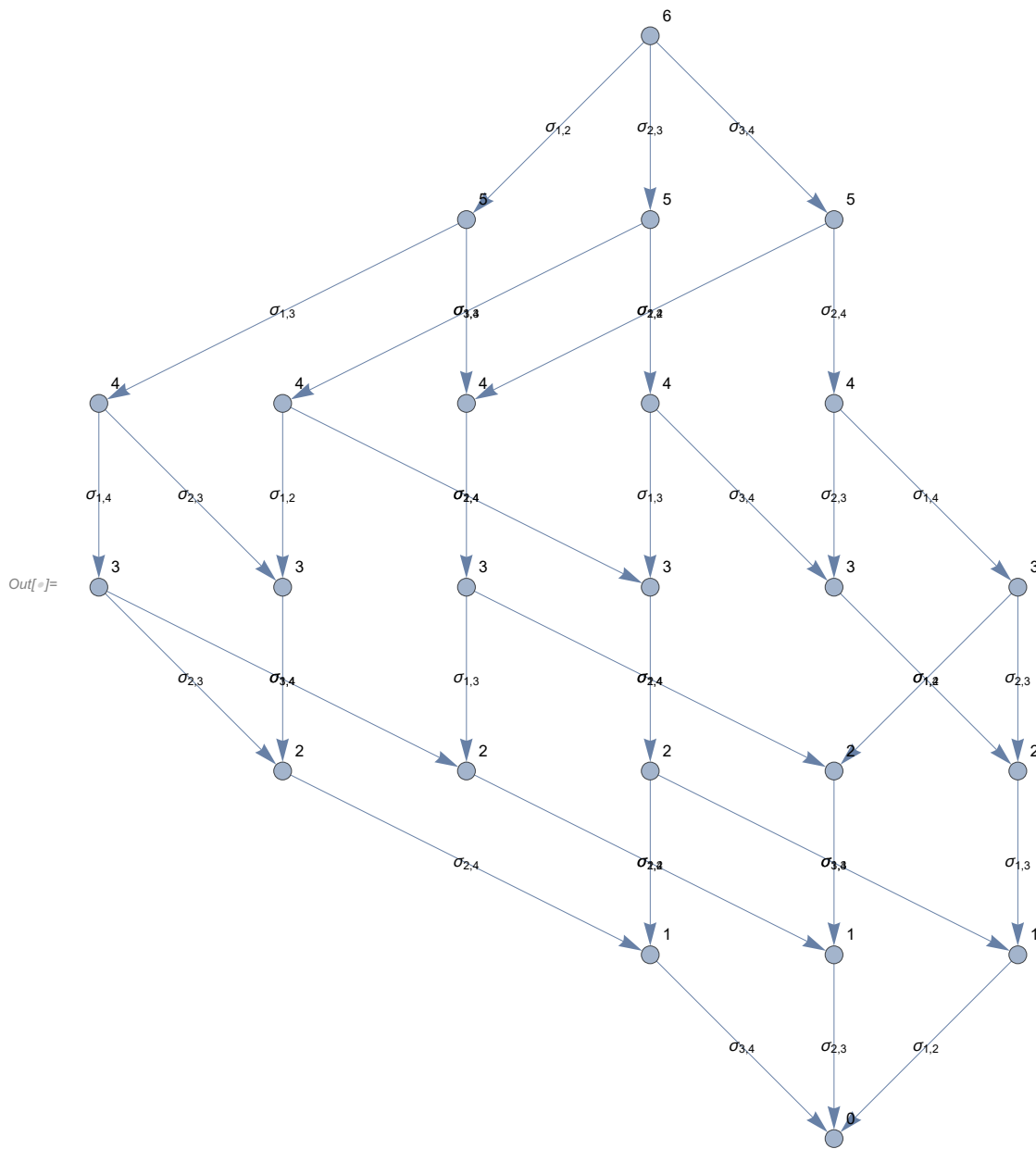
$In[\bullet]:=$ **ExtractionGraph[VPB[5, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{1,4}$, $\sigma_{1,5}$, $\sigma_{2,3}$, $\sigma_{2,4}$, $\sigma_{2,5}$, $\sigma_{3,4}$, $\sigma_{3,5}$, $\sigma_{4,5}$]]**

$Out[\bullet]=$
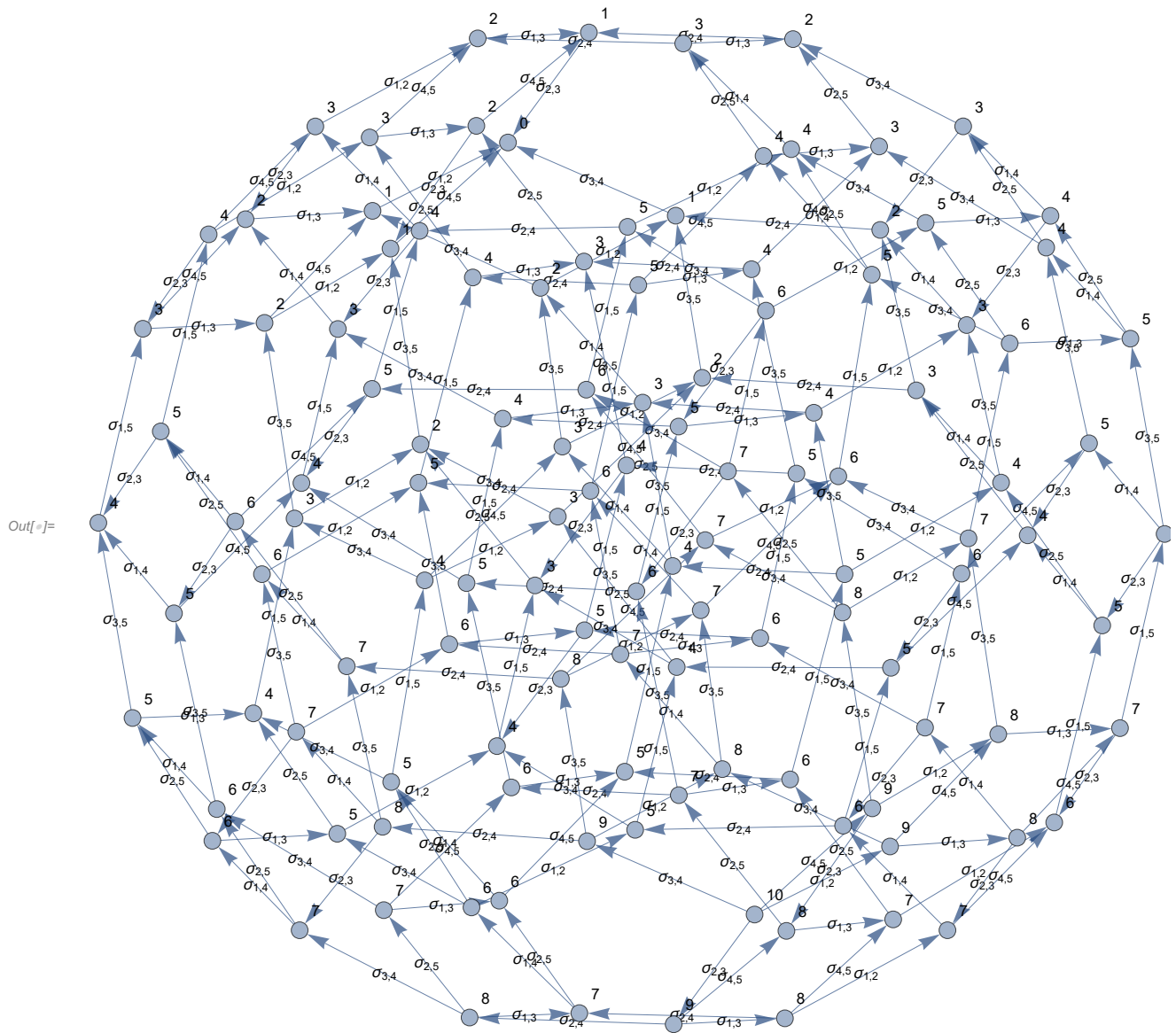
$In[\bullet]:=$ **ExtractionGraph /@ {chb1, chb2, chb3, chb4}**

$Out[\bullet]=$ $\Big\{$ 10 $\xrightarrow{\sigma_{1;2}}$ 6 $\xrightarrow{\sigma_{2,1}}$ 2 $\xrightarrow{\sigma_{1,2}}$ 1 $\xrightarrow{\sigma_{1,3}}$ 0,   6 $\xrightarrow{\sigma_{1,3}}$ 5 $\xrightarrow{\sigma_{1,2}}$ 3 $\xrightarrow{\sigma_{2,1}}$ 1 $\xrightarrow{\sigma_{1,2}}$ 0,

10 $\xrightarrow{\overline{\sigma}_{1;2}}$ 6 $\xrightarrow{\overline{\sigma}_{2,1}}$ 2 $\xrightarrow{\overline{\sigma}_{1,2}}$ 1 $\xrightarrow{\overline{\sigma}_{1,3}}$ 0,   6 $\xrightarrow{\overline{\sigma}_{1,3}}$ 5 $\xrightarrow{\overline{\sigma}_{1,2}}$ 3 $\xrightarrow{\overline{\sigma}_{2,1}}$ 1 $\xrightarrow{\overline{\sigma}_{1,2}}$ 0 $\Big\}$
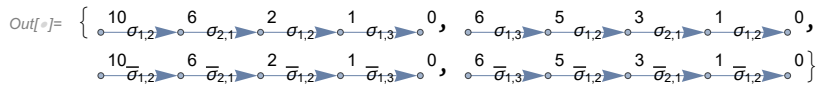
In[ ]:= **ExtractionGraph$\left[$VPB$\left[3, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,3}, \sigma_{1,2}\right]\right]$**

Out[ ]=
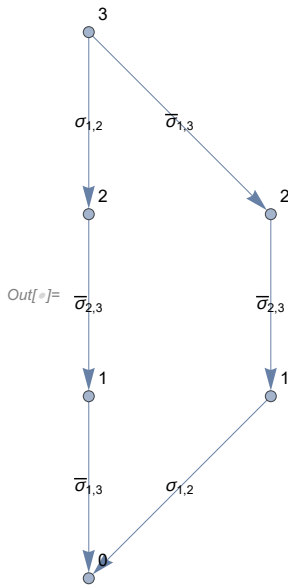
In[ ]:=

```
VPB[BR[n_, is_List]] := VPB[n, Module[{π, i},
    π = Range[n];
    Sequence @@ Table[
      If[i > 0,
        π[[{i, i + 1}]] = π[[{i + 1, i}]]; σ_{π[[i+1]],π[[i]]},
        π[[{-i, -i + 1}]] = π[[{-i + 1, -i}]]; σ̄_{π[[-i]],π[[-i+1]]}
      ],
      {i, is}
    ]]];
VD[br_BR] := VD[VPB@br]
```

In[ ]:= **VPB[BR[3, {1, 2, 1, 2, 1, 2}]]**

Out[ ]= VPB$\left[3, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}, \sigma_{2,1}, \sigma_{3,1}, \sigma_{3,2}\right]$

In[ ]:= **ExtractionGraph[BR[3, {2, 1, 2}]]**

Out[ ]=
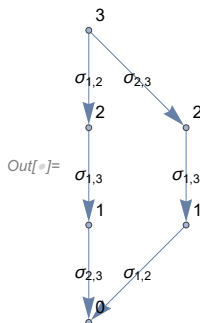
In[ ]:= **VPB@BR[7, {2, 6, 4, -2, -3, -3, -1, -3}]**

Out[ ]= VPB$\left[7, \sigma_{2,3}, \sigma_{6,7}, \sigma_{4,5}, \overline{\sigma}_{2,3}, \overline{\sigma}_{5,3}, \overline{\sigma}_{3,5}, \overline{\sigma}_{2,1}, \overline{\sigma}_{5,3}\right]$

In[ ]:= **ExtractionGraph@BR[7, {2, 6, 4, -2, -3, -3, -1, -3}]**

Out[ ]=
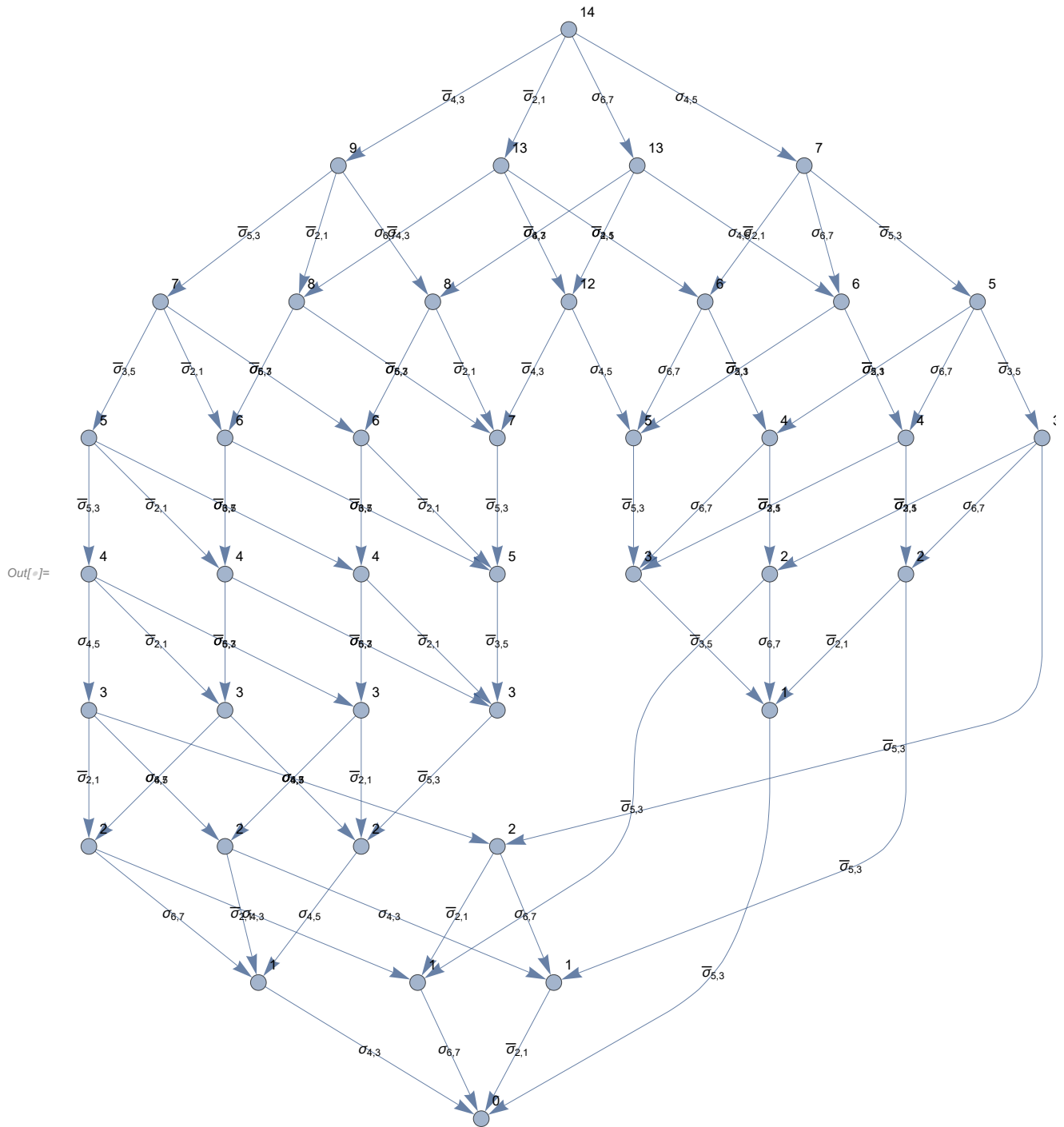
$In[\circ]:=$ **ExtractionGraph@BR[5, {2, 4, -2, -3, -3, -1, -3}]**

$Out[\circ]=$
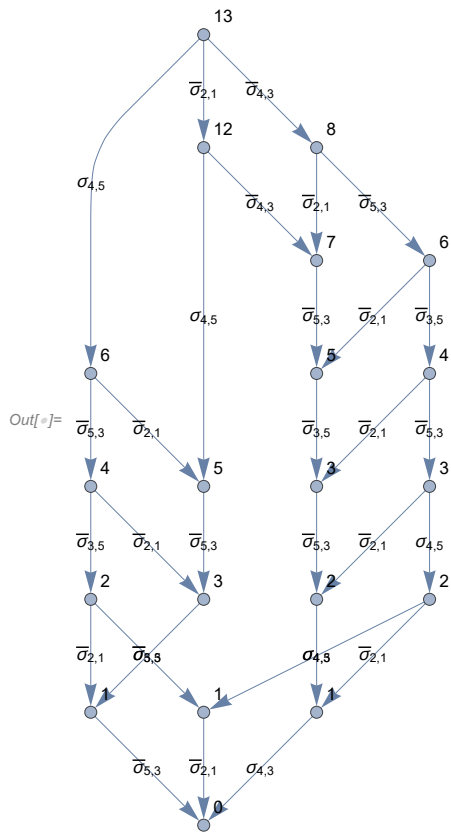
$In[\circ]:=$ **VPB@BR[5, {2, 4, -2, -3, -3, -1, -3}]**

$Out[\circ]=$ $\text{VPB}\left[5,\ \sigma_{2,3},\ \sigma_{4,5},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{5,3},\ \overline{\sigma}_{3,5},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{5,3}\right]$

$In[•]:=$ **ExtractionGraph@VPB** $\left[5,\ \sigma_{2,3},\ \sigma_{4,5},\ \overline{\sigma}_{2,3},\ \overline{\sigma}_{5,3},\ \overline{\sigma}_{3,5},\ \overline{\sigma}_{2,1},\ \overline{\sigma}_{5,3}\right]$
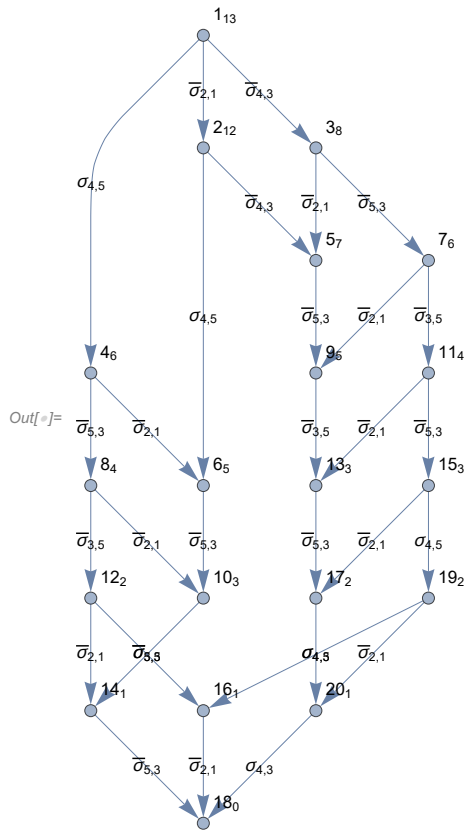
In[ ]:= **ExtractionGraph@VPB[3, bp[2, 3], bm[3, 1], bm[1, 3], bm[1, 3]]**
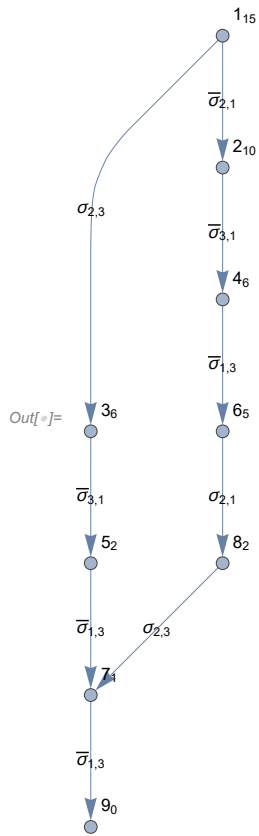
Out[ ]=

1_{15}

$\overline{\sigma}_{2,1}$

2_{10}

$\sigma_{2,3}$

$\overline{\sigma}_{3,1}$

4_6

$\overline{\sigma}_{1,3}$

3_6 6_5

$\overline{\sigma}_{3,1}$ $\sigma_{2,1}$

5_2 8_2

$\overline{\sigma}_{1,3}$ $\sigma_{2,3}$

7_1

$\overline{\sigma}_{1,3}$

9_0

In[ ]:= **ExtractionGraph@VPB[3, bp[2, 3], bm[3, 1], bm[1, 3]]**

Out[ ]=

$In[\circ]:=$ **ExtractionGraph@VPB[3, bp[2, 3], bm[3, 1], bm[1, 3], bm[3, 1]]**

$Out[\circ]=$
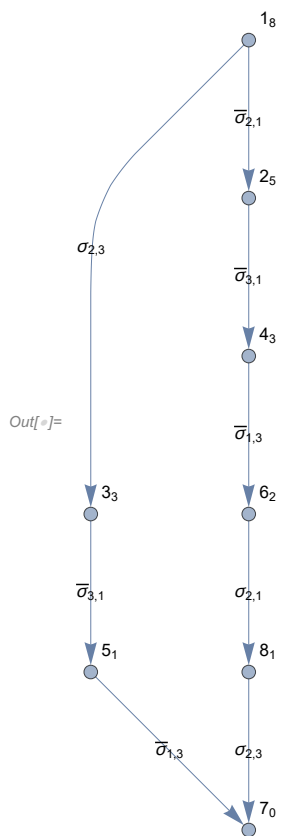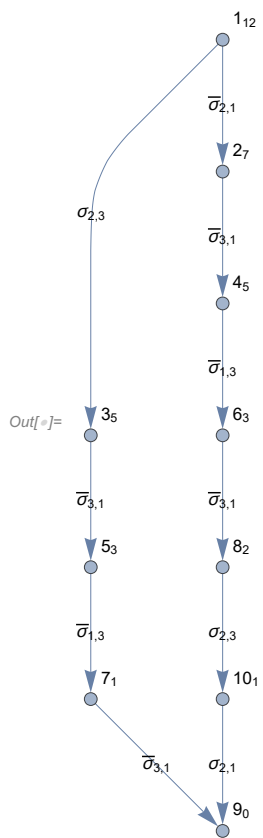
$In[\circ]:=$ **ExtractionGraph@VPB[3, bp[2, 3], bm[3, 1], bm[1, 3], bm[3, 1], bm[1, 3]]**
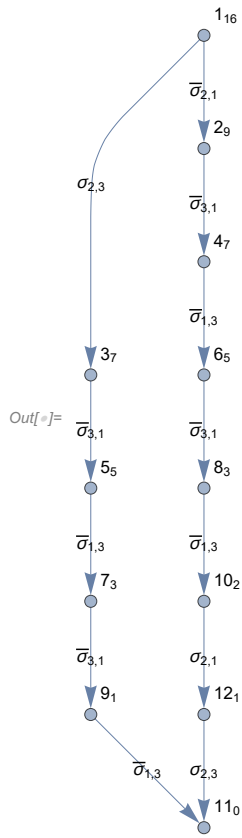
$Out[\circ]=$

$In[\circ]:=$ **RandomBraid[$n\_$, $m\_$] := BR[n, Table[RandomChoice[Range[n - 1] $\bigcup$ (-Range[n - 1])], {m}]]**

$In[\circ]:=$ **RandomBraid[4, 8]**

$Out[\circ]=$ BR[4, {2, -2, 3, -1, -3, -2, 1, 2}]

In[ ]:= **ExtractionGraph[RandomBraid[5, 10]]**
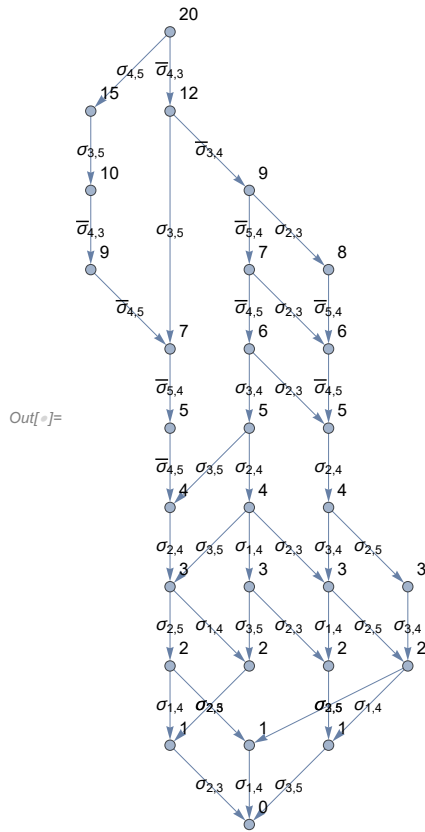
Out[ ]=

In[ ]:= **While[True, Print[ExtractionGraph[**
    Echo@RandomBraid[RandomChoice[{3, 4, 5, 6, 7}], RandomChoice[{8, 10, 12, 16, 20}]]]]]

» BR[7, {1, -4, 2, 3, 5, -6, -3, -4, 5, 4}]
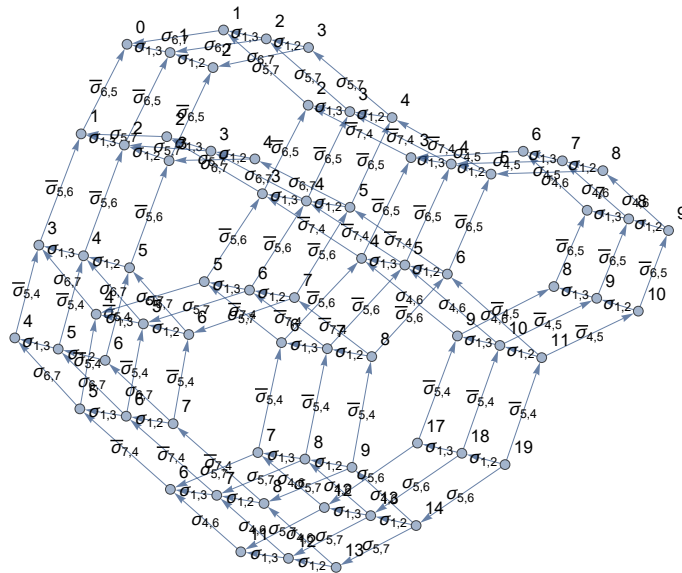
» BR[3, {2, 1, -2, 2, -2, 1, 2, -2, -1, -1, 1, 2}]
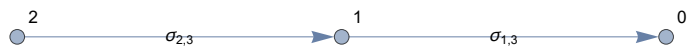
» BR[6, {−5, −4, −5, 5, −4, 1, 4, −3, 2, 5, −1, −3, 4, −2, −2, 5, −2, −5, −2, 2}]
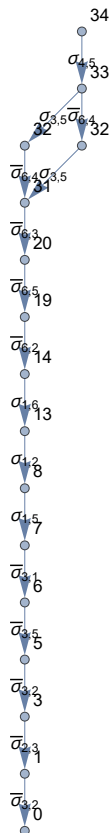
» BR[4, {2, 2, −2, 1, −3, 2, −2, −2, 3, −2, 2, −3}]
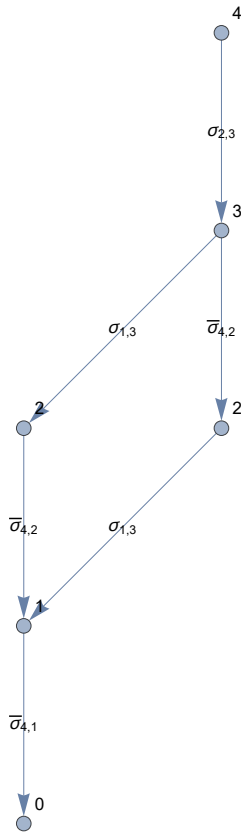
» BR[5, {3, -4, 2, 1, -2, 1, -2, -4, 4, -2, 2, -2}]
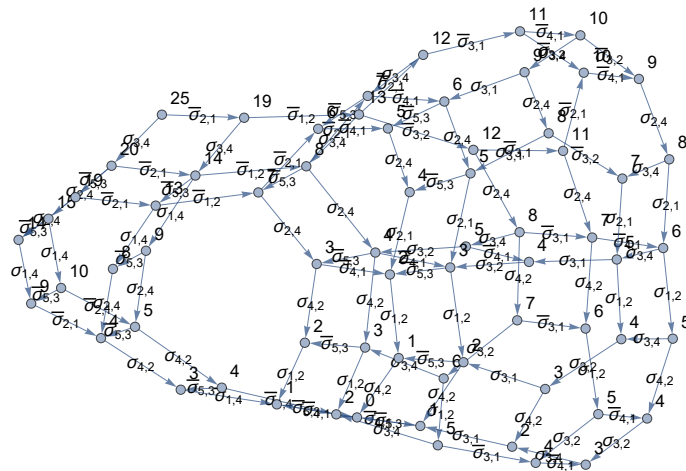
» BR[4, {3, 3, 1, -3, 1, 2, 3, 1}]
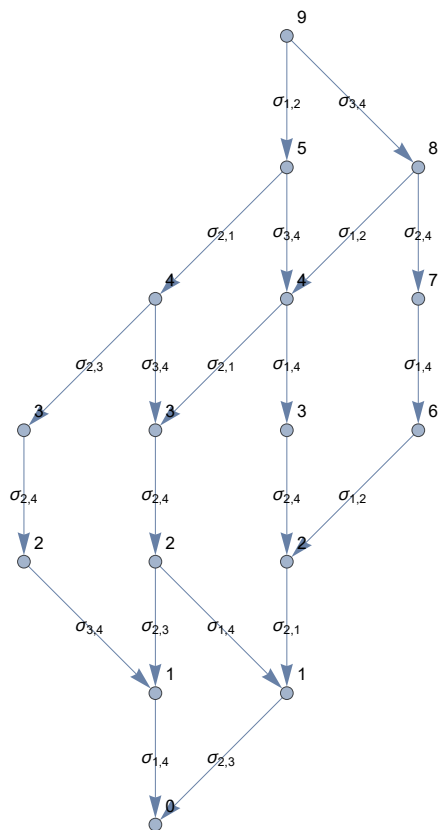
» BR[5, {3, -4, -1, 1, -3, -4, 4, 4, -1, 4, -1, 4, 3, 4, -2, 1}]
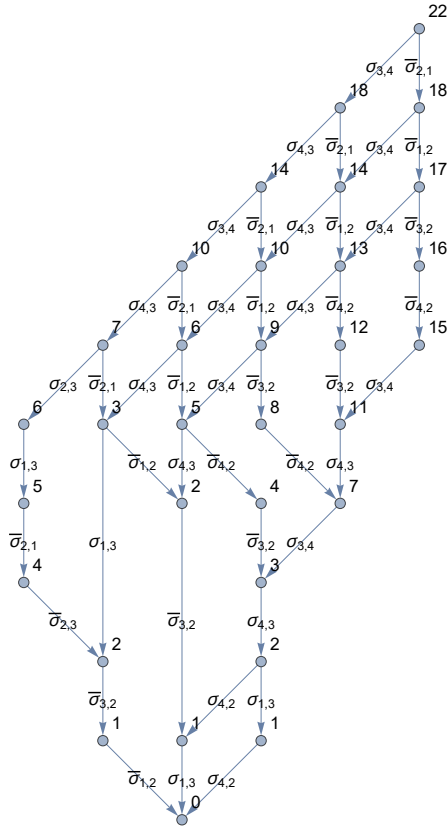
» $\mathrm{BR}[7, \{4, -1, -2, -5, 3, -3, 4, 6, 4, 2, 1, 5, -4, -5, 2, -3, -6, 1, -1, 1\}]$
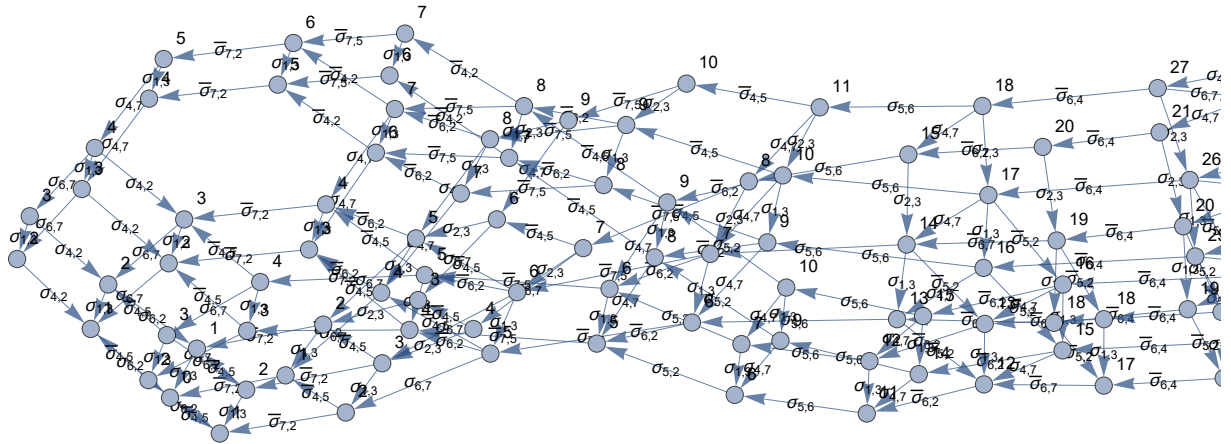
» $\mathrm{BR}[7, \{2, 6, 4, -2, -3, -3, -1, -3\}]$
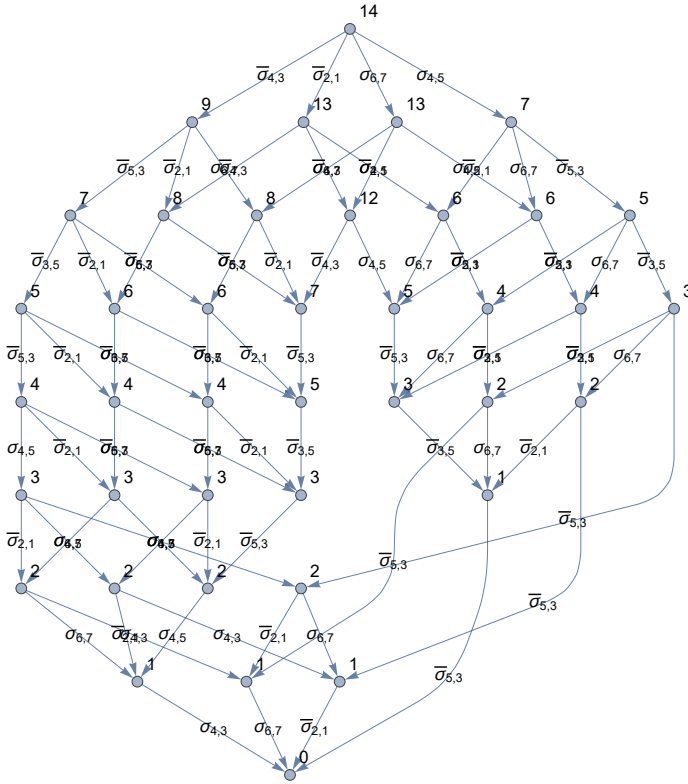
» BR[7, {−1, −3, 3, 3, −4, −3, −3, 6, −4, −6, −2, 4, 3, −2, 1, −1, 1, −4, −3, −2}]
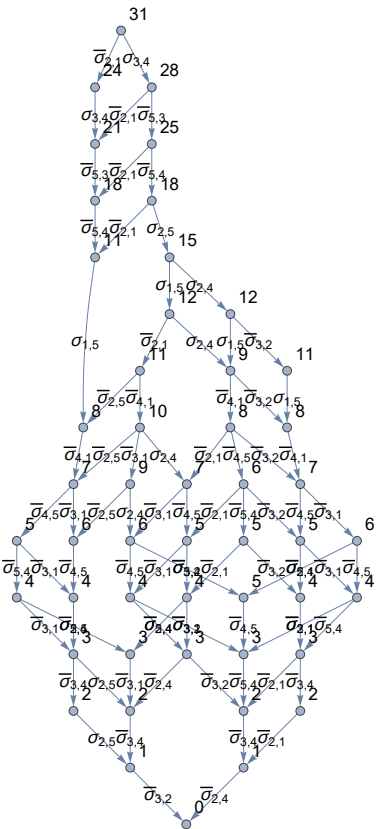
» BR[6, {1, 5, −3, −2, −1, 5, −3, −2, 2, −2, −2, 2, −2, −2, 5, −3}]
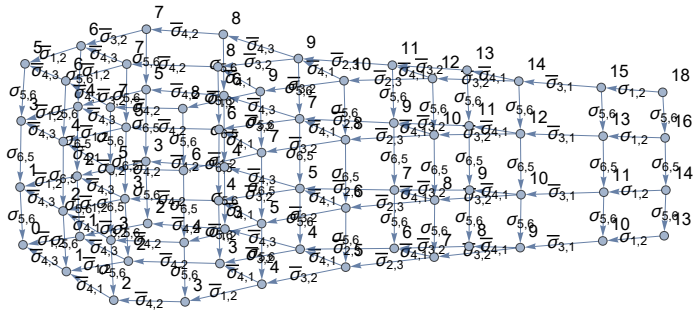
» BR[6, {2, 1, −2, 4, 4, −1, −5, 3}]

» BR[4, {2, 1, 1, 1, −3, −3, −1, −2, 3, 1, 2, 1, −1, 1, 1, 3}]

33

$\overline{\sigma}_{2,1}$
30

$\sigma_{2,3}$ $\sigma_{1,3}$
24

$\sigma_{2,3}$ $\sigma_{3,1}$
20   21   21

$\overline{\sigma}_{4,2}$ $\sigma_{1,3}$ $\sigma_{2,1}$ $\sigma_{2,1}$
17   14   18

$\sigma_{3,4}$ $\sigma_{1,3}$ $\overline{\sigma}_{4,2}$ $\sigma_{3,1}$ $\sigma_{2,3}$
16   11   11

$\sigma_{1,4}$ $\sigma_{1,4}$ $\sigma_{3,1}$ $\overline{\sigma}_{4,2}$
15   10   8

$\sigma_{1,3}$ $\sigma_{3,4}$ $\sigma_{3,4}$
9   5

$\sigma_{3,1}$ $\sigma_{1,4}$ $\overline{\sigma}_{2,3}$
4   4

$\overline{\sigma}_{2,3}$ $\sigma_{1,4}$ $\sigma_{4,2}$
3   3

$\sigma_{1,2}$ $\sigma_{1,2}$
2   2

$\sigma_{1,3}$ $\sigma_{4,2}$ $\sigma_{1,4}$
1   1

$\sigma_{4,2}$ $\sigma_{1,3}$
0

» BR[6, {3, -3, 4, 5, 1, -3, 3, -4, -1, -2}]

4
  $\overline{\sigma}_{3,2}$ $\overline{\sigma}_{6,5}$

3   3

  $\overline{\sigma}_{6,5}$ $\overline{\sigma}_{3,2}$ $\sigma_{4,6}$

2   2

  $\sigma_{4,6}$ $\overline{\sigma}_{3,2}$ $\sigma_{4,5}$

1   1

  $\sigma_{4,5}$ $\overline{\sigma}_{3,2}$

0
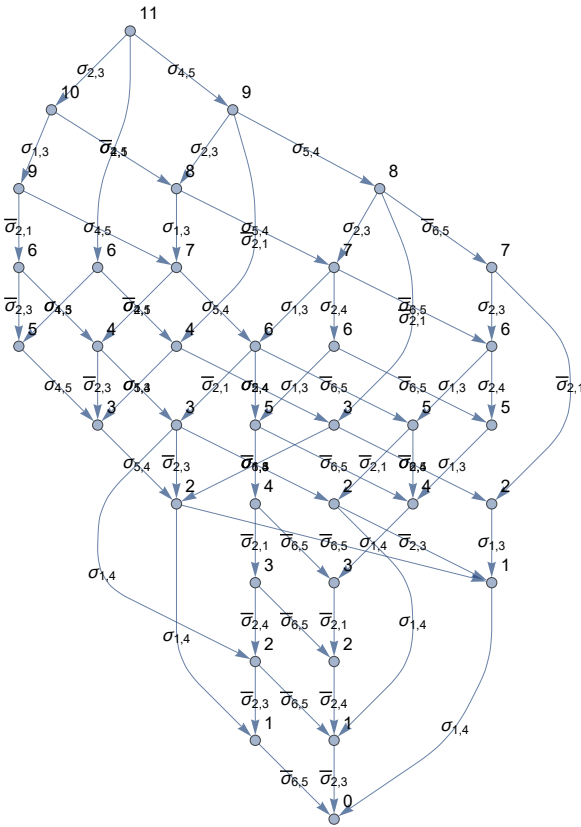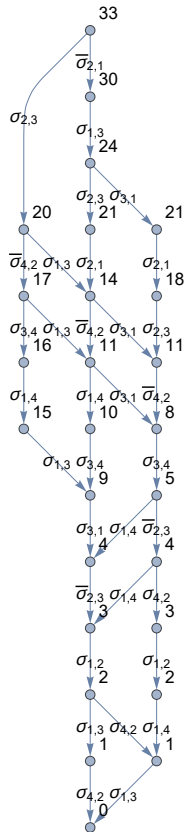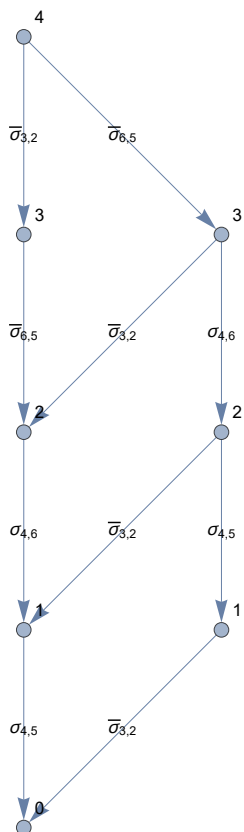
» BR[7, {4, 5, 6, -6, -4, -1, 2, 1, -2, -2, -5, -6, -4, 3, 1, -2, -6, 2, -5, -6}]
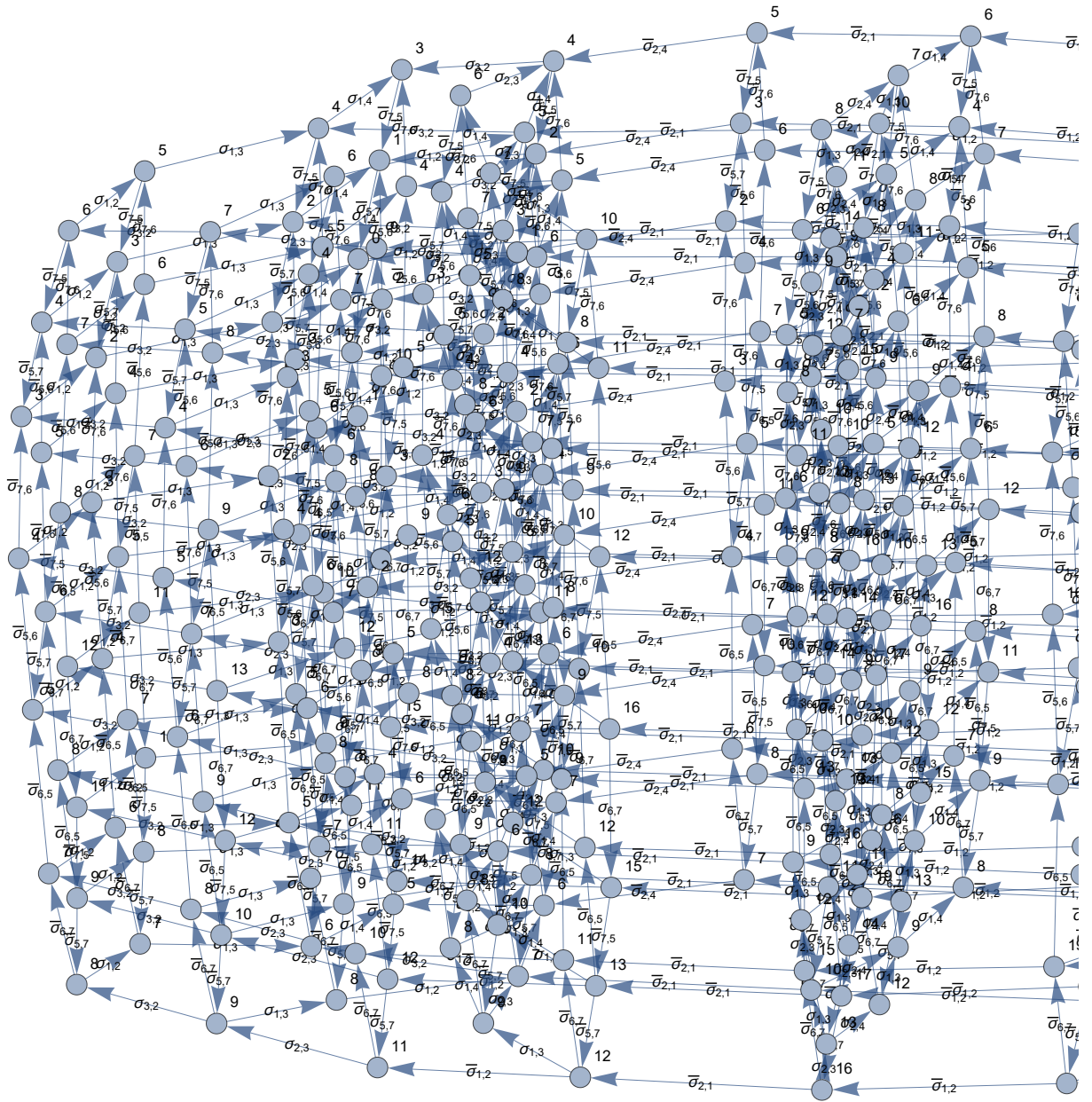
» BR[3, {2, -1, 1, -2, -1, 2, -1, -2, -2, -1}]
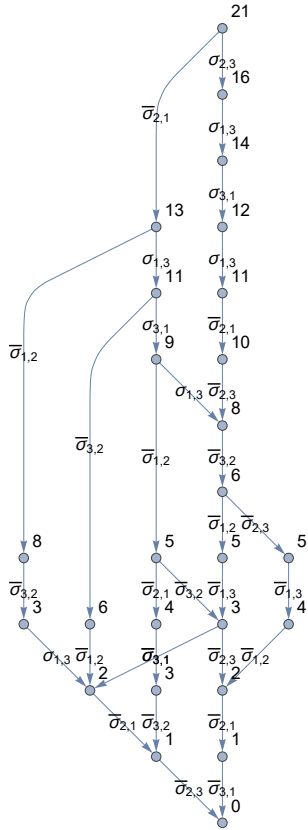
» `BR[6, {3, -3, 2, -2, 5, 3, 3, -4, 5, 3, 1, 3, -4, -3, 1, -3}]`

Out[ ]= `$Aborted`

In[ ]:= **`BR[2, {1, 1, 1, 1, 1}] // ExtractionGraph`**
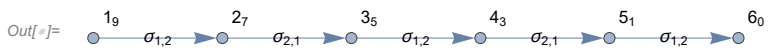
Out[ ]=

In[ ]:= **BR[3, {2, -1, -1}] // ExtractionGraph**

Out[ ]=

In[ ]:= **BR[3, {2, -1, -1, -1}] // ExtractionGraph**

Out[ ]=

1_{12}

$\overline{\sigma}_{2,1}$

2_7

$\overline{\sigma}_{3,1}$

4_5

$\overline{\sigma}_{1,3}$

$\sigma_{2,3}$

3_5 6_3

$\overline{\sigma}_{3,1}$ $\overline{\sigma}_{3,1}$

5_3 8_2

$\overline{\sigma}_{1,3}$ $\sigma_{2,3}$

7_1 10_1

$\overline{\sigma}_{3,1}$ $\sigma_{2,1}$

9_0

$In[\circ]:=$ **BR[3, {2, 1, 1, 1}] // ExtractionGraph**

$Out[\circ]=$

1_{10}

$\overline{\sigma}_{2,1}$

2_7

$\sigma_{2,3}$

$\sigma_{1,3}$

4_5

$\sigma_{3,1}$

3_5 6_3

$\sigma_{1,3}$ $\sigma_{2,1}$ $\sigma_{1,3}$

5_3 9_2 8_2

$\sigma_{3,1}$ $\sigma_{2,3}$ $\sigma_{2,3}$

7_1 11_1

$\sigma_{1,3}$ $\sigma_{2,1}$

10_0
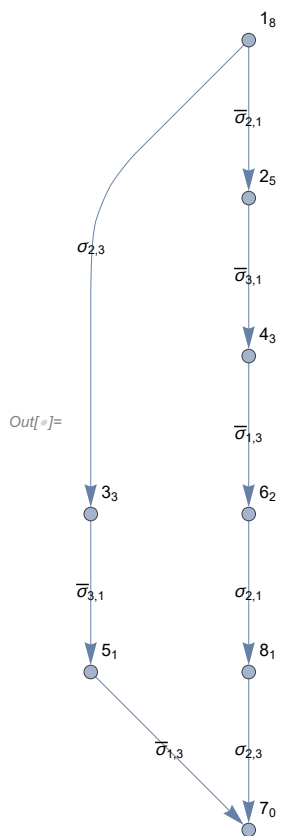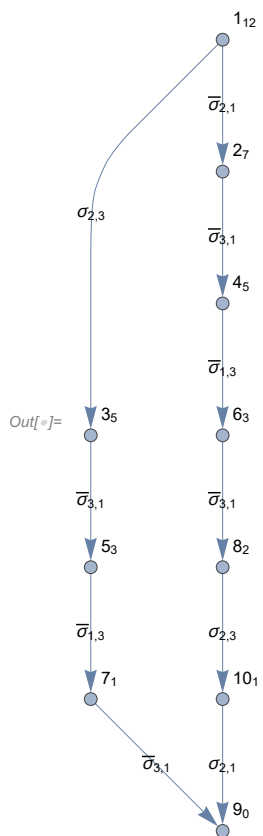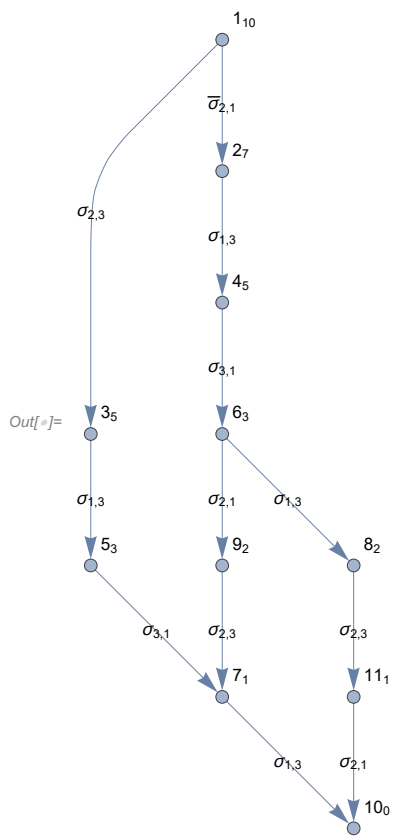
In[ ]:= **BR[3, {2, -1, -1, -1, -1, -1, -1, -1}] // ExtractionGraph**

Out[ ]=
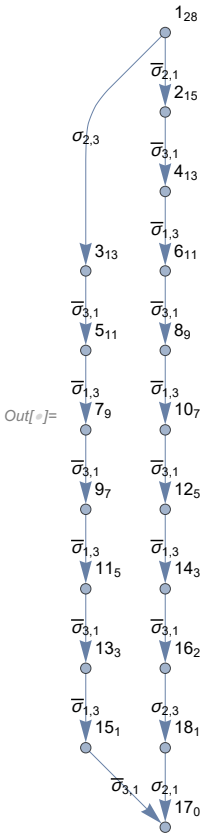
In[ ]:= **BR[8, {1, 3, 5, 7}] // ExtractionGraph**

Out[ ]=
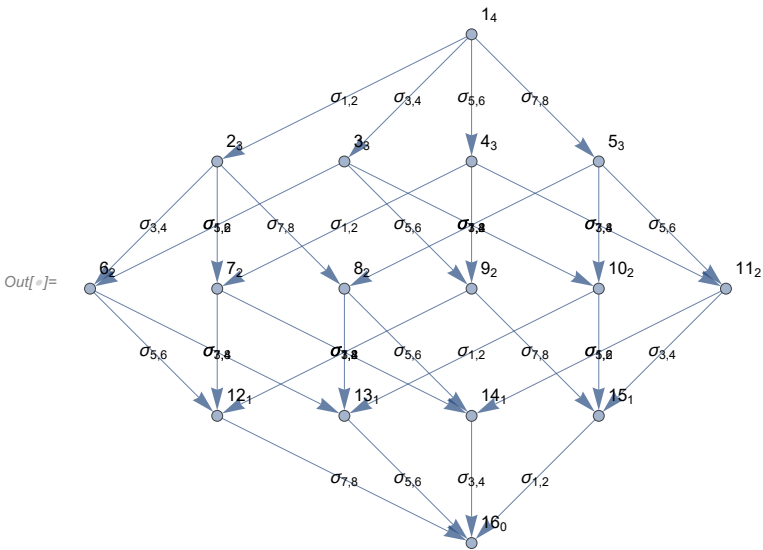
In[ ]:= **BR[3, {1, 2, 1}] // ExtractionGraph**

Out[ ]=
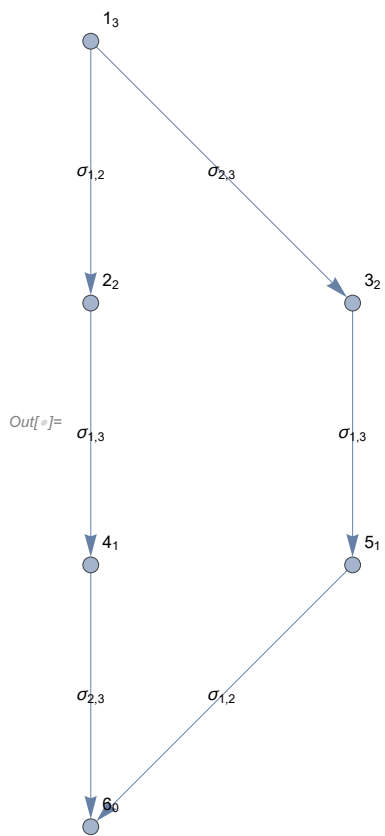
In[ ]:= **BR[4, {1, 2, 3, 1, 2, 1}] // ExtractionGraph**

Out[ ]=
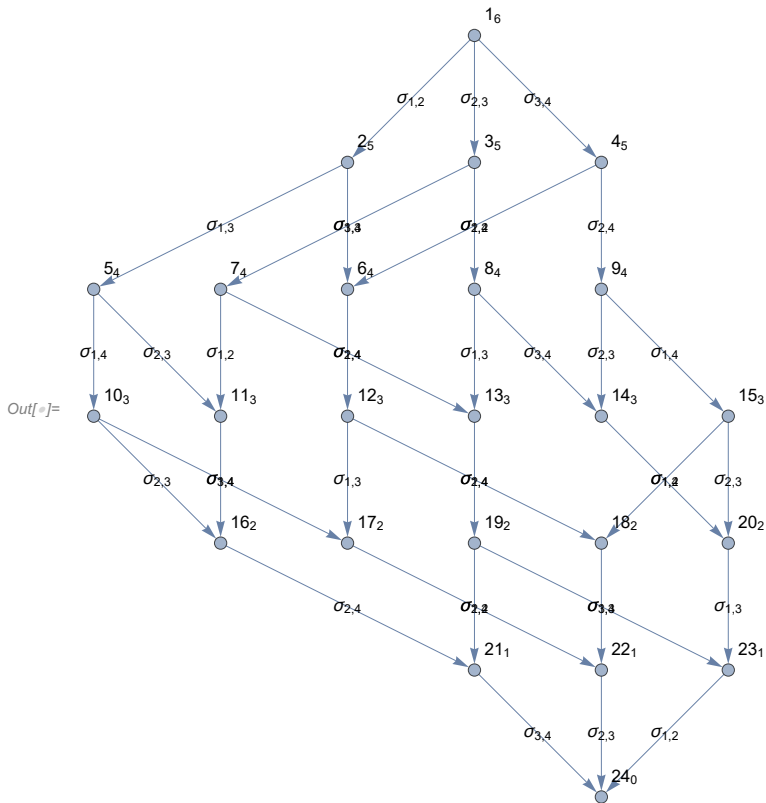
In[ ]:= **BR[5, {1, 2, 3, 4, 1, 2, 3, 1, 2, 1}] // ExtractionGraph // VertexList // Length**

*Out[ ]=* 120

In[ ]:= **BR[5, {1, 2, 3, 4, 1, 2, 3, 1, 2, 1}] // ExtractionGraph**
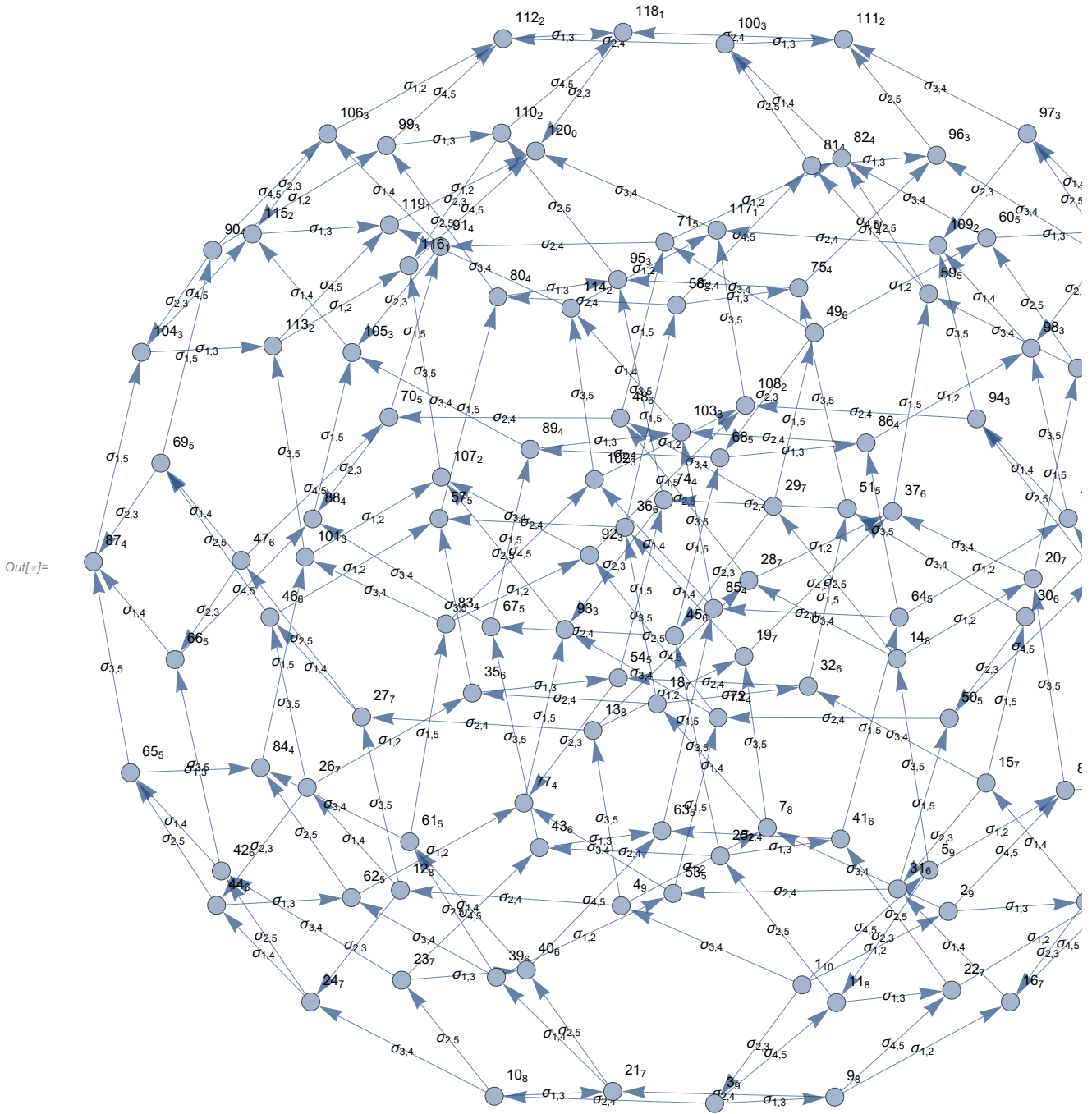
Out[ ]=

In[ ]:= **BR[6, {1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 1, 2, 1}] // ExtractionGraph**

Out[ ]=
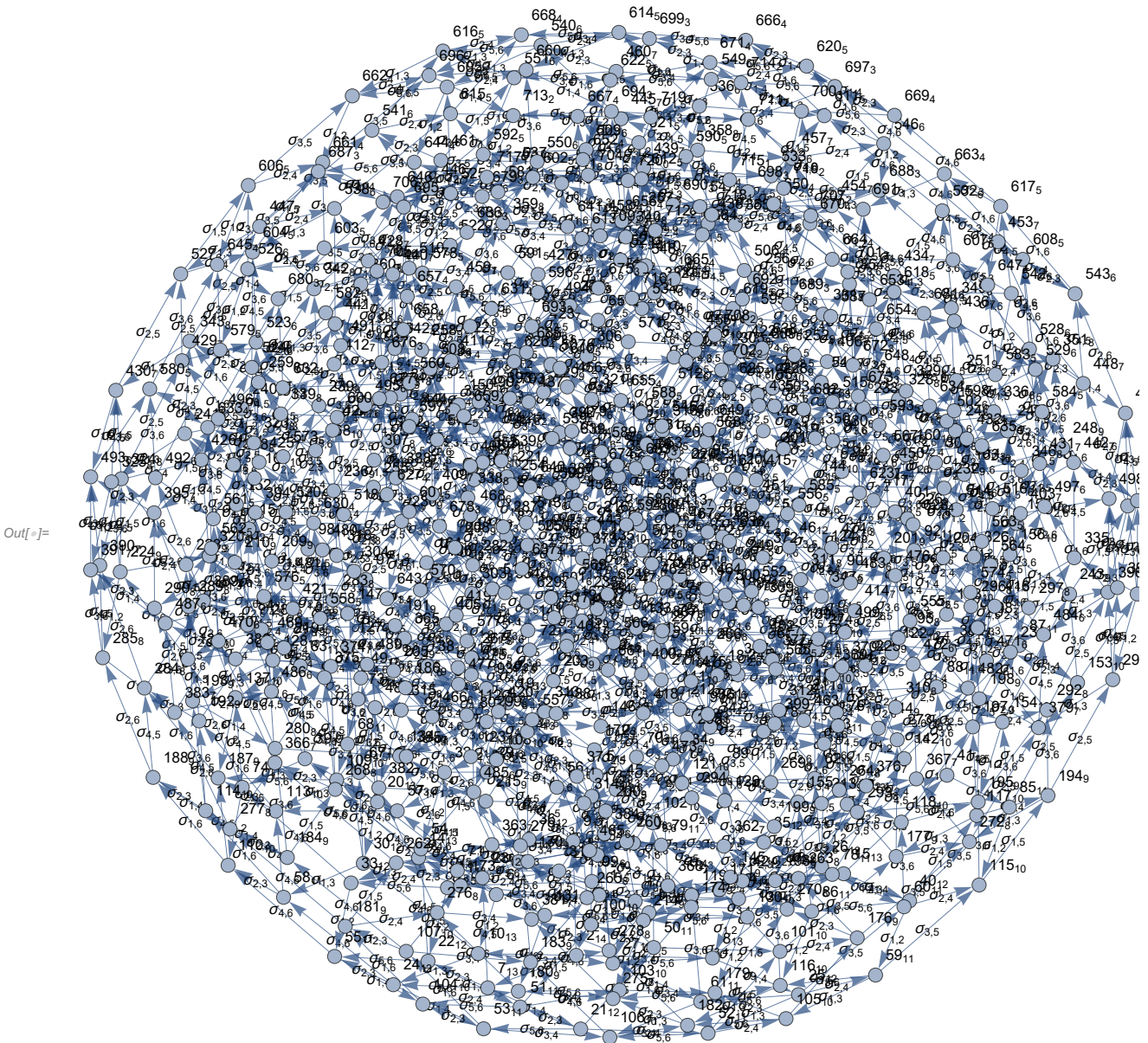
In[ ]:= **BR[6, {1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 1, 2, 1}] // ExtractionGraph // VertexList // Length**

Out[ ]= 720

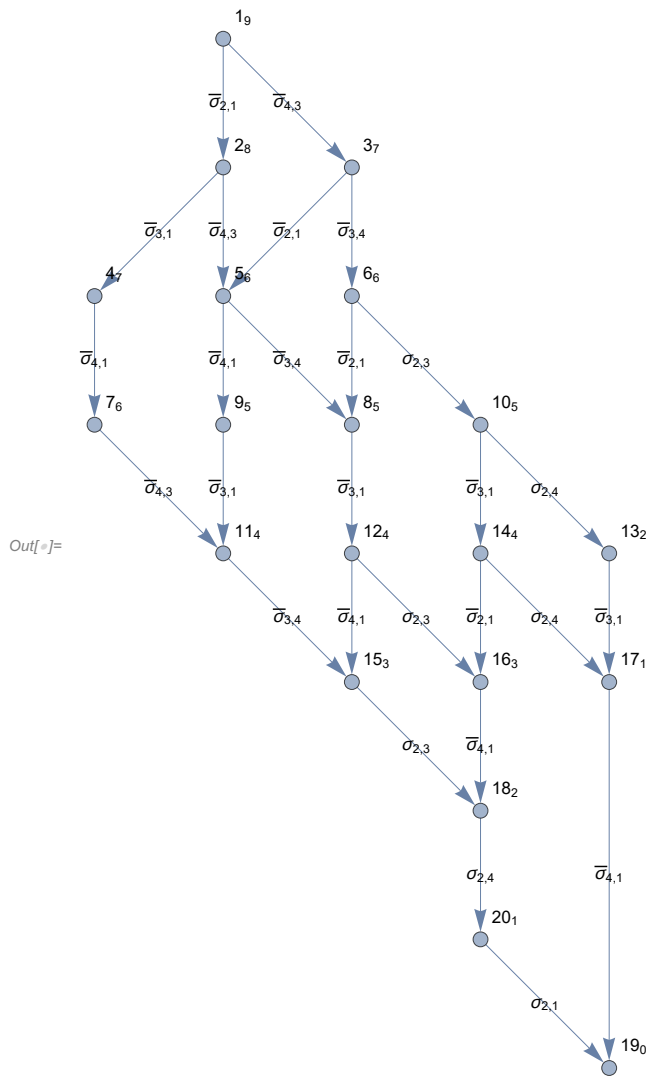*In[ ]:=* **BR[4, {3, 1, -2, 1, -2, 1, -2}] // ExtractionGraph**

1_{41}

$\sigma_{1,2}$ $\sigma_{3,4}$

2_{32} 3_{24}

$\overline{\sigma}_{3,1}$ $\sigma_{3,4}$ $\sigma_{1,2}$

4_{25} 5_{15}

$\overline{\sigma}_{3,2}$ $\overline{\sigma}_{4,1}$

6_{18} 7_{18}

$\overline{\sigma}_{4,1}$ $\overline{\sigma}_{3,2}$ $\sigma_{3,4}$ $\overline{\sigma}_{4,1}$

9_{11} 10_{15}

$\sigma_{2,4}$ $\sigma_{3,1}$

13_{12} 8_8

$\sigma_{2,4}$ $\overline{\sigma}_{3,2}$ $\sigma_{3,1}$ $\sigma_{2,4}$

16_9 11_5

Out[ ]=

$\overline{\sigma}_{3,4}$ $\overline{\sigma}_{1,2}$

12_8 19_6

$\overline{\sigma}_{1,2}$ $\overline{\sigma}_{3,4}$ $\sigma_{3,1}$ $\overline{\sigma}_{1,2}$

15_5 23_5

$\sigma_{4,1}$ $\sigma_{4,1}$ $\sigma_{3,2}$

18_4 25_4 14_2

$\overline{\sigma}_{2,4}$ $\sigma_{3,1}$ $\overline{\sigma}_{3,4}$ $\sigma_{3,2}$ $\sigma_{4,1}$

21_3 22_3 17_1

$\sigma_{3,1}$ $\overline{\sigma}_{2,4}$

24_2

$\sigma_{3,2}$ $\overline{\sigma}_{2,4}$

26_1

$\sigma_{3,4}$

20_0
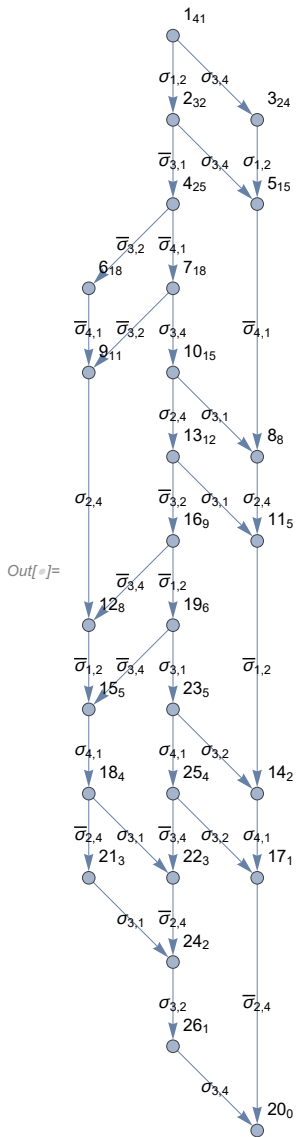
In[ ]:= **BR[4, {2, 3, -1, -1, -1}] // ExtractionGraph**

Out[ ]=
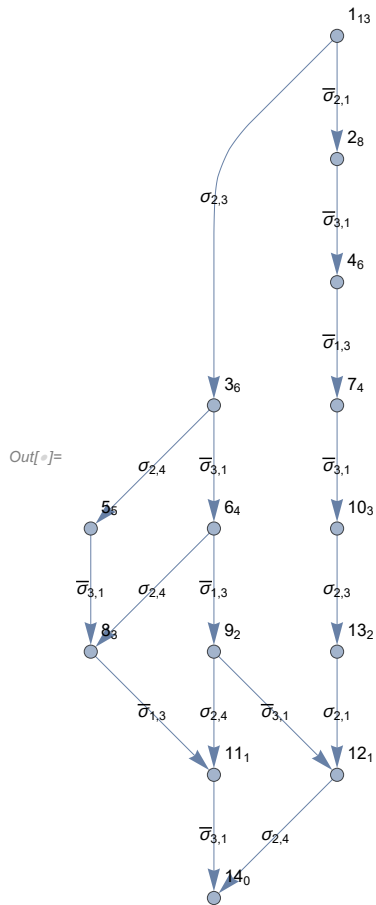
In[ ]:= **BR[4, {2, 3, -2, -2, -2}] // ExtractionGraph**

Out[ ]=
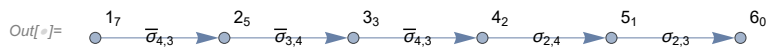
In[ ]:= **BR[4, {2, 3, -1, -2, -1, -1}] // ExtractionGraph**

Out[ ]=

In[ ]:= **BR[4, {2, 3, -2, -2, -2, 1, 2}] // ExtractionGraph**

Out[ ]=
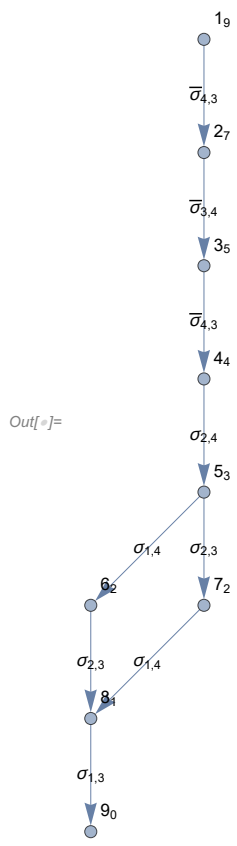
In[ ]:= **BR[3, {-2, -2, -2, 1, 2}] // ExtractionGraph**

Out[ ]=

$In[*]:=$ **BR[4, {2, 3, -2, -2, -2, 1, 2, -3}] // ExtractionGraph**

$Out[*]=$

$In[\bullet]:=$ **BR[4, {2, 3, -2, -2, -2, 1, 2, 3}] // ExtractionGraph**

$Out[\bullet]=$
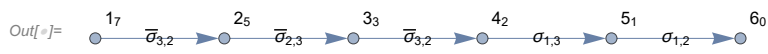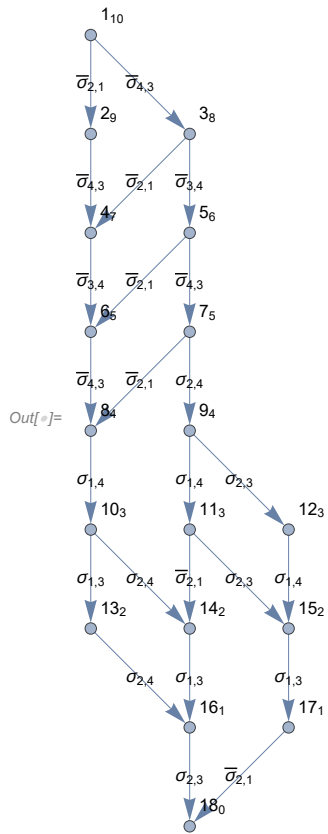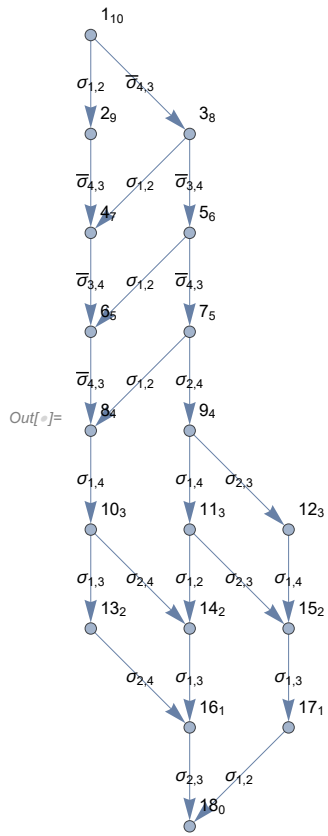
In[ ]:= **{BR[4, {3, -2, -2, -2, 1, 2, 3}] // ExtractionGraph,**
     BR[4, {2, 3, -2, -2, -2, 1, 2}] // ExtractionGraph}

Out[ ]= $\Big\{$

$\Big\}$
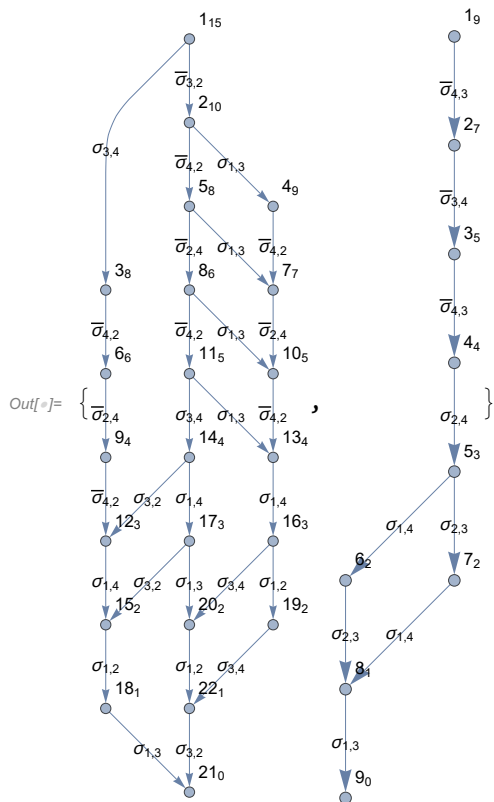
In[ ]:= **BR[4, {2, 3, -2, -2, -2, 1, 2, 3} // Reverse] // ExtractionGraph**

Out[ ]=