OVER THEN UNDER TANGLES

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Sorry, I liked original abstract better.

ABSTRACT. ~~The~~ Over then under (OU) tangle diagrams provide a new framework for knot theory. To illustrate the power of this way of thinking we show that every braid has a unique minimal OU diagram. Pretentious. As a corollary we produce a new type of graphs that that are canonically associated to braids. These results are also generalized to virtual braids.

Many techniques in knot theory can be understood in terms of OU diagrams, even though for knots such diagrams may not exist in the literal sense. We argue that these ideas shed new light upon subjects such as the Drinfel'd double construction of quantum group theory and the quantization of Lie bialgebras.

CONTENTS

1. INTRODUCTION

Brilliant wrong ideas should not be buried or forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong.

In this paper we introduce *Over then Under* (OU) tangles, a class of oriented tangles in which each strand travels through all of its ~~under-~~ over crossings before any of its ~~over-~~ under crossings:

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Make a footnote. The reader should know they don't have to really understand that.

see Figure 1 for some examples and Definition 2.1 for details. This is equivalent to the notion of *ascending* tangles in [ABMW1, Definition 4.15], also called *sorted* in [AM, Definition 1.7] in the context of welded homotopy links.

The key, but incorrect, observation at the core of this paper – explained in Section 2 – is that every tangle can be brought to OU-form using a sequence of *glide moves*: specific isotopies designed to eliminate any "forbidden sequences" of crossings along a strand. The argument is compelling, and has sweeping consequences, including the – clearly false – corollary that every knot is trivial. Indeed On closer look, one notices that in certain special cases of a strand crossing itself, the glide moves fail.

There is, however, much to salvage from the failure of the gliding idea: the argument of Section 2 holds for braids, and every braid – when considered as a tangle – has a unique OU form. Hence, the OU form is a **separating braid invariant**. We also prove that in fact, tangles which can be brought to OU form are precisely braids, using the identification of the braid group with the mapping class group of a punctured disc (see eg [BB, Theorem 1], also explained in Section 3.)

not here.

Even better, the gliding argument extends to virtual braids to show that every virtual braid has a unique OU form when it is regarded as a virtual tangle. With extra work we find that this OU form is a **complete invariant for virtual braids**. This is the subject of Section 4.

what about sec. 5?

In Section 6 we present Mathematica implementations, including tabulations of virtual pure braids and classical braids.

In Section 7 we review a range of other instances in the literature where "OU ideas" play a role: the Drinfeld's double construction in quantum groups, a classification of welded homotopy links by Audoux and Meilhan [AM], Enriquez's work on the quantization of Lie bialgebras [En1, En2], and earlier work of the authors.

All tangle diagrams in this paper are *open and oriented*: Their components are always oriented intervals and never circles. For simplicity and definiteness, all tangles in this paper are unframed: we allow all Reidemeister 1 (R1) moves, though this is not strictly necessary and similar results also hold in the framed case.

2. OU Tangles and Gliding

Definition 2.1. An Over-then-Under (OU) tangle diagram is a tangle whose strands complete all of their over crossings before any of their under crossings, and an OU tangle is an oriented tangle that can be represented by an OU tangle diagram.

In detail, an OU tangle diagram is an oriented tangle diagram each of whose strands can be divided in two by a "transition point", sometimes indicated with a bow tie symbol $\bowtie$, such that in the first part (before the transition) it is the "over" strand in every crossing it goes through, and in the second part (after the transition) it is the "under" strand in every crossing it goes through, so a journey through each strand looks like an OO…O($\bowtie$)UU…U sequence of crossings. Some examples are shown in Figure 1.

Remark 2.2. Loosely, an OU tangle is the "opposite" of an alternating tangle: crossings along each strand read OOOUUU rather than OUOUOU.

The following Fheorem (false theorem), while unfortunately not true, illustrates the idea and potential of *gliding*:

Fheorem 2.3 (Gliding). *Every tangle is an OU tangle.*

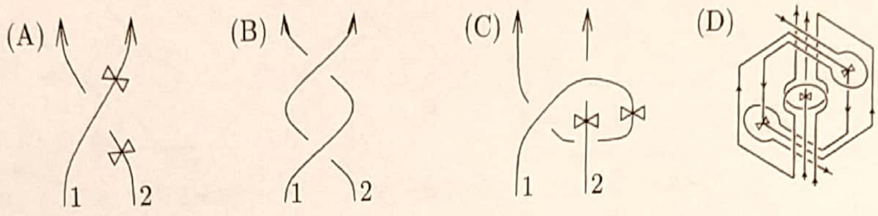


FIGURE 1. The tangle diagram (A) is OU as strand 1 is all "over" (so it has an empty "U" part) and strand 2 is all "under" (so it has an empty "O" part). The tangle diagram (B) is not OU: strand 1 is O then U, but strand 2 is U then O. Yet the tangle represented by (B) is OU because it is also represented by (C), which is OU. The diagram (D) is again OU; which familiar tangle does it represent?

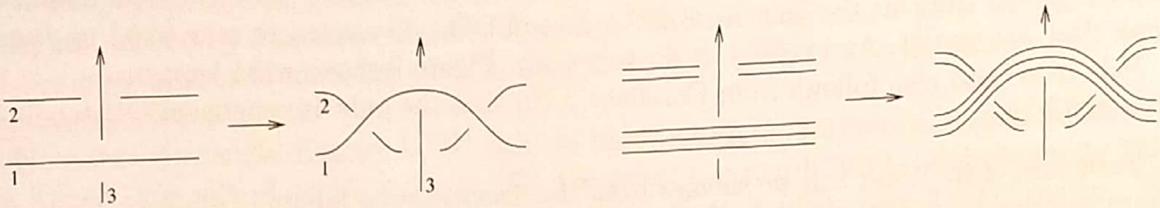


FIGURE 2. Glide moves between two crossings and bulk glide moves.

Froof. As in Figure 2, the froof is frivial. Assume first that strands 1 and 2 are already in OU form (meaning, all their O crossings come before all their U ones) but strand 3 still needs fixing, because at some point it goes through two crossings, first under and then over, as on the left of Figure 2. Simply glide strand 1 forward along and over 3 and glide strand 2 back and under 3 as in Figure 2, and the UO interval along 3 is fixed, and nothing is broken on strands 1 and 2 — strand 1 was over and remains over (more precisely, the part of strand 1 that is shown here is the "O" part), and strand 2 is under and remains under.

In fact, it doesn't matter if strands 1 and 2 are already in OU form because as shown in the second part of Figure 2, glide moves can be performed "in bulk". All that the fixing of strand 3 does to strands 1 and 2 is to replace an O by an OOO on strand 1 and a U by a UUU on strand 2, and this does not increase their complexity as UU...UOO...O sequences can be fixed in one go using bulk glide moves. □

Forollary 2.4. *All long knots are trivial.*

Froof. It is clear that any OU tangle on a single strand is trivial, for it must be descending, as in Figure ↙. □

For the froof of the gliding Fheorem (2.3) is False

Discussion 2.5. Forollary 2.4 is clearly false. ~~For the Gliding Fheorem (2.3) is a Theorem with its T replaced with an F and its froof is a spoof with a leaky Halmos.~~ Indeed, while everything we said about glide moves holds true, there is another way a strand may be U and then O: the U and O may be parts of a single crossing, as on the right, instead of belonging to two distinct crossings, as in the left hand side of the glide move.

It is tempting to dismiss this with "it's only a Reidemeister 1 (R1) issue, so one may glide all kinks to the tail of a strand and count them at the end". Except the same issue can arise in "bulk" UU...UOO...O situations (as now on the right), where it cannot be easily dismissed. One may attempt to resolve the UUOO

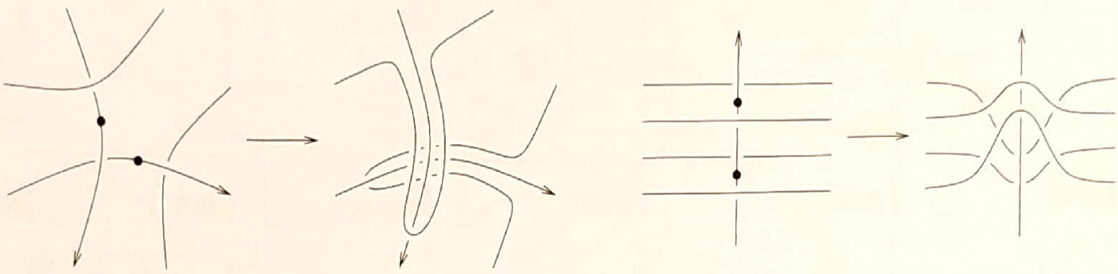


FIGURE 4. Two possibilities for "interacting" UO intervals (each marked with a • symbol).

Glide moves and bulk glide moves as in Figure 2 do not change the acyclicity of a tangle diagram. Indeed by simple inspection the possible transits of a cascade path through either of the sides of a glide move are $1 \to 1$, $1 \to 2$, $1 \to 3$, $2 \to 2$, $3 \to 2$, and $3 \to 3$, with numbering as in Figure 2.

without breaking the OU property.

Note that if a tangle diagram is OU then no Reidemeister 3 (R3) moves can be performed on it — if one side of an R3 move is OU, the other necessarily isn't. This suggests that perhaps an OU form of a tangle diagram is unique up to Reidemeister 2 (R2) moves. We aim to prove this next.

Theorem 3.3. *A tangle diagram D can be made OU using glide moves if and only if it is acyclic, and in that case, the resulting OU tangle diagram, which we call $\Gamma(D)$, is uniquely determined.*

Proof. In an acyclic tangle diagram the U and the O of a UO interval cannot belong to the same crossing (or else an Escher waterfall is present) so the number of UO intervals can be reduced using bulk glide moves as in the Froof of the Gliding Fheorem (2.3). By the observation above, the resulting diagram is still acyclic so the process can be continued.

For the "only if" part, note that OU diagrams are acyclic so anything linked to OU diagrams by glide moves must be acyclic too.

Now to show that $\Gamma(D)$ is unique, observe that when UO intervals are apart from each other, their fixing is clearly independent. It remains to see what happens when UO intervals are adjacent, and there are only two distinct cases to consider. Both of these cases are shown in Figure 4 along with their OU fixes, which are clearly independent of the order in which the glide moves are performed. □

Corollary 3.4. *The stacking product followed by Γ makes OU tangle diagrams into a monoid.* □

Definition 3.5. A tangle diagram is called reduced if its crossing number cannot be reduced using only R1 and R2 moves.

Corollary 3.6. *The map Γ descends to a well-defined map $\bar{\Gamma}$ from "acyclic tangle diagrams modulo Reidemeister moves that preserve the acyclic property" into "reduced OU tangle diagrams".*

In this diagram everything was already defined or is the obvious virtual analog of its counterpart in the classical case and does not need a definition, except that we give a special name, the *Chterental map* $Ch := \bar{\Gamma}_v \circ \bar{\iota}_v$, to the composition along the bottom. Yet in contrast with the Classical Isomorphism Theorem (3.8) we have the theorem below, which is due to Oleg Chterental [Ch1, Ch2]: (Though our version is formulated differently; see a comparison at Discussion 5.1

Theorem 4.1. *(Chterental, [Ch1, Ch2], alternative proof below)* $Ch = \bar{\Gamma}_v \circ \bar{\iota}_v$*, and hence* ι_v*, is injective but not surjective.*

Hence the following corollaries hold true:

Corollary 4.2. *(Chterental, [Ch1, Ch2]). Ch is a complete invariant of virtual pure braids.*[5] □

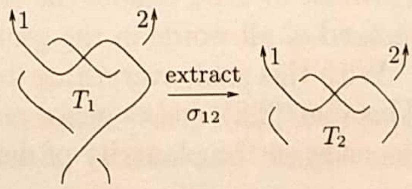
Corollary 4.3. *(Chterental, [Ch1, Ch2]). The two actions of virtual pure braids on reduced virtual OU diagrams are simple but not transitive.* □

Corollary 4.4. *(Chterental, [Ch1, Ch2]). Not all virtual OU tangles are equivalent to virtual pure braids.* □

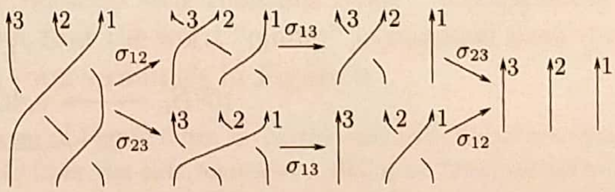
Discussion 4.5. The rest of this section is devoted to a proof of Chterental's Theorem (4.1). The idea is to "extract" as much of a virtual braid out of a virtual OU tangle T as possible, by extracting one braid generator at a time while reducing the complexity of what remains of T. The process won't always invert Ch (for Ch is not invertible), yet it will invert Ch on the image of virtual braids, which is enough. The main tools will be the Division Lemma (4.14) which gives a necessary and sufficient condition for the extraction of one braid generator, and the Diamond Lemma (4.16), which will guarantee that this extraction process always terminates with a well-defined answer.

Definition 4.6. If $T \in v\mathcal{AC}_n$ is a virtual acyclic tangle, let $\xi(T)$ denote the crossing number of $\bar{\Gamma}_v(T)$, its R1- and R2-reduced OU form (not counting virtual crossings, of course). We say that a virtual braid $\beta \in v\mathcal{PB}_n$ divides a virtual acyclic tangle $T \in v\mathcal{AC}_n$, and write $\beta \mid T$, if when β is extracted out of T, this reduces the crossing number. In other words, if $\xi(\beta^{-1}T) < \xi(T)$. In that case, we call $\beta^{-1}T$ the quotient of T by β.

Example 4.7. The figure on the right shows two virtual OU tangles, T_1 and T_2. We have that $\sigma_{12} \mid T_1$ and $\sigma_{12}^{-1}T_1 = T_2$. On the other hand, T_2 is not divisible by anything, as it can be readily verified that $\xi(T_2) = 2$ while $\xi(\sigma_{12}^{\pm1}T_2) > 2$ and $\xi(\sigma_{21}^{\pm1}T_2) > 2$.



Example 4.8. The figure on the right shows in its left part the Garside "positive half twist" braid on 3 strands, which happens to be OU in its given presentation, fit within a hexagon summarizing its five divisors σ_{12}, σ_{23}, $\sigma_{12}\sigma_{13}$, $\sigma_{23}\sigma_{13}$, $\sigma_{12}\sigma_{13}\sigma_{23} = \sigma_{23}\sigma_{13}\sigma_{12}$, and the five resulting quotients. This hexagon is also an example of an extraction graph; see Discussion 5.10.



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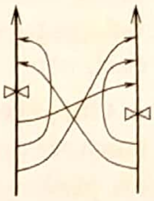
[5] An earlier separation result for virtual braids is in [GP].

that space is studied using various "Heads then Tails" techniques, which in the language of the current paper, correspond to UO presentations (not OU, but of course, it's essentially the same). See especially [BN7, Section 2.4].



An even earlier occurrence of OU ideas, in the associated graded $\mathcal{A}^v$ context for virtual tangles, occurs in a very well-hidden way within Enriquez' work on quantization of Lie bialgebras [En1, En2]. For example, his "universal algebras" [En2, Section 1.3.2] are isomorphic to the space $\mathcal{A}^v_{OU}$ of arrow diagrams as on the right, in which all arrow tails occur before all arrow heads (that's OU!), and is endowed with the product that $\mathcal{A}^v_{OU}$ inherits from the stacking product of $\mathcal{A}^v$ (which is the analogue of the product used in our paper). We are afraid that there aren't excellent introductions available on $\mathcal{A}^v$ and its relationship with virtual tangles. Hopefully we will write one one day. Until then, some information is in [BD2] and in lecture series such as [BN2, BN3]. We also hope to one day explain the Enriquez work as the construction of a "homomorphic expansion" [BD1] for the space of virtual OU / acyclic tangles.

If $\mathfrak{g} = \mathfrak{a}^\star \bowtie \mathfrak{a}$ is the double of a Lie bialgebra $\mathfrak{a}$, there is a standard interpretation of $\mathcal{A}^v$ as a space of formulas for elements in tensor powers $\mathcal{U}(\mathfrak{g})^{\otimes n}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Within this context, arrow tails (or "O") correspond to $\mathfrak{a}^\star$ and arrow heads (or "U") correspond to $\mathfrak{a}$, and the O then U theme of this paper corresponds to the "polarization" isomorphism $\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{a}^\star) \otimes \mathcal{U}(\mathfrak{a})$, which is a consequence of the PBW theorem. In itself, the polarization isomorphism is central to all approaches to the quantization of Lie bialgebras [EK, Se].

A. Referees,

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