Pensieve header: Programs for β-calculus, development notebook.

KnotTheory

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<< KnotTheory`
```

KnotTheory

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Loading KnotTheory` version of February 5, 2013, 3:48:46.4762.
Read more at http://katlas.org/wiki/KnotTheory.
```

Initialization

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], (t | T)_s_ :> s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :> s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

Program

```
⟨μ_⟩ := μ /. t_ → 1;
tm_x_y_→z_[β_] := βCollect[β /. {t_x|y → t_z, T_x|y → T_z}];
hm_x_y_→z_[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_x|y → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_x_y_[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
         + γ / ϵ h_y              + δ - γ * β / ϵ
    ] // βCollect];
gm_x_y_→z_[β_] := β // sw_xy // hm_xy→z // tm_xy→z;
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
(R^+)_x_y_ := B[1, (T_x - 1) t_x h_y];
(R^-)_x_y_ := B[1, ((T_x)^-1 - 1) t_x h_y];
```

tm

```
{β = B[ω, Sum[α_{2 i+j-6} t_i h_j, {i, 1, 4}, {j, 5, 6}]],
  O_1 = β // tm_{12→1} // tm_{13→1},
  O_2 = β // tm_{23→2} // tm_{12→1},
  O_1 == O_2
 } // ColumnForm
```

tm

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

True

hm

```
{β = B[ω, Sum[α_{4 i+j-6} t_i h_j, {i, 1, 2}, {j, 3, 6}]],
   O_1 = β // hm_{34→3} // hm_{35→3},
   O_2 = β // hm_{45→4} // hm_{34→3};
   O_1 == O_2
  } /. α_i_ :→ î // ColumnForm
```

hm

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 & h_6 \\ t_1 & \hat{1} & \hat{2} & \hat{3} & \hat{4} \\ t_2 & \hat{5} & \hat{6} & \hat{7} & \hat{8} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_3 & h_6 \\ t_1 & \hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{3}+\hat{1}\,\hat{3}+\hat{2}\,\hat{3}+\hat{1}\,\hat{2}\,\hat{3}+\hat{2}\,\hat{5}+\hat{3}\,\hat{5}+\hat{2}\,\hat{3}\,\hat{5}+\hat{3}\,\hat{6}+\hat{1}\,\hat{3}\,\hat{6}+\hat{3}\,\hat{5}\,\hat{6} & \hat{4} \\ t_2 & \hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{5}\,\hat{6}+\hat{7}+\hat{1}\,\hat{7}+\hat{2}\,\hat{7}+\hat{1}\,\hat{2}\,\hat{7}+\hat{5}\,\hat{7}+\hat{2}\,\hat{5}\,\hat{7}+\hat{6}\,\hat{7}+\hat{1}\,\hat{6}\,\hat{7}+\hat{5}\,\hat{6}\,\hat{7} & \hat{8} \end{pmatrix}$$

True

htt

```
{β = B[ω, Sum[α_{2 i+j-5} t_i h_j, {i, 1, 3}, {j, 4, 5}]],
 O_1 = β // tm_{12→1} // sw_{14},
 O_2 = β // sw_{24} // sw_{14} // tm_{12→1};
 O_1 == O_2}
```

htt

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_1+\alpha_3) & h_4 & h_5 \\ t_1 & \frac{(\alpha_1+\alpha_3)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} & \frac{(\alpha_2+\alpha_4)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} \\ t_3 & \frac{\alpha_5}{1+\alpha_1+\alpha_3} & \frac{-\alpha_2\,\alpha_5-\alpha_4\,\alpha_5+\alpha_6+\alpha_1\,\alpha_6+\alpha_3\,\alpha_6}{1+\alpha_1+\alpha_3} \end{pmatrix}, \text{True}\right\}$$

hht

```
{β = B[ω, Sum[α_{3 i+j-5} t_i h_j, {i, 1, 2}, {j, 3, 5}]],
   O_1 = β // hm_{34→3} // sw_13 // βCollect,
   O_2 = β // sw_13 // sw_14 // hm_{34→3} // βCollect;
   O_1 == O_2
  } /. α_i_ :→ î // ColumnForm
```

hht

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 \\ t_1 & \hat{1} & \hat{2} & \hat{3} \\ t_2 & \hat{4} & \hat{5} & \hat{6} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right) & h_3 & h_5 \\ t_1 & \frac{\left(1+\hat{1}+\hat{4}\right)\left(\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{\hat{3}\left(1+\hat{1}+\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \\ t_2 & \frac{\hat{4}+\hat{5}+\hat{1}\,\hat{5}+\hat{4}\,\hat{5}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{-\hat{3}\,\hat{4}-\hat{3}\,\hat{5}-\hat{1}\,\hat{3}\,\hat{5}-\hat{3}\,\hat{4}\,\hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{2}\,\hat{6}+\hat{1}\,\hat{2}\,\hat{6}+\hat{2}\,\hat{4}\,\hat{6}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \end{pmatrix}$$

True

R3

```
{(R⁻)_51 (R⁻)_62 (R⁺)_34 // gm_{14→1} // gm_{25→2} // gm_{36→3},
  (R⁺)_61 (R⁻)_24 (R⁻)_35 // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```

R3

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}\right\}$$

8_17-1

```
β = (R⁻)_{12,1} (R⁻)_27 (R⁻)_83 (R⁻)_{4,11} (R⁺)_{16,5} (R⁺)_{6,13} (R⁺)_{14,9} (R⁺)_{10,15}
```

8_17-1

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+T_4}{T_4} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_6 & 0 \\ t_8 & 0 & -\frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{10} \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+T_{14} & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+T_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

8_17-2

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_17-2

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1\,T_{16}}{T_1^2} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+T_1)\,T_{14}\,\left(T_1^3+T_{16}^2\right)}{T_1^2\,T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & -\frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,T_{14}\,T_{16}}{T_1\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,T_{14}}{T_1^2+T_{16}-T_1\,T_{16}} & -1+T_1 \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 \\ t_{14} & \frac{(-1+T_{14})\,\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,(-1+T_{14})\,T_{16}}{T_1\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & -\frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,(-1+T_{14})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \\ t_{16} & \frac{T_1\,(-1+T_{16})}{T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,T_1\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & -\frac{(-1+T_1)^2\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \end{pmatrix}$$

8_17-3

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

8_17-3

$$\begin{pmatrix} -\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^3} \\ t_1 \end{pmatrix}$$

8_17-4

```
Alexander[Knot[8, 17]][X]
```

8_17-4

KnotTheory::loading : Loading precomputed data in PD4Knots`.

8_17-4

$$11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8\,X+4\,X^2-X^3$$

Recycling

StandardAlexander

```
{{1, 0, 0, 0, 0, T - 1, 0, -T},
 {-1, T, 0, 0, 0, 0, 1 - T, 0},
 {0, -1, T, 0, 1 - T, 0, 0, 0},
 {T - 1, 0, -T, 1, 0, 0, 0, 0},
 {0, 1 - T, 0, -1, T, 0, 0, 0},
 {0, 0, 0, 0, -T, 1, 0, T - 1},
 {0, 0, 1 - T, 0, 0, -1, T, 0},
 {0, 0, 0, T - 1, 0, 0, -T, 1}}[[1 ;; 7, 1 ;; 7]] // Det
```

StandardAlexander

$$-1+4\,T-8\,T^2+11\,T^3-8\,T^4+4\,T^5-T^6$$

Work in Progress

```
GD[K_] := GD @@ (
    PD[K] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]],
       Ar[l, i, +1], Ar[j, i, -1]
      ]
  )
```

```
βZ[L_] := Module[
  {skel, β, s, k},
  skel = Skeleton[L];
  β = Times @@ GD[L] /. {Ar[x_, y_, +1] :> (R⁺)xy, Ar[x_, y_, -1] :> (R⁻)xy};
  Do[
   Do[
    β = β // gmskel[[s,1]], skel[[s,k]]→skel[[s,1]],
    {k, 2, Length[skel[[s]]]}
   ],
   {s, Length[skel]}
  ];
  β
 ]
```

```
βZ[Knot[8, 17]][[1]]
```

$$-\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^2}$$

```
Factor[βZ[#][[1]] / Alexander[#][T1]] & /@ AllKnots[{3, 8}]
```

$$\left\{\frac{1}{T_1},\ T_1,\ \frac{1}{T_1^2},\ \frac{1}{T_1^2},\ 1,\ 1,\ 1,\ \frac{1}{T_1^3},\ \frac{1}{T_1^3},\ T_1^4,\ T_1^4,\ \frac{1}{T_1^3},\ \frac{1}{T_1},\ T_1^2,\ \frac{1}{T_1},\ \frac{1}{T_1},\right.$$
$$\left.T_1,\ T_1,\ T_1^3,\ \frac{1}{T_1},\ T_1,\ T_1,\ T_1,\ T_1,\ \frac{1}{T_1},\ T_1,\ T_1,\ \frac{1}{T_1},\ \frac{1}{T_1^3},\ \frac{1}{T_1},\ T_1,\ 1,\ T_1^4,\ 1,\ \frac{1}{T_1}\right\}$$

```
βCollect[Bu[ω_, λ_, μ_]] := Bu[
    βSimp[ω],
    Collect[λ, h_, βSimp],
    Collect[μ, h_, Collect[#, t_, βSimp] &]
   ];
Bu[ηs_List, B[ω_, μ_]] := Module[{λ},
    λ = (1 + Coefficient[μ, #] /. t_ → 1) & /@ ηs;
    Bu[ω,
      Thread[ηs → λ],
      -μ + (ηs /. ha_ :> ta ha).λ
     ] // βCollect
   ];
B[Bu[ω_, λ_, μ_]] := 0;
```

```
β0 = βZ[L = Link["L6a5"]]
```

$$\begin{pmatrix} \frac{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)}{T_1^2\,T_5^2\,T_9^2} & h_1 & h_5 \\ t_1 & -\frac{(-1+T_1)\,(1-T_1-T_5-T_9+T_5\,T_9+T_1\,T_5\,T_9)}{T_5\,(-1+T_1+T_5)\,T_9\,(-1+T_1+T_9)} & -\frac{(-1+T_1)\,T_1}{(-1+T_1+T_5)\,(-1+T_1+T_ \\ t_5 & -\frac{(-1+T_5)\,(-T_1-T_5+T_1\,T_5+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)} & -\frac{(-1+T_5)\,\left(-1+2\,T_1-T_1^2+T_5-T_1\,T_5+2\,T_9-2\,T_1\,T_9-T_5\,T_9-T_1 \\ t_9 & -\frac{(-1+T_9)\,(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)} & -\frac{(-1+T_9)\,(-T_5+T_1\,T_5-T_9+T_1\,T \\ (-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1 \end{pmatrix}$$

```
Bu[{h1, h5, h9}, β0]
```

$$\begin{aligned} &\text{Bu}\left[\frac{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)}{T_1^2\,T_5^2\,T_9^2},\ \left\{h_1 \to \frac{1}{T_5\,T_9},\ h_5 \to \frac{1}{T_1\,T_9},\ h_9 \to \frac{1}{T_1\,T_5}\right\},\right. \\ &\quad h_9\left(\frac{t_1\,(-1+T_1)}{-1+T_1+T_9} + \frac{t_5\,(-1+T_5)\,T_9}{(-1+T_1+T_9)\,(-1+T_5+T_9)} + \frac{t_9\,T_9^2}{(-1+T_1+T_9)\,(-1+T_5+T_9)}\right) + \\ &\quad h_1\left(\frac{t_1\,T_1^2}{(-1+T_1+T_5)\,(-1+T_1+T_9)} + \frac{t_5\,(-1+T_5)\,(-T_1-T_5+T_1\,T_5+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)} + \right. \\ &\qquad \left.\frac{t_9\,(-1+T_9)\,(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)}\right) + \\ &\quad h_5\left(\frac{t_1\,(-1+T_1)\,T_1}{(-1+T_1+T_5)\,(-1+T_1+T_9)} + \frac{t_9\,(-1+T_9)\,(-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)} + \right. \\ &\qquad \left.\left.\left(t_5\left(-1+T_1+T_5-T_1\,T_5-T_5^2+T_1\,T_5^2+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5\,T_9+T_5^2\,T_9\right)\right)\Big/ \right. \right. \\ &\qquad \left.\left.\left((-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)\right)\right)\right] \end{aligned}$$

```
βMVA[Bu[ω_, λ_, μ_]] := Module[
   {lbls, mat},
   lbls = Rest[First /@ λ];
   mat = Outer[
     Coefficient[μ - lbls.(lbls /. ha_ :> ta), #1 * #2] &,
     lbls, lbls /. ha_ :> ta
    ];
   Print[mat];
   Print[{ω, Det[mat]}];
   ω * Det[mat] / (1 - λ[[1, 1]] /. hi_ → Ti) // Factor
  ];
βMVA[L_Link] := βMVA[Bu[h# & /@ (First /@ Skeleton[L]), βZ[L]]]

βMVA[L]
```

```
{h5, h9}
```

$$\left\{\left\{-1+\frac{-1+T_1+T_5-T_1\,T_5-T_5^2+T_1\,T_5^2+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5\,T_9+T_5^2\,T_9}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)},\right.\right.$$
$$\left.\frac{(-1+T_9)\,(-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)}{(-1+T_1+T_5)\,(-1+T_1+T_9)\,(-1+T_5+T_9)}\right\},$$
$$\left\{\frac{(-1+T_5)\,T_9}{(-1+T_1+T_9)\,(-1+T_5+T_9)},\,-1+\frac{T_9^2}{(-1+T_1+T_9)\,(-1+T_5+T_9)}\right\}\}$$

βMVA[L = Link["L8a16"]]

$$\{\{-1-\big(T_{11}\,\big(1-2\,T_1+T_1^2-T_5+3\,T_1\,T_5-2\,T_1^2\,T_5-T_1\,T_5^2+T_1^2\,T_5^2-T_{11}+2\,T_1\,T_{11}-T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+$$
$$3\,T_1^2\,T_5\,T_{11}+2\,T_1\,T_5^2\,T_{11}-2\,T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\big)\big)\,/$$
$$\big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+$$
$$4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+$$
$$T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+$$
$$2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\big),$$
$$\big(T_5\,(-1+T_{11})\,\big(1-2\,T_1+T_1^2+T_1\,T_5-T_1^2\,T_5+T_1\,T_{11}-T_1^2\,T_{11}-3\,T_1\,T_5\,T_{11}+2\,T_1^2\,T_5\,T_{11}+$$
$$T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\big)\big)\,/$$
$$\big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+$$
$$4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+$$
$$T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+$$
$$2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\big)\big\},$$
$$\big\{\big((-1+T_5)\,T_{11}\,\big(1-2\,T_1+T_1^2-T_5+2\,T_1\,T_5-T_1^2\,T_5-T_{11}+2\,T_1\,T_{11}-T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+$$
$$3\,T_1^2\,T_5\,T_{11}+T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\big)\big)\,/$$
$$\big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+$$
$$4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+$$
$$T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+$$
$$2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\big),$$
$$-1-\big(T_5\,\big(1-2\,T_1+T_1^2-T_5+2\,T_1\,T_5-T_1^2\,T_5-T_{11}+3\,T_1\,T_{11}-2\,T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+3\,T_1^2\,T_5\,T_{11}+$$
$$T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}-T_1\,T_{11}^2+T_1^2\,T_{11}^2+2\,T_1\,T_5\,T_{11}^2-2\,T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\big)\big)\,/$$
$$\big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+$$
$$4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+$$
$$T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+$$
$$2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\big)\}\}$$

$$\Big\{\frac{1}{T_1\,T_5\,T_{11}}\Big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\Big),$$

$$-\Big((-1+T_5)\,T_5\,(-1+T_{11})\,T_{11}\Big(1-2\,T_1+T_1^2+T_1\,T_5-T_1^2\,T_5+T_1\,T_{11}-T_1^2\,T_{11}-3\,T_1\,T_5\,T_{11}+2\,T_1^2\,T_5\,T_{11}+T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\Big)\Big(1-2\,T_1+T_1^2-T_5+2\,T_1\,T_5-T_1^2\,T_5-T_{11}+2\,T_1\,T_{11}-T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+3\,T_1^2\,T_5\,T_{11}+T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\Big)\Big)\Big/\Big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\Big)^2+\Big(-1-\Big(T_5\Big(1-2\,T_1+T_1^2-T_5+2\,T_1\,T_5-T_1^2\,T_5-T_{11}+3\,T_1\,T_{11}-2\,T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+3\,T_1^2\,T_5\,T_{11}+T_1\,T_5^2\,T_{11}-T_1^2\,T_5^2\,T_{11}-T_1\,T_{11}^2+T_1^2\,T_{11}^2+2\,T_1\,T_5\,T_{11}^2-2\,T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\Big)\Big)\Big/\Big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\Big)\Big)^2+\Big(-1-\Big(T_{11}\Big(1-2\,T_1+T_1^2-T_5+3\,T_1\,T_5-2\,T_1^2\,T_5-T_1\,T_5^2+T_1^2\,T_5^2-T_{11}+2\,T_1\,T_{11}-T_1^2\,T_{11}+T_5\,T_{11}-5\,T_1\,T_5\,T_{11}+3\,T_1^2\,T_5\,T_{11}+2\,T_1\,T_5^2\,T_{11}-2\,T_1^2\,T_5^2\,T_{11}+T_1\,T_5\,T_{11}^2-T_1^2\,T_5\,T_{11}^2-T_1\,T_5^2\,T_{11}^2+T_1^2\,T_5^2\,T_{11}^2\Big)\Big)\Big/\Big(1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\Big)\Big)\Big\}$$

$$-\frac{(-1+T_1)\,(-1+T_5)\,(-1+T_{11})\,(1+T_5\,T_{11})}{T_1\,T_5\,T_{11}}$$

βZ[L]

$$\left(\frac{1-2\,T_1+T_1^2-2\,T_5+4\,T_1\,T_5-2\,T_1^2\,T_5+T_5^2-2\,T_1\,T_5^2+T_1^2\,T_5^2-2\,T_{11}+4\,T_1\,T_{11}-2\,T_1^2\,T_{11}+4\,T_5\,T_{11}-10\,T_1\,T_5\,T_{11}+6\,T_1^2\,T_5\,T_{11}-2\,T_5^2\,T_{11}+8\,T_1\,T_5^2\,T_{11}-5\,T_1^2\,T_5^2\,T_{11}-T_1\,T_5^3\,T_{11}+T_1^2\,T_5^3\,T_{11}+T_{11}^2-2\,T_1\,T_{11}^2+T_1^2\,T_{11}^2-2\,T_5\,T_{11}^2+8\,T_1\,T_5\,T_{11}^2-5\,T_1^2\,T_5\,T_{11}^2+T_5^2\,T_{11}^2-8\,T_1\,T_5^2\,T_{11}^2+6\,T_1^2\,T_5^2\,T_{11}^2+2\,T_1\,T_5^3\,T_{11}^2-2\,T_1^2\,T_5^3\,T_{11}^2-T_1\,T_5\,T_{11}^3+T_1^2\,T_5\,T_{11}^3+2\,T_1\,T_5^2\,T_{11}^3-2\,T_1^2\,T_5^2\,T_{11}^3-T_1\,T_5^3\,T_{11}^3+T_1^2\,T_5^3\,T_{11}^3\right)$$

Simplify[1/βMVA[#] (MultivariableAlexander[#][T] /. T[i_] :> T$_{\text{Skeleton[#][[i,1]]}}$)] & /@ AllLinks[8]

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$Aborted

test = $-\left(\sqrt{T_1}\ (-1+T_5)\ (-1+T_{11})\right)\Big/$
$\Big((-1+T_5)^2\ (-1+T_{11})^2+T_1^2\ (-1+T_5)\ (-1+T_{11})\ \left(1-T_{11}+T_5^2\ (-1+T_{11})\ T_{11}-\right.$
$\left.T_5\ \left(1-3\ T_{11}+T_{11}^2\right)\right)-T_1\ \left(2\ (-1+T_{11})^2+T_5^3\ (-1+T_{11})^2\ T_{11}+T_5\right.$
$\left.\left.\left(-4+10\ T_{11}-8\ T_{11}^2+T_{11}^3\right)-2\ T_5^2\ \left(-1+4\ T_{11}-4\ T_{11}^2+T_{11}^3\right)\right)\right)^2$

$-\left(\sqrt{T_1}\ (-1+T_5)\ (-1+T_{11})\right)\Big/\Big((-1+T_5)^2\ (-1+T_{11})^2+$
$T_1^2\ (-1+T_5)\ (-1+T_{11})\ \left(1-T_{11}+T_5^2\ (-1+T_{11})\ T_{11}-T_5\ \left(1-3\ T_{11}+T_{11}^2\right)\right)-T_1\ \left(2\ (-1+T_{11})^2+\right.$
$\left.\left.T_5^3\ (-1+T_{11})^2\ T_{11}+T_5\ \left(-4+10\ T_{11}-8\ T_{11}^2+T_{11}^3\right)-2\ T_5^2\ \left(-1+4\ T_{11}-4\ T_{11}^2+T_{11}^3\right)\right)\right)^2$

AllLinks[8][[16]]

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Link[8, Alternating, 16]
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MultivariableAlexander[L][T]

$$\frac{(-1+T[1])\ (-1+T[2])\ (-1+T[3])\ (1+T[2]\ T[3])}{\sqrt{T[1]}\ T[2]\ T[3]}$$