Pensieve header: Programs for β-calculus, development notebook.

KnotTheory

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<< KnotTheory`
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KnotTheory

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Loading KnotTheory` version of February 5, 2013, 3:48:46.4762.
Read more at http://katlas.org/wiki/KnotTheory.
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Initialization

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], (t | T)_s_ :→ s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :→ s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :→ βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

Program

```
⟨μ_⟩ := μ /. t_ → 1;
tm_{x_,y_→z_}[β_] := βCollect[β /. {t_{x|y} → t_z, T_{x|y} → T_z}];
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{x_,y_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
          + γ / ϵ h_y               + δ - γ * β / ϵ
    ] // βCollect];
gm_{x_,y_→z_}[β_] := β // sw_{xy} // hm_{xy→z} // tm_{xy→z};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
(R^+)_{x_,y_} := B[1, (T_x - 1) t_x h_y];
(R^-)_{x_,y_} := B[1, ((T_x)^-1 - 1) t_x h_y];
```

tm

$$\{\beta = \mathbf{B}[\omega, \mathbf{Sum}[\alpha_{2\,i+j-6}\, t_i\, h_j,\ \{i, 1, 4\},\ \{j, 5, 6\}]],$$
$$O_1 = \beta\ //\ tm_{12\to 1}\ //\ tm_{13\to 1},$$
$$O_2 = \beta\ //\ tm_{23\to 2}\ //\ tm_{12\to 1},$$
$$O_1 == O_2$$
$$\}\ //\ \mathbf{ColumnForm}$$

tm

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$
$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$
$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$
True

hm

$$\{\beta = \mathbf{B}[\omega, \mathbf{Sum}[\alpha_{4\,i+j-6}\, t_i\, h_j,\ \{i, 1, 2\},\ \{j, 3, 6\}]],$$
$$O_1 = \beta\ //\ hm_{34\to 3}\ //\ hm_{35\to 3},$$
$$O_2 = \beta\ //\ hm_{45\to 4}\ //\ hm_{34\to 3};$$
$$O_1 == O_2$$
$$\}\ /.\ \alpha_{i\_} \mapsto \hat{i}\ //\ \mathbf{ColumnForm}$$

hm

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 & h_6 \\ t_1 & \hat{1} & \hat{2} & \hat{3} & \hat{4} \\ t_2 & \hat{5} & \hat{6} & \hat{7} & \hat{8} \end{pmatrix}$$
$$\begin{pmatrix} \omega & h_3 & h_6 \\ t_1 & \hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{3}+\hat{1}\,\hat{3}+\hat{2}\,\hat{3}+\hat{1}\,\hat{2}\,\hat{3}+\hat{2}\,\hat{5}+\hat{3}\,\hat{5}+\hat{2}\,\hat{3}\,\hat{5}+\hat{3}\,\hat{6}+\hat{1}\,\hat{3}\,\hat{6}+\hat{3}\,\hat{5}\,\hat{6} & \hat{4} \\ t_2 & \hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{5}\,\hat{6}+\hat{7}+\hat{1}\,\hat{7}+\hat{2}\,\hat{7}+\hat{1}\,\hat{2}\,\hat{7}+\hat{5}\,\hat{7}+\hat{2}\,\hat{5}\,\hat{7}+\hat{6}\,\hat{7}+\hat{1}\,\hat{6}\,\hat{7}+\hat{5}\,\hat{6}\,\hat{7} & \hat{8} \end{pmatrix}$$
True

htt

$$\{\beta = \mathbf{B}[\omega, \mathbf{Sum}[\alpha_{2\,i+j-5}\, t_i\, h_j,\ \{i, 1, 3\},\ \{j, 4, 5\}]],$$
$$O_1 = \beta\ //\ tm_{12\to 1}\ //\ sw_{14},$$
$$O_2 = \beta\ //\ sw_{24}\ //\ sw_{14}\ //\ tm_{12\to 1};$$
$$O_1 == O_2\}$$

htt

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_1+\alpha_3) & h_4 & h_5 \\ t_1 & \frac{(\alpha_1+\alpha_3)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} & \frac{(\alpha_2+\alpha_4)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} \\ t_3 & \frac{\alpha_5}{1+\alpha_1+\alpha_3} & \frac{-\alpha_2\,\alpha_5-\alpha_4\,\alpha_5+\alpha_6+\alpha_1\,\alpha_6+\alpha_3\,\alpha_6}{1+\alpha_1+\alpha_3} \end{pmatrix}, \text{True}\right\}$$

hht

$$\begin{aligned}&\Big\{\beta = \mathbf{B}\big[\omega,\ \mathbf{Sum}\big[\alpha_{3\,i+j-5}\,t_i\,h_j,\ \{i,1,2\},\ \{j,3,5\}\big]\big],\\ &\quad O_1 = \beta\ //\ hm_{34\to 3}\ //\ sw_{13}\ //\ \beta\mathrm{Collect},\\ &\quad O_2 = \beta\ //\ sw_{13}\ //\ sw_{14}\ //\ hm_{34\to 3}\ //\ \beta\mathrm{Collect};\\ &\quad O_1 == O_2\\ &\ \Big\}\ /.\ \alpha_{i\_} :\to \hat{i}\ //\ \mathrm{ColumnForm}\end{aligned}$$

hht

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 \\ t_1 & \hat{1} & \hat{2} & \hat{3} \\ t_2 & \hat{4} & \hat{5} & \hat{6} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right) & h_3 & h_5 \\ t_1 & \frac{\left(1+\hat{1}+\hat{4}\right)\left(\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{\hat{3}\left(1+\hat{1}+\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \\ t_2 & \frac{\hat{4}+\hat{5}+\hat{1}\,\hat{5}+\hat{4}\,\hat{5}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{-\hat{3}\,\hat{4}-\hat{3}\,\hat{5}-\hat{1}\,\hat{3}\,\hat{5}-\hat{3}\,\hat{4}\,\hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{2}\,\hat{6}+\hat{1}\,\hat{2}\,\hat{6}+\hat{2}\,\hat{4}\,\hat{6}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \end{pmatrix}$$

True

R3

$$\begin{aligned}&\{(R^-)_{51}\,(R^-)_{62}\,(R^+)_{34}\ //\ gm_{14\to 1}\ //\ gm_{25\to 2}\ //\ gm_{36\to 3},\\ &\ (R^+)_{61}\,(R^-)_{24}\,(R^-)_{35}\ //\ gm_{14\to 1}\ //\ gm_{25\to 2}\ //\ gm_{36\to 3}\}\end{aligned}$$

R3

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix},\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}\right\}$$

8_17-1

$$\beta = (R^-)_{12,1}\,(R^-)_{27}\,(R^-)_{83}\,(R^-)_{4,11}\,(R^+)_{16,5}\,(R^+)_{6,13}\,(R^+)_{14,9}\,(R^+)_{10,15}$$

8_17-1

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+T_4}{T_4} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_6 & 0 \\ t_8 & 0 & -\frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{10} \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+T_{14} & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+T_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

8_17-2

$$\mathrm{Do}[\beta = \beta\ //\ gm_{1k\to 1},\ \{k, 2, 10\}];\ \beta$$

8_17-2

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1\,T_{16}}{T_1^2} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+T_1)\,T_{14}\,(T_1^3+T_{16}^2)}{T_1^2\,T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})} & -\frac{(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1\,T_{16})} & \frac{(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}}{T_1^2+T_{16}-T_1\,T_{16}} & -1+T_1 \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 \\ t_{14} & \frac{(-1+T_{14})\,(-T_1+T_1^2+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})} & \frac{(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1\,T_{16})} & -\frac{(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \\ t_{16} & \frac{T_1\,(-1+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})} & \frac{(-1+T_1)\,T_1\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & -\frac{(-1+T_1)^2\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \end{pmatrix}$$

8_17-3

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

8_17-3

$$\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} \\ t_1 \end{pmatrix}$$

8_17-4

```
Alexander[Knot[8, 17]][X]
```

8_17-4

KnotTheory::loading : Loading precomputed data in PD4Knots`.

8_17-4

$$11 - \frac{1}{X^3} + \frac{4}{X^2} - \frac{8}{X} - 8X + 4X^2 - X^3$$

Recycling

StandardAlexander

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & T-1 & 0 & -T \\ -1 & T & 0 & 0 & 0 & 0 & 1-T & 0 \\ 0 & -1 & T & 0 & 1-T & 0 & 0 & 0 \\ T-1 & 0 & -T & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-T & 0 & -1 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -T & 1 & 0 & T-1 \\ 0 & 0 & 1-T & 0 & 0 & -1 & T & 0 \\ 0 & 0 & 0 & T-1 & 0 & 0 & -T & 1 \end{pmatrix} \texttt{[[1 ;; 7, 1 ;; 7]] // Det}$$

StandardAlexander

$$-1 + 4T - 8T^2 + 11T^3 - 8T^4 + 4T^5 - T^6$$

The Borromean Link

$$\beta = (R^-)_{r,6}\,(R^+)_{2,4}\,(R^-)_{g,9}\,(R^+)_{5,7}\,(R^-)_{b,3}\,(R^+)_{8,1}$$

$$\begin{pmatrix} 1 & h_1 & h_3 & h_4 & h_6 & h_7 & h_9 \\ t_2 & 0 & 0 & -1+T_2 & 0 & 0 & 0 \\ t_5 & 0 & 0 & 0 & 0 & -1+T_5 & 0 \\ t_8 & -1+T_8 & 0 & 0 & 0 & 0 & 0 \\ t_b & 0 & -\frac{-1+T_b}{T_b} & 0 & 0 & 0 & 0 \\ t_g & 0 & 0 & 0 & 0 & 0 & -\frac{-1+T_g}{T_g} \\ t_r & 0 & 0 & 0 & -\frac{-1+T_r}{T_r} & 0 & 0 \end{pmatrix}$$

```
Do[β = β // gm_{rk→r}, {k, 1, 3}];
Do[β = β // gm_{gk→g}, {k, 4, 6}];
Do[β = β // gm_{bk→b}, {k, 7, 9}];
β /. {T_c_ :→ c}
```

$$\begin{pmatrix} \frac{1-2b+b^2-2g+3bg-b^2g+g^2-bg^2-2r+3br-b^2r+3gr-3bgr+b^2gr-g^2r+bg^2r+r^2-br^2-gr^2+bgr^2}{bgr} & & \\ t_b & & \overline{1-2b+b^2-2g+3bg-b^2g+g^2-bg^2-} \\ t_g & & -\overline{1-2b+b^2-2g+3bg-b^2g+g^2-bg^2} \\ t_r & & \overline{1-2b+b^2-2g+3bg-b^2g+g^2-bg^2-} \end{pmatrix}$$

Work in Progress

```
βZ[L_] := Module[{s, β, c, k},
  s = Skeleton[L];
  β = Times @@ PD[L] /. X[i_, j_, k_, l_] :→ If[
      PositiveQ[X[i, j, k, l]],
      (R⁺)_{l,i}, (R⁻)_{j,i}];
  Do[β = β // gm_{s[[c,1]], s[[c,k]]→s[[c,1]]},
   {c, Length[s]}, {k, 2, Length[s[[c]]]}];
  β]
```

```
βZ[Knot[8, 17]] // First
```

$$-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^2}$$

```
Factor[βZ[#][[1]] / Alexander[#][T₁]] & /@ AllKnots[{3, 8}]
```

$$\left\{\frac{1}{T_1}, T_1, \frac{1}{T_1^2}, \frac{1}{T_1^2}, 1, 1, 1, \frac{1}{T_1^3}, \frac{1}{T_1^3}, T_1^4, T_1^4, \frac{1}{T_1^3}, \frac{1}{T_1}, T_1^2, \frac{1}{T_1}, \frac{1}{T_1}, T_1, T_1, T_1^3, \frac{1}{T_1}, T_1, T_1, T_1, T_1, \frac{1}{T_1}, T_1, T_1, \frac{1}{T_1}, \frac{1}{T_1^3}, \frac{1}{T_1}, T_1, 1, T_1^4, 1, \frac{1}{T_1}\right\}$$

```
βMVA[L_Link] := Module[{ηs, ω, μ, M},
  {ω, μ} = List @@ βZ[L];
  ηs = Rest[h_# & /@ (First /@ Skeleton[L])];
  M = Outer[
    Coefficient[μ - (μ /. t_ → 1 /. h_a_ :→ t_a h_a), #1 * #2] &,
    ηs, ηs /. h_a_ :→ t_a];
  Factor[ω Det[M] / (1 - T_{Skeleton[L][[1,1]]})]]
```

βMVA[Link["L8a16"]]

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$-\frac{(-1+T_1)\ (-1+T_5)\ (-1+T_{11})\ (1+T_5\ T_{11})}{T_1\ T_5\ T_{11}}$$

Simplify[$\frac{1}{\beta\text{MVA}[\#]}$(MultivariableAlexander[#][T] /. T[i_] :> T$_{\text{Skeleton}[\#][[i,1]]}$)] & /@ AllLinks[8]

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{\sqrt{T_1}\ T_5^{3/2},\ \frac{\sqrt{T_1}}{\sqrt{T_5}},\ T_1^{3/2}\ T_5^{3/2},\ \sqrt{T_1}\ T_5^{3/2},\ \frac{T_1^{3/2}}{\sqrt{T_5}},\ \frac{T_1^{3/2}}{\sqrt{T_5}},\ T_1^{3/2}\ T_5^{7/2},\ \frac{T_1}{T_7},\ T_1\ T_7,\ T_1^2\ T_7^3,\right.$$
$$T_1^2\ T_7^3,\ T_1^{5/2}\ T_9^{5/2},\ T_1^{5/2}\ T_9^{5/2},\ T_1^{5/2}\ T_9^{5/2},\ -T_1^{3/2}\ T_5^{3/2}\ \sqrt{T_9},\ -\sqrt{T_1},\ -T_1^{3/2}\ T_5^2\ T_{11}^2,\ -\frac{\sqrt{T_1}}{T_{11}^2},$$
$$-\frac{\sqrt{T_1}\ T_5}{T_{11}},\ -\frac{T_1^{3/2}\ \sqrt{T_5}}{T_{13}^{3/2}},\ T_1^{3/2}\ T_5^{3/2}\ T_9^{3/2}\ T_{13}^{3/2},\ T_1^{3/2}\ T_5^{3/2},\ \sqrt{T_1}\ \sqrt{T_5},\ -T_1^{3/2}\ T_5^2\ T_{11}^2,$$
$$\left.-\sqrt{T_1}\ T_5^2\ T_{11},\ -\sqrt{T_1}\ T_5^{3/2}\ T_{11}^{3/2},\ -T_1^{3/2}\ T_5^{5/2}\ \sqrt{T_{13}},\ \frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}\ \sqrt{T_{13}}},\ \sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_9}\ \sqrt{T_{13}}\right\}$$