Pensieve header: Programs for β-calculus.

KnotTheory

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<< KnotTheory`
```

KnotTheory

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Loading KnotTheory` version of February 5, 2013, 3:48:46.4762.
Read more at http://katlas.org/wiki/KnotTheory.
```

Initialization

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_s_ :> s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :> s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

Program

```
⟨μ_⟩ := μ /. t_ → 1;
tm_x_y_→z_[β_] := βCollect[β /. {t_x|y → t_z, T_x|y → T_z}];
hm_x_y_→z_[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_x|y → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_x_y_[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
         + γ / ϵ h_y              + δ - γ * β / ϵ
    ] // βCollect];
gm_x_y_→z_[β_] := β // sw_xy // hm_xy→z // tm_xy→z;
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
(R^+)_x_y_ := B[1, (T_x - 1) t_x h_y];
(R^-)_x_y_ := B[1, ((T_x)^-1 - 1) t_x h_y];
```

tm

```
{β = B[ω, Sum[α_{2 i+j-6} t_i h_j, {i, 1, 4}, {j, 5, 6}]],
  O_1 = β // tm_{12→1} // tm_{13→1},
  O_2 = β // tm_{23→2} // tm_{12→1},
  O_1 == O_2} // ColumnForm
```

tm

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_5 & h_6 \\ t_1 & \alpha_1+\alpha_3+\alpha_5 & \alpha_2+\alpha_4+\alpha_6 \\ t_4 & \alpha_7 & \alpha_8 \end{pmatrix}$$

True

hm

```
{β = B[ω, Sum[α_{4 i+j-6} t_i h_j, {i, 1, 2}, {j, 3, 6}]],
   O_1 = β // hm_{34→3} // hm_{35→3},
   O_2 = β // hm_{45→4} // hm_{34→3};
   O_1 == O_2} /. α_i_ :> î // ColumnForm
```

hm

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 & h_6 \\ t_1 & \hat{1} & \hat{2} & \hat{3} & \hat{4} \\ t_2 & \hat{5} & \hat{6} & \hat{7} & \hat{8} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_3 & h_6 \\ t_1 & \hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{3}+\hat{1}\,\hat{3}+\hat{2}\,\hat{3}+\hat{1}\,\hat{2}\,\hat{3}+\hat{2}\,\hat{5}+\hat{3}\,\hat{5}+\hat{2}\,\hat{3}\,\hat{5}+\hat{3}\,\hat{6}+\hat{1}\,\hat{3}\,\hat{6}+\hat{3}\,\hat{5}\,\hat{6} & \hat{4} \\ t_2 & \hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{5}\,\hat{6}+\hat{7}+\hat{1}\,\hat{7}+\hat{2}\,\hat{7}+\hat{1}\,\hat{2}\,\hat{7}+\hat{5}\,\hat{7}+\hat{2}\,\hat{5}\,\hat{7}+\hat{6}\,\hat{7}+\hat{1}\,\hat{6}\,\hat{7}+\hat{5}\,\hat{6}\,\hat{7} & \hat{8} \end{pmatrix}$$

True

htt

```
{β = B[ω, Sum[α_{2 i+j-5} t_i h_j, {i, 1, 3}, {j, 4, 5}]],
 O_1 = β // tm_{12→1} // sw_{14},
 O_2 = β // sw_{24} // sw_{14} // tm_{12→1};
 O_1 == O_2}
```

htt

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_1+\alpha_3) & h_4 & h_5 \\ t_1 & \frac{(\alpha_1+\alpha_3)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} & \frac{(\alpha_2+\alpha_4)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} \\ t_3 & \frac{\alpha_5}{1+\alpha_1+\alpha_3} & \frac{-\alpha_2\,\alpha_5-\alpha_4\,\alpha_5+\alpha_6+\alpha_1\,\alpha_6+\alpha_3\,\alpha_6}{1+\alpha_1+\alpha_3} \end{pmatrix}, \text{True}\right\}$$

hht

```
{β = B[ω, Sum[α_{3 i+j-5} t_i h_j, {i, 1, 2}, {j, 3, 5}]],
   O_1 = β // hm_{34→3} // sw_13 // βCollect,
   O_2 = β // sw_13 // sw_14 // hm_{34→3} // βCollect;
   O_1 == O_2
  } /. α_i_ :→ î // ColumnForm
```

hht

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 \\ t_1 & \hat{1} & \hat{2} & \hat{3} \\ t_2 & \hat{4} & \hat{5} & \hat{6} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right) & h_3 & h_5 \\ t_1 & \frac{\left(1+\hat{1}+\hat{4}\right)\left(\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{\hat{3}\left(1+\hat{1}+\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \\ t_2 & \frac{\hat{4}+\hat{5}+\hat{1}\,\hat{5}+\hat{4}\,\hat{5}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{-\hat{3}\,\hat{4}-\hat{3}\,\hat{5}-\hat{1}\,\hat{3}\,\hat{5}-\hat{3}\,\hat{4}\,\hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{2}\,\hat{6}+\hat{1}\,\hat{2}\,\hat{6}+\hat{2}\,\hat{4}\,\hat{6}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \end{pmatrix}$$

True

R3

```
{(R^-)_51 (R^-)_62 (R^+)_34 // gm_{14→1} // gm_{25→2} // gm_{36→3},
  (R^+)_61 (R^-)_24 (R^-)_35 // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```

R3

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}\right\}$$

8_17-1

```
β = (R^-)_{12,1} (R^-)_27 (R^-)_83 (R^-)_{4,11} (R^+)_{16,5} (R^+)_{6,13} (R^+)_{14,9} (R^+)_{10,15}
```

8_17-1

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+T_4}{T_4} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_6 & 0 \\ t_8 & 0 & -\frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{10} \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+T_{14} & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+T_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

8_17-2

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_17-2

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1\,T_{16}}{T_1^2} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+T_1)\,T_{14}\,\left(T_1^3+T_{16}^2\right)}{T_1^2\,T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & -\frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,T_{14}\,T_{16}}{T_1\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,T_{14}}{T_1^2+T_{16}-T_1\,T_{16}} & -1+T_1 \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 \\ t_{14} & \frac{(-1+T_{14})\,\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,(-1+T_{14})\,T_{16}}{T_1\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & -\frac{(-1+T_1)\,\left(1-T_1+T_1^2\right)\,(-1+T_{14})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \\ t_{16} & \frac{T_1\,(-1+T_{16})}{T_{12}\,\left(T_1^2+T_{16}-T_1\,T_{16}\right)} & \frac{(-1+T_1)\,T_1\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & -\frac{(-1+T_1)^2\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}} & 0 \end{pmatrix}$$

8_17-3

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

8_17-3

$$\left(-\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^3}\right)$$

8_17-4

```
Alexander[Knot[8, 17]][X]
```

8_17-4

KnotTheory::loading : Loading precomputed data in PD4Knots`.

8_17-4

$$11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8\,X+4\,X^2-X^3$$

betaZ

```
βZ[L_] := Module[{s, β, c, k},
  s = Skeleton[L];
  β = Times @@ PD[L] /. X[i_, j_, k_, l_] :> If[
       PositiveQ[X[i, j, k, l]],
       (R⁺)_{l,i}, (R⁻)_{j,i}];
  Do[β = β // gm_{s[[c,1]], s[[c,k]]→s[[c,1]]},
   {c, Length[s]}, {k, 2, Length[s[[c]]]}];
  β]
```

```
βZ[Knot[8, 17]] // First
```

$$-\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^2}$$

TestAllKnots

```
Factor[βZ[#][[1]] / Alexander[#][T_1]] & /@ AllKnots[{3, 8}]
```

TestAllKnots

$$\left\{\frac{1}{T_1},\ T_1,\ \frac{1}{T_1^2},\ \frac{1}{T_1^2},\ 1,\ 1,\ 1,\ \frac{1}{T_1^3},\ \frac{1}{T_1^3},\ T_1^4,\ T_1^4,\ \frac{1}{T_1^3},\ \frac{1}{T_1},\ T_1^2,\ \frac{1}{T_1},\ \frac{1}{T_1},\right.$$
$$\left.T_1,\ T_1,\ T_1^3,\ \frac{1}{T_1},\ T_1,\ T_1,\ T_1,\ T_1,\ \frac{1}{T_1},\ T_1,\ T_1,\ \frac{1}{T_1},\ \frac{1}{T_1^3},\ \frac{1}{T_1},\ T_1,\ 1,\ T_1^4,\ 1,\ \frac{1}{T_1}\right\}$$

```
Factor[βZ[#][[1]] / Alexander[#][T1]] & /@ AllKnots[{3, 11}] // Union
```

KnotTheory::loading : Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory::credits :

The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

$$\left\{1, \frac{1}{T_1^5}, \frac{1}{T_1^4}, \frac{1}{T_1^3}, \frac{1}{T_1^2}, \frac{1}{T_1}, T_1, T_1^2, T_1^3, T_1^4, T_1^5, T_1^6, T_1^7\right\}$$

betaMVA

```
βMVA[L_Link] := Module[{ηs, ω, μ, M},
  {ω, μ} = List @@ βZ[L];
  ηs = Rest[h# & /@ (First /@ Skeleton[L])];
  M = Outer[
    Coefficient[μ - (μ /. t_ → 1 /. ha_ :→ ta ha), #1 * #2] &,
    ηs, ηs /. ha_ :→ ta];
  Factor[ω Det[M] / (1 - TSkeleton[L][[1,1]])]]
```

BorromeanMVA

```
βMVA[Link["L6a4"]]
```

BorromeanMVA

KnotTheory::loading : Loading precomputed data in PD4Links`.

BorromeanMVA

$$-\frac{(-1+T_1)\,(-1+T_5)\,(-1+T_9)}{T_1\,T_5}$$

TestAllLinks

```
Factor[1/βMVA[#] (MultivariableAlexander[#][T] /. T[i_] :→ TSkeleton[#][[i,1]])] & /@
 AllLinks[{2, 7}]
```

TestAllLinks

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

TestAllLinks

$$\left\{T_1^2\,T_3,\ T_1^{3/2}\,T_5^{3/2},\ \sqrt{T_1}\ T_5^{3/2},\ T_1^{3/2}\sqrt{T_5},\ T_1^2\,T_7^2,\ T_1^2\,T_7^2,\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{3/2}},\right.$$

$$\left.\sqrt{T_1}\ \sqrt{T_5},\ T_1^{3/2}\,T_5^{7/2},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ T_1\,T_7^2,\ \frac{1}{T_7},\ -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ T_1^{3/2}\,T_5^{7/2},\ \sqrt{T_1}\ T_5^{5/2}\right\}$$

```
Factor[1/βMVA[#] (MultivariableAlexander[#][T] /. T[i_] :→ TSkeleton[#][[i,1]])] & /@
  AllLinks[{2, 11}] // Union
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

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Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

General::stop : Further output of Infinity::indet will be suppressed during this calculation. ≫

$$\Big\{1,\ \text{Indeterminate},\ -\sqrt{T_1},\ T_1,\ -T_1^{3/2},\ T_1^2,\ T_1^2\,T_3,\ \frac{\sqrt{T_1}}{T_5^{7/2}},\ \frac{\sqrt{T_1}}{T_5^{5/2}},\ \frac{T_1^{3/2}}{T_5^{5/2}},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ \frac{T_1^{3/2}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5},$$

$$\frac{\sqrt{T_1}}{\sqrt{T_5}},\ \frac{T_1^{3/2}}{\sqrt{T_5}},\ \sqrt{T_1}\ \sqrt{T_5},\ T_1^{3/2}\sqrt{T_5},\ -\sqrt{T_1}\ T_5,\ -T_1^{3/2}\,T_5,\ \sqrt{T_1}\ T_5^{3/2},\ T_1^{3/2}\,T_5^{3/2},\ -\sqrt{T_1}\ T_5^2,$$

$$-T_1^{3/2}\,T_5^2,\ \sqrt{T_1}\ T_5^{5/2},\ T_1^{3/2}\,T_5^{5/2},\ \sqrt{T_1}\ T_5^{7/2},\ T_1^{3/2}\,T_5^{7/2},\ \sqrt{T_1}\ T_5^{9/2},\ T_1^{3/2}\,T_5^{9/2},\ T_1^{3/2}\,T_5^{11/2},\ \frac{1}{T_7^3},\ \frac{1}{T_7^2},$$

$$\frac{T_1}{T_7^2},\ \frac{1}{T_7},\ \frac{T_1}{T_7},\ \frac{T_1^2}{T_7},\ T_7,\ T_1\,T_7,\ T_1^2\,T_7,\ T_7^2,\ T_1\,T_7^2,\ T_1^2\,T_7^2,\ T_7^3,\ T_1\,T_7^3,\ T_1^2\,T_7^3,\ T_1\,T_7^4,\ T_1^2\,T_7^4,\ T_1^2\,T_7^5,$$

$$-\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{7/2}},\ \frac{1}{\sqrt{T_1}\ T_9^{5/2}},\ \frac{\sqrt{T_1}}{T_9^{5/2}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{5/2}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_9^{5/2}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{T_9^{5/2}},\ \frac{1}{\sqrt{T_1}\ T_9^{3/2}},\ \frac{\sqrt{T_1}}{T_9^{3/2}},$$

$$\frac{T_1^{3/2}}{T_9^{3/2}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{3/2}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_9^{3/2}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{T_9^{3/2}},\ \frac{1}{\sqrt{T_1}\ \sqrt{T_9}},\ \frac{\sqrt{T_1}}{\sqrt{T_9}},\ \frac{T_1^{3/2}}{\sqrt{T_9}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}},$$

$$-\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{\sqrt{T_9}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{\sqrt{T_9}},\ \sqrt{T_1}\ \sqrt{T_9},\ T_1^{3/2}\sqrt{T_9},\ T_1^{5/2}\sqrt{T_9},\ -T_1^{3/2}\,T_5^{3/2}\sqrt{T_9},$$

$$\frac{T_9^{3/2}}{\sqrt{T_1}},\ \sqrt{T_1}\ T_9^{3/2},\ T_1^{3/2}\,T_9^{3/2},\ T_1^{5/2}\,T_9^{3/2},\ -\sqrt{T_1}\ \sqrt{T_5}\ T_9^{3/2},\ -T_1^{3/2}\sqrt{T_5}\ T_9^{3/2},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2},$$

$$\sqrt{T_1}\ T_9^{5/2},\ T_1^{3/2}\,T_9^{5/2},\ T_1^{5/2}\,T_9^{5/2},\ -\sqrt{T_1}\ \sqrt{T_5}\ T_9^{5/2},\ -T_1^{3/2}\sqrt{T_5}\ T_9^{5/2},\ -\sqrt{T_1}\ T_5^{3/2}\,T_9^{5/2},$$

$$-T_1^{3/2}\,T_5^{3/2}\,T_9^{5/2},\ T_1^{3/2}\,T_9^{7/2},\ T_1^{5/2}\,T_9^{7/2},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{7/2},\ T_1^{5/2}\,T_9^{9/2},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{9/2},\ -\frac{\sqrt{T_1}}{T_{11}^4},$$

$$-\frac{\sqrt{T_1}}{T_{11}^3},\ -\frac{\sqrt{T_1}\ T_5}{T_{11}^3},\ \frac{1}{T_1\,T_{11}^2},\ -\frac{\sqrt{T_1}}{T_{11}^2},\ -\frac{T_1^{3/2}}{T_{11}^2},\ -\frac{\sqrt{T_1}\ T_5}{T_{11}^2},\ \frac{1}{T_1^{3/2}\,T_{11}^{3/2}},\ \frac{1}{\sqrt{T_1}\ T_{11}^{3/2}},\ \frac{T_1^{5/2}}{T_{11}^{3/2}},$$

$$-\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_{11}^{3/2}},\ \frac{1}{T_{11}},\ -\frac{\sqrt{T_1}}{T_{11}},\ -\frac{T_1^{3/2}}{T_{11}},\ -\frac{\sqrt{T_1}\ T_5}{T_{11}},\ -\frac{T_1^{3/2}\,T_5}{T_{11}},\ -\frac{\sqrt{T_1}\ T_5^2}{T_{11}},\ \frac{1}{\sqrt{T_1}\ \sqrt{T_{11}}},$$

$$\frac{\sqrt{T_1}}{\sqrt{T_{11}}},\ \frac{T_1^{3/2}}{\sqrt{T_{11}}},\ \frac{T_1^{5/2}}{\sqrt{T_{11}}},\ \frac{T_1^{7/2}}{\sqrt{T_{11}}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{\sqrt{T_{11}}},\ \frac{\sqrt{T_{11}}}{T_1^{3/2}},\ \frac{\sqrt{T_{11}}}{\sqrt{T_1}},\ \sqrt{T_1}\ \sqrt{T_{11}},\ T_1^{3/2}\sqrt{T_{11}},$$

$$-\sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_{11}},\ \sqrt{T_1}\ T_5^2\sqrt{T_{11}},\ T_{11},\ -\sqrt{T_1}\ T_{11},\ -T_1^{3/2}\,T_{11},\ T_1^2\,T_{11},\ -\sqrt{T_1}\ T_5\,T_{11},$$

$$-T_1^{3/2}\,T_5\,T_{11},\ -\sqrt{T_1}\ T_5^2\,T_{11},\ -T_1^{3/2}\,T_5^2\,T_{11},\ \frac{T_{11}^{3/2}}{\sqrt{T_1}},\ \sqrt{T_1}\ T_{11}^{3/2},\ T_1^{3/2}\,T_{11}^{3/2},\ T_1^{5/2}\,T_{11}^{3/2},\ T_1^{7/2}\,T_{11}^{3/2},$$

$$-\sqrt{T_1}\ \sqrt{T_5}\ T_{11}^{3/2},\ \sqrt{T_1}\ T_5\,T_{11}^{3/2},\ -\sqrt{T_1}\ T_5^{3/2}\,T_{11}^{3/2},\ T_1\,T_{11}^2,\ -T_1^{3/2}\,T_{11}^2,\ -\sqrt{T_1}\ T_5\,T_{11}^2,\ -T_1^{3/2}\,T_5\,T_{11}^2,$$

$$-\sqrt{T_1}\ T_5^2\,T_{11}^2,\ -T_1^{3/2}\,T_5^2\,T_{11}^2,\ \frac{T_{11}^{5/2}}{\sqrt{T_1}},\ T_1^{3/2}\,T_{11}^{5/2},\ -\sqrt{T_1}\ \sqrt{T_5}\ T_{11}^{5/2},\ -\sqrt{T_1}\ T_5^{3/2}\,T_{11}^{5/2},\ T_1^2\,T_{11}^3,\ T_1^3\,T_{11}^3,$$

$$-T_1^{3/2}\,T_5\,T_{11}^3,\ -T_1^{3/2}\,T_5^2\,T_{11}^3,\ \sqrt{T_1}\ T_{11}^{7/2},\ T_1^{7/2}\,T_{11}^{7/2},\ T_1^3\,T_{11}^4,\ -T_1^{3/2}\,T_5^2\,T_{11}^4,\ -\frac{1}{T_{13}^{7/2}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{7/2}},$$

$$-\frac{T_1\,T_7}{T_{13}^{7/2}},\ -\frac{1}{T_{13}^{5/2}},\ -\frac{T_1}{T_{13}^{5/2}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}\ T_{13}^{5/2}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_{13}^{5/2}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{T_{13}^{5/2}},\ -\frac{T_1\,T_7}{T_{13}^{5/2}},\ -\frac{T_1^2\,T_7}{T_{13}^{5/2}},\ -\frac{T_1\,T_7^2}{T_{13}^{5/2}},$$

$$-\frac{1}{T_{13}^{3/2}},\ -\frac{T_1}{T_{13}^{3/2}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}\ T_{13}^{3/2}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}\ T_{13}^{3/2}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_{13}^{3/2}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{T_{13}^{3/2}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{T_{13}^{3/2}},$$

$$-\frac{T_1^{3/2}\,T_5^{5/2}}{T_{13}^{3/2}},\ -\frac{T_7}{T_{13}^{3/2}},\ -\frac{T_1\,T_7}{T_{13}^{3/2}},\ \frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}\ T_{13}^{3/2}},\ \frac{T_1^{3/2}\,T_5^{3/2}\sqrt{T_9}}{T_{13}^{3/2}},\ -\frac{1}{\sqrt{T_{13}}},\ -\frac{T_1}{\sqrt{T_{13}}},\ -\frac{T_1^2}{\sqrt{T_{13}}},$$

$$-\frac{\sqrt{T_1}}{\sqrt{T_5}\ \sqrt{T_{13}}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_{13}}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_{13}}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{\sqrt{T_{13}}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{\sqrt{T_{13}}},\ -\frac{T_1^{3/2}\,T_5^{5/2}}{\sqrt{T_{13}}},\ -\frac{T_7}{\sqrt{T_{13}}},$$

$$-\frac{T_1\,T_7}{\sqrt{T_{13}}},\ -\frac{T_7^2}{\sqrt{T_{13}}},\ \frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}\ \sqrt{T_{13}}},\ \frac{\sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_9}}{\sqrt{T_{13}}},\ \frac{T_1^{3/2}\,T_5^{3/2}\sqrt{T_9}}{\sqrt{T_{13}}},\ -T_1\sqrt{T_{13}},\ -T_1^2\sqrt{T_{13}},$$

$$-\frac{\sqrt{T_1}\ \sqrt{T_{13}}}{\sqrt{T_5}},\ -\sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_{13}},\ -T_1^{3/2}\sqrt{T_5}\ \sqrt{T_{13}},\ -\sqrt{T_1}\ T_5^{3/2}\sqrt{T_{13}},\ -T_1^{3/2}\,T_5^{3/2}\sqrt{T_{13}},$$

$$-\sqrt{T_1}\ T_5^{5/2}\sqrt{T_{13}},\ -T_1^{3/2}\,T_5^{5/2}\sqrt{T_{13}},\ -T_7\sqrt{T_{13}},\ -T_1\,T_7\sqrt{T_{13}},\ -T_1^2\,T_7\sqrt{T_{13}},\ -T_1\,T_7^2\sqrt{T_{13}},$$

$$\sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_9}\ \sqrt{T_{13}},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\sqrt{T_{13}},\ -\sqrt{T_1}\ T_{13},\ -\sqrt{T_1}\ T_5\,T_{13},\ -\sqrt{T_1}\ \sqrt{T_5}\ T_{13}^{3/2},$$

$$-T_1^{3/2}\sqrt{T_5}\ T_{13}^{3/2},\ -\sqrt{T_1}\ T_5^{3/2}\,T_{13}^{3/2},\ -T_1^{3/2}\,T_5^{3/2}\,T_{13}^{3/2},\ -\sqrt{T_1}\ T_5^{5/2}\,T_{13}^{3/2},\ -T_1^{3/2}\,T_5^{5/2}\,T_{13}^{3/2},$$

$$-\sqrt{T_1}\ \sqrt{T_7}\ T_{13}^{3/2},\ -T_1\,T_7\,T_{13}^{3/2},\ -T_1\,T_7^2\,T_{13}^{3/2},\ -T_1^2\,T_7^2\,T_{13}^{3/2},\ \frac{\sqrt{T_1}\ \sqrt{T_5}\ T_{13}^{3/2}}{\sqrt{T_9}},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,T_{13}^{3/2},$$

$$-\sqrt{T_1}\ T_5^2\,T_{13}^2,\ -T_1^{3/2}\,T_5^{3/2}\,T_{13}^{5/2},\ -T_1^{3/2}\,T_5^{5/2}\,T_{13}^{5/2},\ -T_1^2\,T_7\,T_{13}^{5/2},\ -T_1^2\,T_7^2\,T_{13}^{5/2},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,T_{13}^{7/2},\ -\frac{T_1^{3/2}}{T_{15}^2},$$

$$\frac{T_1^{3/2}\sqrt{T_5}}{T_{15}^2},\ -\frac{T_1^{3/2}\,T_5}{T_{15}^2},\ \frac{T_1^{3/2}\sqrt{T_5}\ T_9}{T_{15}^2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}\,T_{15}^{3/2}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_{15}^{3/2}},\ -\frac{T_1^{3/2}}{T_{15}},\ -\frac{T_1^{3/2}\,T_5}{T_{15}},\ \frac{T_1^{3/2}\,T_5^{3/2}}{T_{15}},$$

$$-\frac{T_1^{3/2}\,T_5^2}{T_{15}},\ -\frac{T_1^2\sqrt{T_7}}{T_{15}},\ \frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9\,T_{15}},\ \frac{\sqrt{T_1}}{\sqrt{T_{11}}\ T_{15}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}\ \sqrt{T_{15}}},\ -\frac{\sqrt{T_1}\ T_5^{3/2}}{\sqrt{T_{15}}},\ -\frac{\sqrt{T_1}\ \sqrt{T_{15}}}{\sqrt{T_5}},$$

$$\sqrt{T_1}\ \sqrt{T_5}\ T_{15},\ -T_1^{3/2}\,T_5^2\,T_{15},\ -T_1^{3/2}\,T_5^3\,T_{15},\ -T_1\,T_7^{5/2}\,T_{15},\ \sqrt{T_1}\ \sqrt{T_5}\ T_9\,T_{15},\ T_1^{3/2}\,T_5^{3/2}\,T_9\,T_{15},$$

$$\sqrt{T_1}\ \sqrt{T_{11}}\ T_{15},\ \sqrt{T_1}\ T_5\sqrt{T_{11}}\ T_{15},\ T_1^{3/2}\,T_5\sqrt{T_{11}}\ T_{15},\ T_1^{3/2}\,T_5^2\sqrt{T_{11}}\ T_{15},\ T_1^{3/2}\,T_5\,T_{11}^{3/2}\,T_{15},$$

$$\sqrt{T_1}\ T_5^2\,T_{11}^{3/2}\,T_{15},\ -\sqrt{T_1}\ \sqrt{T_5}\ T_{15}^{3/2},\ -\sqrt{T_1}\ T_5^{3/2}\,T_{15}^{3/2},\ -\sqrt{T_1}\ T_5^{5/2}\,T_{15}^{3/2},\ -T_1^{3/2}\,T_5^2\,T_{15}^2,$$

$$-T_1^{3/2}\,T_5^3\,T_{15}^2,\ T_1^{3/2}\,T_5^{3/2}\,T_9\,T_{15}^2,\ T_1^{3/2}\,T_5^{3/2}\,T_9^2\,T_{15}^2,\ T_1^{3/2}\,T_5^2\,T_{11}^{3/2}\,T_{15}^2,\ -\frac{T_1^{3/2}}{\sqrt{T_5}\,T_{17}^{3/2}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{T_{17}^{3/2}},$$

$$\frac{\sqrt{T_1}\,\sqrt{T_5}}{T_9^{3/2}\,T_{17}^{3/2}},\ \frac{T_1^{3/2}\,T_5^{3/2}\,\sqrt{T_9}}{T_{17}^{3/2}},\ \frac{\sqrt{T_1}}{T_{11}^2\,T_{17}^{3/2}},\ -\frac{\sqrt{T_1}}{T_{17}},\ -\frac{\sqrt{T_1}}{T_5^2\,T_{17}},\ -\frac{\sqrt{T_1}}{T_5\,T_{17}},\ -\frac{T_1^{3/2}\,T_5}{T_{17}},\ -\frac{T_1^{3/2}\,\sqrt{T_5}}{\sqrt{T_{17}}},$$

$$\frac{\sqrt{T_1}\,T_5}{\sqrt{T_{17}}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{\sqrt{T_{17}}},\ -\frac{T_1^{3/2}\,T_5^{5/2}}{\sqrt{T_{17}}},\ \frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_9}\,\sqrt{T_{17}}},\ \frac{T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}}{\sqrt{T_{17}}},\ \frac{\sqrt{T_1}\,T_5}{T_{11}\,\sqrt{T_{17}}},\ \frac{T_1^{3/2}\,T_5\,T_{11}}{\sqrt{T_{17}}},$$

$$\frac{T_1^{3/2}\,T_5^2\,T_{11}}{\sqrt{T_{17}}},\ -\frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_9}\,\sqrt{T_{13}}\,\sqrt{T_{17}}},\ -\frac{T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,\sqrt{T_{13}}}{\sqrt{T_{17}}},\ -\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_{17}},\ -T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_{17}},$$

$$-T_1^{3/2}\,T_5^{3/2}\,\sqrt{T_{17}},\ \sqrt{T_1}\,T_5^2\,\sqrt{T_{17}},\ -T_1^{3/2}\,T_5^{5/2}\,\sqrt{T_{17}},\ \frac{\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_{17}}}{\sqrt{T_9}},\ \sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_9}\,\sqrt{T_{17}},$$

$$T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,\sqrt{T_{17}},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{5/2}\,\sqrt{T_{17}},\ \frac{\sqrt{T_1}\,\sqrt{T_{17}}}{T_{11}},\ -\frac{\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_{13}}\,\sqrt{T_{17}}}{\sqrt{T_9}},$$

$$-\sqrt{T_1}\,T_{17},\ -\sqrt{T_1}\,T_5\,T_{17},\ -\sqrt{T_1}\,T_5^2\,T_{17},\ -T_1^{3/2}\,T_5^{3/2}\,T_{17}^{3/2},\ -T_1^{3/2}\,T_5^{7/2}\,T_{17}^{3/2},\ T_1^{3/2}\,T_5\,T_{11}\,T_{17}^{3/2},$$

$$T_1^{3/2}\,T_5^2\,T_{11}^2\,T_{17}^{3/2},\ -T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2}\,T_{13}^{3/2}\,T_{17}^{3/2},\ -\sqrt{T_1}\,T_5\,T_{17}^2,\ -\sqrt{T_1}\,T_5^3\,T_{17}^2,\ -\frac{\sqrt{T_1}}{T_5^{5/2}\,T_{19}^{3/2}},$$

$$-\frac{T_1^{3/2}}{T_5^{3/2}\,T_{19}^{3/2}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}\,T_{19}^{3/2}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}\,T_{19}^{3/2}},\ -\frac{\sqrt{T_1}\,T_5^{3/2}}{T_{19}^{3/2}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{T_{19}^{3/2}},\ -\frac{T_1^{3/2}\,T_5^{5/2}}{T_{19}^{3/2}},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}\,\sqrt{T_{19}}},$$

$$-\frac{T_1^{3/2}}{\sqrt{T_5}\,\sqrt{T_{19}}},\ -\frac{\sqrt{T_1}\,\sqrt{T_5}}{\sqrt{T_{19}}},\ -\frac{T_1^{3/2}\,\sqrt{T_5}}{\sqrt{T_{19}}},\ -\frac{\sqrt{T_1}\,T_5^{3/2}}{\sqrt{T_{19}}},\ -\frac{T_1^{3/2}\,T_5^{3/2}}{\sqrt{T_{19}}},\ -\frac{\sqrt{T_1}\,T_5^{5/2}}{\sqrt{T_{19}}},\ -\frac{T_1^{3/2}\,T_5^{5/2}}{\sqrt{T_{19}}},$$

$$-\frac{T_1^{3/2}\,T_5^{7/2}}{\sqrt{T_{19}}},\ \frac{T_1^{3/2}\,\sqrt{T_5}}{\sqrt{T_9}\,\sqrt{T_{19}}},\ \frac{\sqrt{T_1}\,T_5^{3/2}}{\sqrt{T_9}\,\sqrt{T_{19}}},\ \frac{T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_9}}{\sqrt{T_{19}}},\ \frac{\sqrt{T_1}\,T_5^{3/2}\,\sqrt{T_9}}{\sqrt{T_{19}}},\ \frac{\sqrt{T_1}}{T_{11}^2\,\sqrt{T_{19}}},$$

$$\frac{T_1^{3/2}}{T_{11}\,\sqrt{T_{19}}},\ \frac{T_1^{3/2}\,T_5\,T_{11}}{\sqrt{T_{19}}},\ -\frac{\sqrt{T_1}\,\sqrt{T_{19}}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}\,\sqrt{T_{19}}}{\sqrt{T_5}},\ -\sqrt{T_1}\,\sqrt{T_5}\,\sqrt{T_{19}},\ -T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_{19}},$$

$$-\sqrt{T_1}\,T_5^{3/2}\,\sqrt{T_{19}},\ -\sqrt{T_1}\,T_5^{5/2}\,\sqrt{T_{19}},\ -\sqrt{T_1}\,T_5^{7/2}\,\sqrt{T_{19}},\ -T_1^{3/2}\,T_5^{7/2}\,\sqrt{T_{19}},\ -T_1^{3/2}\,T_5^{9/2}\,\sqrt{T_{19}},$$

$$\frac{T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_{19}}}{\sqrt{T_9}},\ T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_9}\,\sqrt{T_{19}},\ T_1^{3/2}\,T_5\,T_{19}^{3/2},\ T_1^{3/2}\,\sqrt{T_5}\,\sqrt{T_9}\,T_{19}^{3/2},\ T_1^{3/2}\,T_5^2\,T_{11}^2\,T_{19}^{3/2}\Big\}$$

Recycling

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & T-1 & 0 & -T \\ -1 & T & 0 & 0 & 0 & 0 & 1-T & 0 \\ 0 & -1 & T & 0 & 1-T & 0 & 0 & 0 \\ T-1 & 0 & -T & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-T & 0 & -1 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -T & 1 & 0 & T-1 \\ 0 & 0 & 1-T & 0 & 0 & -1 & T & 0 \\ 0 & 0 & 0 & T-1 & 0 & 0 & -T & 1 \end{pmatrix}\texttt{[[1 ;; 7, 1 ;; 7]] // Det}$$

$$-1+4\,T-8\,T^2+11\,T^3-8\,T^4+4\,T^5-T^6$$