Initialization

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], (t | T)_s_ :↦ s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :↦ s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :↦ βForm[β];
Format[β_B, StandardForm] := βForm[β];
B /: B[ω1_, β1_] == B[ω2_, β2_] := (ω1 == ω2) && (β1 == β2);
```

Program

```
⟨μ_⟩ := μ /. t_ → 1;
tm_{x_,y_→z_}[β_] := β /. {t_{x|y} → t_z, T_{x|y} → T_z};
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{x_,y_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
         + γ / ϵ h_y                + δ - γ * β / ϵ
    ] // βCollect];
gm_{x_,y_→z_}[β_] := β // sw_{x,y} // hm_{x,y→z} // tm_{x,y→z};
tΔ_{x_→y_,z_}[β_] := β /. {t_x → t_y + t_z, T_x → T_y T_z};
hΔ_{x_→y_,z_}[β_] := β /. {h_x → h_y + h_z};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{x_,y_} := B[1, (T_x - 1) t_x h_y];
Rm_{x_,y_} := B[1, (T_x^{-1} - 1) t_x h_y];
```

Demo

```
{β = B[ω, Sum[α10 i+j ti hj, {i, {1, 2, 3}}, {j, {4, 5}}]],
 β1 = β // tm1,2→1 // sw1,4;
 β2 = β // sw2,4 // sw1,4 // tm1,2→1,
 FullSimplify[β1] == FullSimplify[β2]
}
```

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_{14}+\alpha_{24}) & h_4 & h_5 \\ t_1 & \frac{(\alpha_{14}+\alpha_{24})\,(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} & \frac{(\alpha_{15}+\alpha_{25})\,(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} \\ t_3 & \frac{\alpha_{34}}{1+\alpha_{14}+\alpha_{24}} & \frac{-\alpha_{15}\,\alpha_{34}-\alpha_{25}\,\alpha_{34}+\alpha_{35}+\alpha_{14}\,\alpha_{35}+\alpha_{24}\,\alpha_{35}}{1+\alpha_{14}+\alpha_{24}} \end{pmatrix}, \text{True}\right\}$$

```
{β = B[ω, a t1 h1 + b t1 h2],
 β // hm1,2→1 // tΔ1→1,2,
 β // tΔ1→1,2 // hm1,2→1
}
```

$$\left\{\begin{pmatrix} \omega & h_1 & h_2 \\ t_1 & a & b \end{pmatrix}, \begin{pmatrix} \omega & h_1 \\ t_1 & a+b+a\,b \\ t_2 & a+b+a\,b \end{pmatrix}, \begin{pmatrix} \omega & h_1 \\ t_1 & a+b+2\,a\,b \\ t_2 & a+b+2\,a\,b \end{pmatrix}\right\}$$

```
Rp1,4 Rm2,3 // gm1,2→1 // gm3,4→3
```

```
( 1 )
```

R3

```
{Rm5,1 Rm6,2 Rp3,4 // gm1,4→1 // gm2,5→2 // gm3,6→3,
  Rp6,1 Rm2,4 Rm3,5 // gm1,4→1 // gm2,5→2 // gm3,6→3}
```

R3

$$\left\{B\left[1, h_1\left(-\frac{t_2\,(-1+T_2)}{T_2}+\frac{t_3\,(-1+T_3)}{T_2}\right)-\frac{h_2\,t_3\,(-1+T_3)}{T_3}\right],\right.$$
$$\left.B\left[1, h_1\left(-\frac{t_2\,(-1+T_2)}{T_2}+t_3\,(-1+T_3)-\frac{t_3\,(-1+T_2)\,(-1+T_3)}{T_2}\right)-\frac{h_2\,t_3\,(-1+T_3)}{T_3}\right]\right\}$$

8_17-1

```
β = Rm12,1 Rm2,7 Rm8,3 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15
```

8_17-1

$$B\left[1, h_7\,t_2\left(-1+\frac{1}{T_2}\right)+h_{11}\,t_4\left(-1+\frac{1}{T_4}\right)+h_{13}\,t_6\,(-1+T_6)+h_3\,t_8\left(-1+\frac{1}{T_8}\right)+\right.$$
$$\left.h_{15}\,t_{10}\,(-1+T_{10})+h_1\,t_{12}\left(-1+\frac{1}{T_{12}}\right)+h_9\,t_{14}\,(-1+T_{14})+h_5\,t_{16}\,(-1+T_{16})\right]$$

8_17-2

```
Do[β = β // gm1,k→1, {k, 2, 10}]; β
```

8_17-2

$$B\left[\frac{T_1^2+T_{16}-T_1\,T_{16}}{T_1^2},\right.$$
$$h_{15}\,t_1\,(-1+T_1)+h_{13}\left(-\frac{t_{14}\,(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})}{T_1^2+T_{16}-T_1\,T_{16}}+\frac{t_1\,(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}}{T_1^2+T_{16}-T_1\,T_{16}}-\right.$$
$$\left.\frac{t_{16}\,(-1+T_1)^2\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}}\right)+h_{11}\left(\frac{t_{16}\,(-1+T_1)\,T_1\,(-1+T_{16})}{T_1^2+T_{16}-T_1\,T_{16}}+\right.$$
$$\left.\frac{t_{14}\,(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1\,T_{16})}-\frac{t_1\,(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1\,T_{16})}\right)+$$
$$h_1\left(-\frac{t_{12}\,(-1+T_{12})}{T_{12}}+\frac{t_{16}\,T_1\,(-1+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})}+\frac{t_{14}\,(-1+T_{14})\,(-T_1+T_1^2+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})}-\right.$$
$$\left.\left.\frac{t_1\,(-1+T_1)\,T_{14}\,(T_1^3+T_{16}^2)}{T_1^2\,T_{12}\,(T_1^2+T_{16}-T_1\,T_{16})}\right)\right]$$

8_17-3

```
Do[β = β // gm1,k→1, {k, 11, 16}]; β
```

8_17-3

$$B\left[-\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^3},\right.$$
$$\left.h_1\left(t_1\,(-1+T_1)-\left(t_1\,(-1+T_1)\left(T_1-4\,T_1^2+8\,T_1^3-11\,T_1^4+8\,T_1^5-4\,T_1^6+T_1^7\right)\right)\Big/\left(T_1\left(1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6\right)\right)\right)\right]$$

8_17-4

```
<< KnotTheory`
Alexander[Knot[8, 17]][T1] // Factor
```

8_17-4

```
Loading KnotTheory` version of March 22, 2011, 21:10:4.67737.
Read more at http://katlas.org/wiki/KnotTheory.
```

8_17-4

KnotTheory::loading : Loading precomputed data in PD4Knots`.

8_17-4

$$-\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^3}$$