Pensieve header: Testing the common program for all w-meta-calculi. Continues pensieve://2014-07/MetaCalculi/.

```
In[∘]:= dir = SetDirectory["C:/drorbn/AcademicPensieve/Projects/MetaCalculi/"];
        << KnotTheory`
        << MetaCalculi.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
MetaCalculi` loading...
```

General

```
SXForm[L = Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

```
SXForm[{Loop[1, 2, 3, 4], Loop[5, 6, 7, 8], Loop[9, 10, 11, 12]},
 Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]
```

```
Z[L]
```

```
dm[9, 12, 9][
 dm[9, 11, 9][dm[9, 10, 9][dm[5, 8, 5][dm[5, 7, 5][dm[5, 6, 5][dm[1, 4, 1][dm[1, 3, 1][
        dm[1, 2, 1][Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]]]]]]]]]
```

α-Calculus

$$\{\mathrm{Xp}_{ab},\ \mathrm{Xm}_{ab}\}\ //\ \mathrm{A}$$

$$\left\{\left(\begin{array}{ccc} 1 & h[a] & h[b] \\ t[a] & 0 & \frac{-1+e^{c_a}}{c_a} \\ A & 1 & e^{c_a} \end{array}\right),\ \left(\begin{array}{ccc} 1 & h[a] & h[b] \\ t[a] & 0 & \frac{e^{-c_a}\left(1-e^{c_a}\right)}{c_a} \\ A & 1 & e^{-c_a} \end{array}\right)\right\}$$

$$\{\mathrm{Xm}_{51}\,\mathrm{Xm}_{62}\,\mathrm{Xp}_{34}\ //\ \mathrm{A}\ //\ \mathrm{dm}[1, 4, 1]\ //\ \mathrm{dm}[2, 5, 2]\ //\ \mathrm{dm}[3, 6, 3],$$
$$\mathrm{Xp}_{61}\,\mathrm{Xm}_{24}\,\mathrm{Xm}_{35}\ //\ \mathrm{A}\ //\ \mathrm{dm}[1, 4, 1]\ //\ \mathrm{dm}[2, 5, 2]\ //\ \mathrm{dm}[3, 6, 3]\}$$

$$\left\{\left(\begin{array}{cccc} 1 & h[1] & h[2] & h[3] \\ t[2] & \frac{e^{-c_2}\left(1-e^{c_2}\right)}{c_2} & 0 & 0 \\ t[3] & \frac{e^{-c_2}\left(-1+e^{c_3}\right)}{c_3} & \frac{e^{-c_3}\left(1-e^{c_3}\right)}{c_3} & 0 \\ A & e^{-c_2+c_3} & e^{-c_3} & 1 \end{array}\right),\ \left(\begin{array}{cccc} 1 & h[1] & h[2] & h[3] \\ t[2] & \frac{e^{-c_2}\left(1-e^{c_2}\right)}{c_2} & 0 & 0 \\ t[3] & \frac{e^{-c_2}\left(-1+e^{c_3}\right)}{c_3} & \frac{e^{-c_3}\left(1-e^{c_3}\right)}{c_3} & 0 \\ A & e^{-c_2+c_3} & e^{-c_3} & 1 \end{array}\right)\right\}$$

$$\alpha = \mathrm{Xm}_{12,1}\,\mathrm{Xm}_{27}\,\mathrm{Xm}_{83}\,\mathrm{Xm}_{4,11}\,\mathrm{Xp}_{16,5}\,\mathrm{Xp}_{6,13}\,\mathrm{Xp}_{14,9}\,\mathrm{Xp}_{10,15}\ \texttt{//}\ \mathrm{A}$$

$$\left(\begin{array}{ccccccccccccc}
1 & h[1] & h[2] & h[3] & h[4] & h[5] & h[6] & h[7] & h[8] & h[9] & h[10] & h[11] & h \\
t[2] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{-c_2}\left(1-e^{c_2}\right)}{c_2} & 0 & 0 & 0 & 0 & \\
t[4] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{-c_4}\left(1-e^{c_4}\right)}{c_4} & \\
t[6] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
t[8] & 0 & 0 & \frac{e^{-c_8}\left(1-e^{c_8}\right)}{c_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
t[10] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
t[12] & \frac{e^{-c_{12}}\left(1-e^{c_{12}}\right)}{c_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
t[14] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{14}}}{c_{14}} & 0 & 0 & \\
t[16] & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{16}}}{c_{16}} & 0 & 0 & 0 & 0 & 0 & 0 & \\
A & e^{-c_{12}} & 1 & e^{-c_8} & 1 & e^{c_{16}} & 1 & e^{-c_2} & 1 & e^{c_{14}} & 1 & e^{-c_4} &
\end{array}\right.$$

$$\alpha = \mathrm{Xm}_{12,1}\,\mathrm{Xm}_{27}\,\mathrm{Xm}_{83}\,\mathrm{Xm}_{4,11}\,\mathrm{Xp}_{16,5}\,\mathrm{Xp}_{6,13}\,\mathrm{Xp}_{14,9}\,\mathrm{Xp}_{10,15}\ \texttt{//}\ \mathrm{A};$$
$$\mathrm{Do}[\alpha = \alpha\ \texttt{//}\ \mathrm{dm}[1, k, 1], \{k, 2, 16\}];\ \alpha$$

$$\left(\begin{array}{cc}
e^{-3c_1}\left(-1+4e^{c_1}-8e^{2c_1}+11e^{3c_1}-8e^{4c_1}+4e^{5c_1}-e^{6c_1}\right) & h[1] \\
t[1] & 0 \\
A & 1
\end{array}\right)$$

Testing R3

$$\{(\mathrm{Xp}_{12}\ \texttt{//}\ \mathrm{A}) ** (\mathrm{Xp}_{13}\ \texttt{//}\ \mathrm{A}) ** (\mathrm{Xp}_{23}\ \texttt{//}\ \mathrm{A}),\ (\mathrm{Xp}_{23}\ \texttt{//}\ \mathrm{A}) ** (\mathrm{Xp}_{13}\ \texttt{//}\ \mathrm{A}) ** (\mathrm{Xp}_{12}\ \texttt{//}\ \mathrm{A})\}$$

$$\left\{\left(\begin{array}{cccc}
1 & h[1] & h[2] & h[3] \\
t[1] & 0 & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\
t[2] & 0 & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \\
A & 1 & e^{c_1} & e^{c_1+c_2}
\end{array}\right), \left(\begin{array}{cccc}
1 & h[1] & h[2] & h[3] \\
t[1] & 0 & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\
t[2] & 0 & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \\
A & 1 & e^{c_1} & e^{c_1+c_2}
\end{array}\right)\right\}$$

Testing tr

$$\alpha 0 = \mathrm{A}\left[\omega,\ h[a]\,\sigma_a + h[b]\,\sigma_b + h[S]\,\sigma_S,\ \{t[a], t[b], t[S]\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h[a], h[b], h[S]\}\right];$$

$$\{t1 = \alpha 0\ \texttt{//}\ \mathrm{dm}[a, b, c]\ \texttt{//}\ \mathrm{tr}[c],\ t2 = \alpha 0\ \texttt{//}\ \mathrm{dm}[b, a, c]\ \texttt{//}\ \mathrm{tr}[c],\ t1 == t2\}$$

$$\left\{\left(\begin{array}{cc}
\omega + \beta\,\omega\,c_c + \gamma\,\omega\,c_c + \beta\,\gamma\,\omega\,c_c^2 - \alpha\,\delta\,\omega\,c_c^2 + \delta\,\omega\,c_c\,\sigma_a + \alpha\,\omega\,c_c\,\sigma_b - \omega\,\sigma_a\,\sigma_b & \\
t[c] & \\
t[S] & \frac{\Xi+\beta\,\Xi\,c_c+\gamma\,\Xi\,c_c-\epsilon\,\phi\,c_c-\theta\,\psi\,c_c+\beta\,\gamma\,\Xi\,c_c^2-\alpha\,\delta\,\Xi\,c}{1+\beta\,c} \\
A &
\end{array}\right.\right.$$

Testing the KV Solution

See the 2014-07 version of this file.

Γ-Calculus

$$\{\mathrm{Xp}_{ab},\ \mathrm{Xm}_{ab}\}\ //\ \Gamma$$

$$\left\{\begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1-T_a \\ s_b & 0 & T_a \\ \Gamma & 1 & T_a \end{pmatrix},\ \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1+T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Gamma & 1 & \frac{1}{T_a} \end{pmatrix}\right\}$$

Meta-Associativity

$$n = 4;\ \gamma 0 = \Gamma\left[\omega,\ \sum_{a=1}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

$$\gamma 0\ //\ \mathrm{dm}[1, 2, 1]\ //\ \mathrm{dm}[1, 3, 1]$$

$$\begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}\right) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

$$\gamma 0\ //\ \mathrm{dm}[2, 3, 2]\ //\ \mathrm{dm}[1, 2, 1]$$

$$\begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}\right) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{43}+}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

$$\left(\gamma 0\ //\ \mathrm{dm}[1, 2, 1]\ //\ \mathrm{dm}[1, 3, 1]\right) == \left(\gamma 0\ //\ \mathrm{dm}[2, 3, 2]\ //\ \mathrm{dm}[1, 2, 1]\right)$$

```
True
```

Γ^{op}

In[∘]:=
```
Γ /: Γ[ω_, σ_, λ_]^op := Module[{S = dL[Γ[ω, σ, λ]], M},
    M = Outer[ΓSimp[(∂_{t#1 h#2} λ)] &, S, S];
    Γ[
     ΓSimp[Det[M] ω], Collect[σ, h_, (1/#) &],
     (t# & /@ S).Inverse[M].(h# & /@ S)
    ]
   ];
```

In[∘]:=
$$n = 3;\ \gamma0 = \Gamma\left[\omega,\ \sum_{a=1}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

Out[∘]=
$$\left(\begin{array}{cccc} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{array}\right)$$

In[∘]:=
```
γ0^op // dm[1, 2, 1]
```

Out[∘]=
$$\left(\begin{array}{cc} -\omega\ (\alpha_{13}\,\alpha_{22}\,\alpha_{31} - \alpha_{12}\,\alpha_{23}\,\alpha_{31} + \alpha_{13}\,\alpha_{32} - \alpha_{13}\,\alpha_{21}\,\alpha_{32} + \alpha_{11}\,\alpha_{23}\,\alpha_{32} - \alpha_{12}\,\alpha_{33} + \alpha_{12}\,\alpha_{21}\,\alpha_{33} - \alpha_{11}\,\alpha_{22}\,\alpha_{33}) & \\ s_1 & \overline{\ -\alpha_{13}\,\alpha_{2} \ } \\ s_3 & \overline{\ \alpha_{13}\,\alpha_{22}\ } \\ \Gamma & \end{array}\right.$$

In[∘]:=
```
Simplify[(γ0^op // dm[1, 2, 1]) == (γ0 // dm[2, 1, 1])^op]
```

Out[∘]=
```
True
```

In[∘]:=
```
{Γ[Xp[i, j]], Γ[Xp[i, j]]^op /. T_i → T_i^-1}
```

Out[∘]=
$$\left\{\left(\begin{array}{ccc} 1 & s_i & s_j \\ s_i & 1 & 1-T_i \\ s_j & 0 & T_i \\ \Gamma & 1 & T_i \end{array}\right),\ \left(\begin{array}{ccc} \frac{1}{T_i} & s_i & s_j \\ s_i & 1 & 1-T_i \\ s_j & 0 & T_i \\ \Gamma & 1 & T_i \end{array}\right)\right\}$$

Cyclicity of tr

$$n = 3;\ \gamma0 = \Gamma\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\left(\begin{array}{cccc} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{array}\right)$$

```
{γ0 // dm[1, 2, 1], γ0 // dm[1, 2, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} -\omega\left(-1+\alpha_{12}\right) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{12}} \\ s_3 & \frac{-\alpha_{31}+\alpha_{12}\,\alpha_{31}-\alpha_{11}\,\alpha_{32}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{32}-\alpha_{33}+\alpha_{12}\,\alpha_{33}}{-1+\alpha_{12}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}\right) & \\ s_3 & \underline{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}} \\ \Gamma & \end{pmatrix}\right.$$

```
{γ0 // dm[2, 1, 1], γ0 // dm[2, 1, 1] // tr[1]}
```

$$\left\{\begin{pmatrix} -\omega\left(-1+\alpha_{21}\right) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{12}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}} & \frac{-\alpha_{13}+\alpha_{13}\,\alpha_{21}-\alpha_{11}\,\alpha_{23}}{-1+\alpha_{21}} \\ s_3 & \frac{-\alpha_{22}\,\alpha_{31}-\alpha_{32}+\alpha_{21}\,\alpha_{32}}{-1+\alpha_{21}} & \frac{-\alpha_{23}\,\alpha_{31}-\alpha_{33}+\alpha_{21}\,\alpha_{33}}{-1+\alpha_{21}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \omega\left(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}\right) & \\ s_3 & \underline{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}} \\ \Gamma & \end{pmatrix}\right.$$

```
(γ0 // dm[1, 2, 1] // tr[1]) == (γ0 // dm[2, 1, 1] // tr[1])
```

```
True
```

Testing the MVA

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\left(\begin{array}{cc} \frac{(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)\,(T_1+T_5-T_1\,T_5+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5\,T_9)}{T_1\,T_5\,T_9} & s_1 \\ s_1 & \frac{T_9\,(1-2\,T_1+T_1^2-T_5+3\,T_1\,T_5-T_1^2\,T_5-T_9+2\,T_1\,T_9-T_1^2\,T_9+T_5\,T_9-2\,T_1\,T_5\,T_9+T_1^2\,T}{(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)\,(T_1+T_5-T_1\,T_5+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5} \\ s_5 & \frac{T_1\,(-1+T_5)\,(1-T_1+T_1\,T_5)\,(-1+T_9)}{(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)\,(T_1+T_5-T_1\,T_5+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5} \\ s_9 & \frac{(-1+T_5)\,T_5\,(-1+T_9)\,(1-2\,T_1-T_9+T_1\,T_9)}{(1-T_1-T_5+T_1\,T_5-T_9+T_1\,T_9+T_5\,T_9)\,(T_1+T_5-T_1\,T_5+T_9-T_1\,T_9-T_5\,T_9+T_1\,T_5} \\ \Gamma & 1 \end{array}\right.$$

```
trZ[L_] := Module[{γ},
  γ = Z[Γ, L];
  Do[
   γ = γ // tr[k],
   {k, Most[γ // dL]}];
  γ[ω] / (T_Last[γ//dL] - 1)]
```

```
trZ[Link["L10a4"]]
```

$$\frac{\left(-1+T_1\right)\left(-1+T_5\right)\left(1-3\,T_5+3\,T_5^2-3\,T_5^3+T_5^4\right)}{T_5^2}$$

```
MultivariableAlexander[Link["L10a4"]][T]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$-\frac{\left(-1+T[1]\right)\left(-1+T[2]\right)\left(1-3\,T[2]+3\,T[2]^2-3\,T[2]^3+T[2]^4\right)}{\sqrt{T[1]}\;T[2]^{5/2}}$$

$$\mathbf{Factor}\left[\frac{(\mathbf{MultivariableAlexander}[\#][T]\ /.\ T[i\_] \mapsto T_{\mathrm{Skeleton}[\#][[i,1]]})}{\mathbf{trZ}[\#]}\right]\ \&\ /@\ \mathbf{AllLinks}[\{2,7\}]$$

$$\left\{-T_1^2\,T_3,\ -T_1^{3/2}\,T_5^{3/2},\ -\sqrt{T_1}\,T_5^{3/2},\ -T_1^{3/2}\,\sqrt{T_5},\ -T_1^2\,T_7^2,\ -T_1^2\,T_7^2,\ \frac{\sqrt{T_1}\,\sqrt{T_9}}{\sqrt{T_5}},\ T_1^{3/2}\,T_5^{3/2}\,T_9^{3/2},\ \frac{\sqrt{T_1}}{\sqrt{T_5}\,\sqrt{T_9}},\right.$$
$$\left.-\frac{1}{\sqrt{T_1}\,\sqrt{T_5}},\ -T_1^{3/2}\,T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1\,T_7^2,\ -\frac{1}{T_7},\ \frac{T_1^{3/2}\,\sqrt{T_5}}{\sqrt{T_9}},\ -\sqrt{T_1}\,T_5^{5/2},\ -\frac{T_5^{3/2}}{\sqrt{T_1}}\right\}$$

The Mirror Properties

$$n=3;\ \gamma 0=\Gamma\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

$$\gamma 0\ //\ \mathbf{Mirror}\ //\ \mathbf{dm}[1,2,1]$$

$$\left(\begin{array}{cc} -\frac{\omega\,(\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}-\alpha_{21}\,\alpha_{33}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}\,\alpha_{33})}{\sigma_1\,\sigma_2\,\sigma_3} & s_1 \\ s_1 & \frac{\alpha_{13}\,\alpha_{32}+\alpha_{33}-\alpha_{12}\,\alpha}{-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}} \\ s_3 & \frac{\alpha_{13}\,\alpha_{22}+\alpha_{23}-\alpha_{12}\,\alpha}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}} \\ \Gamma & \frac{1}{\sigma_1\,\sigma_2} \end{array}\right.$$

$$\gamma 0\ //\ \mathbf{dm}[1,2,1]\ //\ \mathbf{Mirror}$$

$$\left(\begin{array}{cc} -\frac{\omega\,(\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}-\alpha_{21}\,\alpha_{33}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}\,\alpha_{33})}{\sigma_1\,\sigma_2\,\sigma_3} & s_1 \\ s_1 & \frac{\alpha_{13}\,\alpha_{32}+\alpha_{33}-\alpha_{12}\,\alpha}{-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}} \\ s_3 & \frac{\alpha_{13}\,\alpha_{22}+\alpha_{23}-\alpha_{12}\,\alpha}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}} \\ \Gamma & \frac{1}{\sigma_1\,\sigma_2} \end{array}\right.$$

$$(\gamma 0\ //\ \mathbf{Mirror}\ //\ \mathbf{dm}[1,2,1]) == (\gamma 0\ //\ \mathbf{dm}[1,2,1]\ //\ \mathbf{Mirror})$$

```
True
```

$$\Big\{\texttt{n = 2; }\gamma 0 = \Gamma\Big[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\Big],$$
$$\texttt{t1} = \gamma 0\ \texttt{// Mirror // dS[1], t2} = \gamma 0\ \texttt{// dS[1] // Mirror, t1 == t2}\Big\}$$

$$\left\{\begin{pmatrix} \omega & s_1 & s_2 \\ s_1 & \alpha_{11} & \alpha_{12} \\ s_2 & \alpha_{21} & \alpha_{22} \\ \Gamma & \sigma_1 & \sigma_2 \end{pmatrix},\ \begin{pmatrix} \frac{\omega\,\alpha_{22}}{\sigma_2} & s_1 & s_2 \\ s_1 & \frac{-\alpha_{12}\,\alpha_{21}+\alpha_{11}\,\alpha_{22}}{\alpha_{22}} & -\frac{\alpha_{21}}{\alpha_{22}} \\ s_2 & \frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{\omega\,\alpha_{22}}{\sigma_2} & s_1 & s_2 \\ s_1 & \frac{-\alpha_{12}\,\alpha_{21}+\alpha_{11}\,\alpha_{22}}{\alpha_{22}} & \frac{\alpha_{21}}{\alpha_{22}} \\ s_2 & -\frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix},\ -\frac{\alpha_{21}}{\alpha_{22}} == \frac{\alpha_{21}}{\alpha_{22}}\ \&\&\ \frac{\alpha_{12}}{\alpha_{22}} == -\frac{\alpha_{12}}{\alpha_{22}}\right\}$$

Column Sums

$$\texttt{Clear}[\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu, \omega];$$
$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$
$$\gamma 0\ \texttt{// dm[a, b, c]}$$

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \\ \Sigma & \sigma_a\,\sigma_b & \sigma_S \end{pmatrix}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \texttt{// Simplify}$$

$$\left\{\frac{(-1+\beta)\,\gamma + (-1+\beta)\,\phi - \alpha\,(\delta+\psi)}{-1+\beta},\ \frac{\epsilon - \beta\,\epsilon + \delta\,\theta + \Xi - \beta\,\Xi + \theta\,\psi}{1-\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \texttt{/.}\ \{\alpha \to \texttt{s1} - \gamma - \phi,\ \delta \to \texttt{s2} - \beta - \psi,\ \Xi \to \texttt{s3} - \theta - \epsilon\}\ \texttt{// Simplify}$$

$$\left\{\frac{\texttt{s1}\,(-\texttt{s2}+\beta) + (-1+\texttt{s2})\,(\gamma+\phi)}{-1+\beta},\ \frac{\texttt{s3}\,(-1+\beta) + \theta - \texttt{s2}\,\theta}{-1+\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\ \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \end{pmatrix}\ \texttt{/.}\ \{\alpha \to \texttt{s1} - \gamma - \phi,\ \delta \to \texttt{s2} - \beta - \psi,\ \Xi \to \texttt{s3} - \theta - \epsilon\}\ \texttt{/.}$$
$$\texttt{s1 | s2 | s3} \to \texttt{1 // Simplify}$$

$$\{1, 1\}$$

Tangle Concatenation; Γ-inversion

$$\mathbf{n = 3;}\ \Big\{\gamma 1 = \Gamma\Big[\omega 1,\ \sum_{a=0}^{n} h_a\, \sigma 1_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \alpha_{10\,a+b}\Big],$$

$$\gamma 2 = \Gamma\Big[\omega 2,\ \sum_{a=0}^{n} h_a\, \sigma 2_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \beta_{10\,a+b}\Big],$$

$$\text{FullStitch}[\gamma 1, \gamma 2],\ \gamma 1 ** \gamma 2,\ \text{FullStitch}[\gamma 1, \gamma 2] == \gamma 1 ** \gamma 2\Big\}$$

$$\left\{\begin{pmatrix} \omega 1 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Sigma & \sigma 1_1 & \sigma 1_2 & \sigma 1_3 \end{pmatrix},\ \begin{pmatrix} \omega 2 & s_1 & s_2 & s_3 \\ s_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ s_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ s_3 & \beta_{31} & \beta_{32} & \beta_{33} \\ \Sigma & \sigma 2_1 & \sigma 2_2 & \sigma 2_3 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega 1\,\omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11} + \alpha_{21}\beta_{12} + \alpha_{31}\beta_{13} & \alpha_{12}\beta_{11} + \alpha_{22}\beta_{12} + \alpha_{32}\beta_{13} & \alpha_{13}\beta_{11} + \alpha_{23}\beta_{12} + \alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21} + \alpha_{21}\beta_{22} + \alpha_{31}\beta_{23} & \alpha_{12}\beta_{21} + \alpha_{22}\beta_{22} + \alpha_{32}\beta_{23} & \alpha_{13}\beta_{21} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31} + \alpha_{21}\beta_{32} + \alpha_{31}\beta_{33} & \alpha_{12}\beta_{31} + \alpha_{22}\beta_{32} + \alpha_{32}\beta_{33} & \alpha_{13}\beta_{31} + \alpha_{23}\beta_{32} + \alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega 1\,\omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11} + \alpha_{21}\beta_{12} + \alpha_{31}\beta_{13} & \alpha_{12}\beta_{11} + \alpha_{22}\beta_{12} + \alpha_{32}\beta_{13} & \alpha_{13}\beta_{11} + \alpha_{23}\beta_{12} + \alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21} + \alpha_{21}\beta_{22} + \alpha_{31}\beta_{23} & \alpha_{12}\beta_{21} + \alpha_{22}\beta_{22} + \alpha_{32}\beta_{23} & \alpha_{13}\beta_{21} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31} + \alpha_{21}\beta_{32} + \alpha_{31}\beta_{33} & \alpha_{12}\beta_{31} + \alpha_{22}\beta_{32} + \alpha_{32}\beta_{33} & \alpha_{13}\beta_{31} + \alpha_{23}\beta_{32} + \alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},\ \text{True}\right\}$$

$$\gamma 1^{-1}$$

$$\begin{pmatrix} \frac{1}{\omega 1} & s_1 & s_2 \\ s_1 & \frac{\alpha_{23}\alpha_{32}-\alpha_{22}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{32}-\alpha_{12}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}} \\ s_2 & \frac{\alpha_{23}\alpha_{31}-\alpha_{21}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{31}-\alpha_{11}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}} \\ s_3 & \frac{\alpha_{22}\alpha_{31}-\alpha_{21}\alpha_{32}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{12}\alpha_{31}-\alpha_{11}\alpha_{32}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}} \\ \Sigma & \frac{1}{\sigma 1_1} & \frac{1}{\sigma 1_2} \end{pmatrix}$$

$$\gamma 1 ** \gamma 1^{-1}$$

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}$$

```
γ1 // dS[1] // dS[2] // dS[3]
```

$$\begin{pmatrix} -\frac{\omega1\,(\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}\,\alpha_{33})}{\sigma1_1\,\sigma1_2\,\sigma1_3} & s_1 \\ s_1 & \frac{\alpha_{23}\,\alpha_{32}-\alpha_{22}\,\alpha_{33}}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}} \\ s_2 & \frac{\alpha_{23}\,\alpha_{31}-\alpha_{21}\,\alpha_{33}}{-\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\alpha_{22}} \\ s_3 & \frac{\alpha_{22}\,\alpha_{31}-\alpha_{21}\,\alpha_{32}}{\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{12}\,\alpha_{21}\,\alpha_{33}-\alpha_{11}\,\alpha_{22}} \\ \Sigma & \frac{1}{\sigma1_1} \end{pmatrix}$$

```
(γ1 // dS[1] // dS[2] // dS[3]) == γ1^-1 // Simplify
```

$$-\frac{\omega1\,(\alpha_{13}\,(\alpha_{22}\,\alpha_{31}-\alpha_{21}\,\alpha_{32})+\alpha_{12}\,(-\alpha_{23}\,\alpha_{31}+\alpha_{21}\,\alpha_{33})+\alpha_{11}\,(\alpha_{23}\,\alpha_{32}-\alpha_{22}\,\alpha_{33}))}{\sigma1_1\,\sigma1_2\,\sigma1_3} == \frac{1}{\omega1}$$

Other

R3

```
{Xm51 Xm62 Xp34 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
 Xp61 Xm24 Xm35 // Γ // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

$\gamma = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // Γ

$$\left(\begin{array}{ccccccccccccccccc}
1 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\
s_1 & \frac{1}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_2 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_3 & 0 & 0 & \frac{1}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T_4}{T_4} & 0 & 0 & 0 & 0 & 0 \\
s_5 & 0 & 0 & 0 & 0 & T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1-T_6 & 0 & 0 & 0 \\
s_7 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_8 & 0 & 0 & \frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1-T_{10} & 0 \\
s_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{T_4} & 0 & 0 & 0 & 0 & 0 \\
s_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
s_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 \\
s_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-T_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
s_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{10} & 0 \\
s_{16} & 0 & 0 & 0 & 0 & 1-T_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\Sigma & \frac{1}{T_{12}} & 1 & \frac{1}{T_8} & 1 & T_{16} & 1 & \frac{1}{T_2} & 1 & T_{14} & 1 & \frac{1}{T_4} & 1 & T_6 & 1 & T_{10} & 1
\end{array}\right)$$

Do[$\gamma = \gamma$ // $dm_{1k\to1}$, {k, 2, 10}]; γ

$$\left(\begin{array}{cccccccc}
\frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2} & s_1 & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\
s_1 & \frac{T_{14}\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)} & \frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14} T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)} & 0 & -\frac{(-1+T_1)\left(1-T_1+T_1^2\right) T_{14}}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 1-T_1 & 0 \\
s_{11} & 0 & \frac{1}{T_1} & 0 & 0 & 0 & 0 & 0 \\
s_{12} & \frac{-1+T_{12}}{T_{12}} & 0 & 1 & 0 & 0 & 0 & 0 \\
s_{13} & 0 & 0 & 0 & T_1 & 0 & 0 & 0 \\
s_{14} & -\frac{(-1+T_{14})\left(-T_1+T_1^2+T_{16}\right)}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)} & -\frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14}) T_{16}}{T_1\left(T_1^2+T_{16}-T_1 T_{16}\right)} & 0 & \frac{(-1+T_1)\left(1-T_1+T_1^2\right)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}} & 1 & 0 & 0 \\
s_{15} & 0 & 0 & 0 & 0 & 0 & T_1 & 0 \\
s_{16} & -\frac{T_1(-1+T_{16})}{T_{12}\left(T_1^2+T_{16}-T_1 T_{16}\right)} & -\frac{(-1+T_1) T_1(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & \frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 & 0 & 1 \\
\Sigma & \frac{T_{14} T_{16}}{T_1^2 T_{12}} & \frac{1}{T_1} & 1 & T_1 & 1 & T_1 & 1
\end{array}\right)$$

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$\gamma = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // Γ;

Do[$\gamma = \gamma$ // dm[1, k, 1], {k, 2, 16}]; γ

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$$\left(\begin{array}{cc}
-\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3} & s_1 \\
s_1 & 1 \\
\Sigma & 1
\end{array}\right)$$

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\left(\begin{array}{cc} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & s_1 \\ s_1 & \frac{T_9\ (1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9+T_1^2 T}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5} \\ s_5 & \frac{T_1\ (-1+T_5)\ (1-T_1+T_1 T_5)\ (-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5} \\ s_9 & \frac{(-1+T_5)\ T_5\ (-1+T_9)\ (1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5} \\ \Sigma & 1 \end{array}\right.$$

```
MVA[Γ, Link["L6a4"]]
```

$$-\frac{(-1+T_1)\ (-1+T_5)\ (-1+T_9)}{T_1 T_5}$$

$$\texttt{Factor}\left[\frac{(\texttt{MultivariableAlexander[\#][T] /. T[}i\texttt{\_]} :\!\to T_{\texttt{Skeleton[\#]}[[i,1]]})}{\texttt{MVA[}\Gamma\texttt{, \#]}}\right]\ \texttt{\& /@ AllLinks[\{2, 8\}]}$$

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\left\{-T_1^2 T_3,\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\ T_5^{3/2},\ -T_1^{3/2} \sqrt{T_5},\ -T_1^2 T_7^2,\ -T_1^2 T_7^2,\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2},\right.$$

$$-\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{3/2}},\ -\sqrt{T_1}\ \sqrt{T_5},\ -T_1^{3/2} T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1 T_7^2,\ -\frac{1}{T_7},\ -\frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{7/2},$$

$$-\sqrt{T_1}\ T_5^{5/2},\ -\sqrt{T_1}\ T_5^{3/2},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\ T_5^{3/2},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{7/2},\ -\frac{T_1}{T_7},$$

$$-T_1 T_7,\ -T_1^2 T_7^3,\ -T_1^2 T_7^3,\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{3/2} T_5^{3/2} \sqrt{T_9},\ -\sqrt{T_1},\ -T_1^{3/2} T_5^2 T_{11}^2,$$

$$-\frac{\sqrt{T_1}}{T_{11}^2},\ -\frac{\sqrt{T_1}\ T_5}{T_{11}},\ -\frac{T_1^{3/2} \sqrt{T_5}}{T_{13}^{3/2}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\ \sqrt{T_5},\ -T_1^{3/2} T_5^2 T_{11}^2,$$

$$\left.-\sqrt{T_1}\ T_5^2 T_{11},\ -\sqrt{T_1}\ T_5^{3/2} T_{11}^{3/2},\ -T_1^{3/2} T_5^{5/2} \sqrt{T_{13}},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}\ \sqrt{T_{13}}},\ -\sqrt{T_1}\ \sqrt{T_5}\ \sqrt{T_9}\ \sqrt{T_{13}}\right\}$$

α↔Γ Convertions

```
{Xp[1, 2] // Γ, Xp[1, 2] // A // Γ}
```

$$\left\{\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right),\ \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right)\right\}$$

```
{Xm[1, 2] // A, Xm[1, 2] // Γ // A}
```

$$\left\{\left(\begin{array}{cc} 1 & h[2] \\ t[1] & \frac{e^{-c_1}\ (1-e^{c_1})}{c_1} \end{array}\right),\ \left(\begin{array}{cc} 1 & h[2] \\ t[1] & \frac{e^{-c_1}\ (1-e^{c_1})}{c_1} \end{array}\right)\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
{γ0, γ0 // A, γ0 // A // Γ, (γ0 // A // Γ) /. {α → 1 - γ - ϕ, δ → 1 - β - ψ, Ξ → 1 - θ - ϵ}}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix},\ \begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \frac{-\alpha+\sigma_a}{c_a} & -\frac{\beta}{c_a} & -\frac{\theta}{c_a} \\ t[b] & -\frac{\gamma}{c_b} & \frac{-\delta+\sigma_b}{c_b} & -\frac{\epsilon}{c_b} \\ t[S] & -\frac{\phi}{c_S} & -\frac{\psi}{c_S} & \frac{-\Xi+\sigma_S}{c_S} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & 1-\alpha-\gamma-\phi+\sigma_a & 1-\beta-\delta-\psi+\sigma_b & 1-\epsilon-\theta-\Xi+\sigma_S \end{pmatrix},\ \begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
{γ0 // dm[a, b, c] // A, γ0 // A // dm[a, b, c]} /.
  {α → 1 - γ - ϕ, δ → 1 - β - ψ, Ξ → 1 - θ - ϵ}
```

$$\left\{\begin{pmatrix} \omega-\beta\,\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\,\phi-\psi+\gamma\,\psi+\phi\,\psi-\sigma_a\,\sigma_b+\beta\,\sigma_a\,\sigma_b}{-c_c+\beta\,c_c} & \frac{\epsilon-\beta\,\epsilon+\theta-\beta\,\theta-\theta\,\psi}{-c_c+\beta\,c_c} \\ t[S] & \frac{\phi-\beta\,\phi+\psi-\gamma\,\psi-\phi\,\psi}{-c_S+\beta\,c_S} & \frac{1-\beta-\epsilon+\beta\,\epsilon-\theta+\beta\,\theta+\theta\,\psi-\sigma_S+\beta\,\sigma_S}{-c_S+\beta\,c_S} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega-\beta\,\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\,\phi-\psi+\gamma\,\psi+\phi\,\psi-\sigma_a\,\sigma_b+\beta\,\sigma_a\,\sigma_b}{-c_c+\beta\,c_c} & \frac{\epsilon-\beta\,\epsilon+\theta-\beta\,\theta-\theta\,\psi}{-c_c+\beta\,c_c} \\ t[S] & \frac{\phi-\beta\,\phi+\psi-\gamma\,\psi-\phi\,\psi}{-c_S+\beta\,c_S} & \frac{1-\beta-\epsilon+\beta\,\epsilon-\theta+\beta\,\theta+\theta\,\psi-\sigma_S+\beta\,\sigma_S}{-c_S+\beta\,c_S} \end{pmatrix}\right\}$$

The KV solution in Γ, starting from α

```
V // A // αCollect[FullSimplify]
```

$$\left(\begin{array}{lll} \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\ t[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ t[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}\,c_1\sqrt{\frac{\text{Sinh}[}{c_2}}}{} \end{array}\right.$$

$$\mathbf{V\ //\ A\ //\ \Gamma\ //}$$
$$\mathbf{\Gamma Collect\left[Assuming\left[T_1 > 0\ \&\&\ T_2 > 0,\ \left(\#\ /.\ \left\{Sinh[x\_] \mapsto \frac{e^{x} - e^{-x}}{2}\right\}\right)\ //\ FullSimplify\right]\ \&\right]}$$

$$\left(\begin{array}{ccc}
\frac{\left(\frac{-1+T_1}{\mathrm{Log}[T_1]}\right)^{1/4}\left(\frac{-1+T_2}{\mathrm{Log}[T_2]}\right)^{1/4}}{\left(\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}\right)^{1/4}} & s_1 & s_2 \\
s_1 & \frac{\mathrm{Log}[T_1]\left(\mathrm{Log}[T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}-\sqrt{\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}}+T_1\sqrt{\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}}\right)}{\mathrm{Log}[T_1 T_2]\ (-1+T_1)\sqrt{\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}}} & -\frac{\mathrm{Log}[T_1]\left(\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\ T_2-\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\right.}{\mathrm{Log}[T_1 T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\frac{-1+T_1}{\mathrm{Log}[T_1}}} \\
s_2 & \frac{\mathrm{Log}[T_2]\left(-T_1\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}+\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}}\right)}{\mathrm{Log}[T_1 T_2]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\sqrt{\frac{-1+T_1 T_2}{\mathrm{Log}[T_1 T_2]}}} & \frac{\mathrm{Log}[T_1]\sqrt{\frac{(-1+T_1)T_1}{\mathrm{Log}[T_1]}}\ T_2+\mathrm{Log}[T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}}{\mathrm{Log}[T_1 T_2]\sqrt{\frac{-1+T_2}{\mathrm{Log}[T_2]}}\sqrt{\frac{-1+T_1 T}{\mathrm{Log}[T_1 T}}} \\
\Sigma & 1 & \sqrt{T_1}
\end{array}\right.$$

$$\mathbf{\Gamma Simp = Assuming\left[T_1 > 1\ \&\&\ T_2 > 1,\ \left(\#\ /.\ \left\{Sinh[x\_] \mapsto \frac{e^{x} - e^{-x}}{2}\right\}\right)\ //\ FullSimplify\right]\ \&;}$$
$$\mathbf{V\ //\ A\ //\ \Gamma}$$

$$\left(\begin{array}{ccc}
\left(\frac{\mathrm{Log}[T_1 T_2]\ (-1+T_1)\ (-1+T_2)}{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ (-1+T_1 T_2)}\right)^{1/4} & s_1 & s_2 \\
s_1 & \frac{\mathrm{Log}[T_1]+\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ T_1\ (-1+T_2)}{(-1+T_1)\ (-1+T_1 T_2)}}}{\mathrm{Log}[T_1 T_2]} & \frac{\mathrm{Log}[T_1]\left(-1+T_2\left(T_1-\sqrt{\frac{\mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]}{\mathrm{Log}[T_1}}\right.\right.}{\mathrm{Log}[T_1 T_2]\ (-1+T_1 T} \\
s_2 & -\frac{\mathrm{Log}[T_2]\left(-1+\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_1 T_2]\ T_1\ (-1+T_2)}{\mathrm{Log}[T_2]\ (-1+T_1)\ (-1+T_1 T_2)}}\right)}{\mathrm{Log}[T_1 T_2]} & \frac{-\mathrm{Log}[T_2]+T_2\left(\mathrm{Log}[T_2]\ T_1+\sqrt{\frac{\mathrm{Log}[T_1]^3\ \mathrm{Log}[T_2]\ (-1+T_1)\ T_1\ (-1+T_1}{\mathrm{Log}[T_1 T_2]\ (-1+T_2)}}\right.}{\mathrm{Log}[T_1 T_2]\ (-1+T_1 T} \\
\Sigma & 1 & \sqrt{T_1}
\end{array}\right.$$

$$\mathbf{\Gamma[V] ** \Gamma[Vi]}$$

$$\left(\begin{array}{cc}
1 & s_1 \\
s_1 & \frac{-\mathrm{Log}[T_1 T_2]+T_2\left(\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ (-1+T_1)\ T_1\ (-1+T_2)}{-1+T_1 T_2}}+T_1\left(\mathrm{Log}[T_1 T_2]+\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ (-1+T_2)}{(-1+T_1)\ T_1\ (-1+T_1 T_2)}}-T_1\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ (-1+T_2)}{(-1+T_1)\ T_1\ (-1+T_1 T_2)}}\right.\right.}{\mathrm{Log}[T_1 T_2]\ (-1+T_1 T_2)} \\
s_2 & \frac{(-1+T_2)\left(-(-1+T_1)\ T_1\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ (-1+T_2)}{(-1+T_1)\ T_1\ (-1+T_1 T_2)}}+\sqrt{\frac{\mathrm{Log}[T_1]\ \mathrm{Log}[T_2]\ \mathrm{Log}[T_1 T_2]\ (-1+T_1)\ T_1\ (-1+T_2)}{-1+T_1 T_2}}\right)}{\mathrm{Log}[T_1 T_2]\ (-1+T_1)\ (-1+T_1 T_2)} \\
\Sigma & 1
\end{array}\right.$$

$$\mathbf{\frac{1}{Log[T_1\,T_2]\ (-1+T_1\,T_2)}\left(-Log[T_1\,T_2]+T_2\left(\sqrt{\frac{Log[T_1]\ Log[T_2]\ Log[T_1\,T_2]\ (-1+T_1)\ T_1\ (-1+T_2)}{-1+T_1\,T_2}}+T_1\left(Log[T_1\,T_2]+\sqrt{\frac{Log[T_1]\ Log[T_2]\ Log[T_1\,T_2]\ (-1+T_2)}{(-1+T_1)\ T_1\ (-1+T_1\,T_2)}}-T_1\sqrt{\frac{Log[T_1]\ Log[T_2]\ Log[T_1\,T_2]\ (-1+T_2)}{(-1+T_1)\ T_1\ (-1+T_1\,T_2)}}\right)\right)\right)\ //\ PowerExpand\ //\ Simplify}$$

1

$$\frac{\left(-1+T_2\right)\left(-\left(-1+T_1\right)T_1\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1\,T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1\,T_2)}}+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1\,T_2]\,(-1+T_1)\,T_1\,(-1+T_2)}{-1+T_1\,T_2}}\right)}{\text{Log}[T_1\,T_2]\left(-1+T_1\right)\left(-1+T_1\,T_2\right)}\ //$$

```
  PowerExpand // Simplify
```

```
0
```

dS and dA for Γ, starting from α

$$\text{Clear}[\alpha,\theta,\phi,\Xi,\omega];$$
$$\gamma0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,t_S\}.\begin{pmatrix}\alpha & \theta\\ \phi & \Xi\end{pmatrix}.\{h_a,h_S\}\right];$$
$$\left(\left(\gamma0\ //\ A\ //\ \text{dS}[a]\ //\ \Gamma\right)==\left(\gamma0\ //\ A\ //\ \text{dA}[a]\ //\ \Gamma\right)\right)\ /.\ \{\alpha\to 1-\phi,\ \Xi\to 1-\theta\}$$

```
True
```

$$\left(\gamma0\ //\ A\ //\ \text{dS}[a]\ //\ \Gamma\right)$$

$$\begin{pmatrix}
\frac{(-1+\phi)\ \omega}{-1+\alpha+\phi-\sigma_a} & s_a & s_S\\
s_a & -\frac{1}{-1+\phi} & -\frac{\theta}{-1+\phi}\\
s_S & \frac{\phi}{-1+\phi} & \frac{-1+\theta+\phi}{-1+\phi}\\
\Sigma & -\frac{1}{-1+\alpha+\phi-\sigma_a} & -\frac{1-\alpha-\theta+\alpha\,\theta-\Xi+\alpha\,\Xi-\phi+\theta\,\phi+\Xi\,\phi+\sigma_a-\theta\,\sigma_a-\Xi\,\sigma_a+\sigma_S-\alpha\,\sigma_S-\phi\,\sigma_S+\sigma_a\,\sigma_S}{-1+\alpha+\phi-\sigma_a}
\end{pmatrix}$$

$$\left(\gamma0\ //\ A\ //\ \text{dA}[a]\ //\ \Gamma\right)\ /.\ \{\phi\to 1-\alpha,\ \Xi\to 1-\theta\}\ //\ \Gamma\text{Collect}$$

$$\begin{pmatrix}
\frac{\alpha\,\omega}{\sigma_a} & s_a & s_S\\
s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha}\\
s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha}\\
\Sigma & \frac{1}{\sigma_a} & \sigma_S
\end{pmatrix}$$

$$\left(\gamma0\ //\ \text{dA}[a]\right)\ /.\ \{\phi\to 1-\alpha,\ \Xi\to 1-\theta\}\ //\ \Gamma\text{Collect}$$

$$\begin{pmatrix}
\frac{\alpha\,\omega}{\sigma_a} & s_a & s_S\\
s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha}\\
s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha}\\
\Sigma & \frac{1}{\sigma_a} & \sigma_S
\end{pmatrix}$$

$$\left(\gamma0\ //\ A\ //\ \text{dA}[a]\ //\ \Gamma\right)==\left(\gamma0\ //\ \text{dA}[a]\right)\ /.\ \{\phi\to 1-\alpha,\ \Xi\to 1-\theta\}\ //\ \text{Simplify}$$

```
True
```

dΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ
                                           ϕ Ξ).{h_a, h_S}];
((γ0 // A // dΔ[a, b, c] // Γ) /. {α → 1 - ϕ, Ξ → 1 - θ}) // ΓCollect
```

$$\left(\begin{array}{cccc}
\omega & s_b & s_c & s_S \\
s_b & -\frac{-\text{Log}[T_b]+\phi\,\text{Log}[T_b]-\text{Log}[T_c]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & -\frac{\text{Log}[T_b]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_b]}{\text{Log}[T_b]+\text{Log}[T_c]} \\
s_c & -\frac{\text{Log}[T_c]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\text{Log}[T_c]-\phi\,\text{Log}[T_c]+\text{Log}[T_b]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_c]}{\text{Log}[T_b]+\text{Log}[T_c]} \\
s_S & \phi & \phi & 1-\theta \\
\Sigma & \sigma_a & \sigma_a & \sigma_S
\end{array}\right)$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ
                                           ϕ Ξ).{h_a, h_S}];
γ0 // A // dΔ[a, b, c] // dS[c] // dm[b, c, a]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity + $\frac{(\Xi-\theta\,\phi-\Xi\,\phi-\sigma_S+\phi\,\sigma_S)\,\text{t}[S]}{-c_S+\phi\,c_S}$ encountered. ≫

$$\left(\frac{-\omega+\phi\,\omega}{-1+\alpha+\phi-\sigma_a}\right)$$

qΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ
                                           ϕ Ξ).{h_a, h_S}];
γ0 // A // qΔ[a, b, c] // Γ
```

$$\left(\begin{array}{cccc}
\omega & s_b & s_c & \\
s_b & -\frac{1-\alpha-\phi+\alpha\,T_c-T_b\,T_c+\phi\,T_b\,T_c+\sigma_a-T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & \frac{\theta\,(-1}{-1} \\
s_c & \frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & -\frac{1-\phi-\alpha\,T_c-T_b\,T_c+\alpha\,T_b\,T_c+\phi\,T_b\,T_c+T_c\,\sigma_a-T_b\,T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\theta\,(-}{-1} \\
s_S & \phi & \phi & 1 \\
\Sigma & 1-\alpha-\phi+\sigma_a & 1-\alpha-\phi+\sigma_a & 1-\theta
\end{array}\right.$$

$$\mathtt{Clear}[\alpha,\theta,\phi,\Xi,\omega];$$
$$\gamma0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,t_S\}.\begin{pmatrix}\alpha&\theta\\ \phi&\Xi\end{pmatrix}.\{h_a,h_S\}\right];$$
$$\gamma0\ //\ q\Delta[a,b,c]$$

$$\left(\begin{array}{cccc}
\omega & s_b & s_c & s_S\\
s_b & \frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & \frac{\theta\,(-1+T_b)\,T_c}{-1+T_b\,T_c}\\
s_c & \frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & \frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\theta\,(-1+T_c)}{-1+T_b\,T_c}\\
s_S & \phi & \phi & \Xi\\
\Sigma & \sigma_a & \sigma_a & \sigma_S
\end{array}\right)$$

$$\mathtt{Clear}[\alpha,\theta,\phi,\Xi,\omega];$$
$$\gamma0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,t_S\}.\begin{pmatrix}\alpha&\theta\\ \phi&\Xi\end{pmatrix}.\{h_a,h_S\}\right];$$
$$\mathtt{Simplify}\Big[\big(\gamma0\ //\ q\Delta[a,b,c]\big)[\![3]\!] == \big(\gamma0\ //\ A\ //\ q\Delta[a,b,c]\ //\ \Gamma\big)[\![3]\!]\ /.\ \{\alpha\to 1-\phi,\ \theta\to 1-\Xi\}\Big]$$

True

qΔ tests for Γ

$$\left\{t1=Xp_{13}\ //\ \Gamma\ //\ q\Delta[1,1,2],\ t2=\big(\epsilon[1]\,Xp_{23}\ //\ \Gamma\big)\ {**}\ \big(\epsilon[2]\,Xp_{13}\ //\ \Gamma\big),\ t1==t2\ //\ \mathtt{Simplify}\right\}$$

$$\left\{\left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & 1 & 0 & -(-1+T_1)\,T_2\\
s_2 & 0 & 1 & 1-T_2\\
s_3 & 0 & 0 & T_1\,T_2\\
\Gamma & 1 & 1 & T_1\,T_2
\end{array}\right),\ \left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & 1 & 0 & -(-1+T_1)\,T_2\\
s_2 & 0 & 1 & 1-T_2\\
s_3 & 0 & 0 & T_1\,T_2\\
\Gamma & 1 & 1 & T_1\,T_2
\end{array}\right),\ \mathrm{True}\right\}$$

$$\left\{t1=Xm_{13}\ //\ \Gamma\ //\ q\Delta[1,1,2],\ t2=\big(\epsilon[2]\,Xm_{13}\ //\ \Gamma\big)\ {**}\ \big(\epsilon[1]\,Xm_{23}\ //\ \Gamma\big),\ t1==t2\right\}$$

$$\left\{\left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & 1 & 0 & \frac{-1+T_1}{T_1}\\
s_2 & 0 & 1 & \frac{-1+T_2}{T_1\,T_2}\\
s_3 & 0 & 0 & \frac{1}{T_1\,T_2}\\
\Gamma & 1 & 1 & \frac{1}{T_1\,T_2}
\end{array}\right),\ \left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & 1 & 0 & \frac{-1+T_1}{T_1}\\
s_2 & 0 & 1 & \frac{-1+T_2}{T_1\,T_2}\\
s_3 & 0 & 0 & \frac{1}{T_1\,T_2}\\
\Gamma & 1 & 1 & \frac{1}{T_1\,T_2}
\end{array}\right),\ \mathrm{True}\right\}$$

$$\left\{t1=Xp_{3,1}\ //\ \Gamma\ //\ q\Delta[1,1,2],\ t2=\big(\epsilon[1]\,Xp_{3,2}\ //\ \Gamma\big)\ {**}\ \big(\epsilon[2]\,Xp_{3,1}\ //\ \Gamma\big),\ t1==t2\right\}$$

$$\left\{\left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & T_3 & 0 & 0\\
s_2 & 0 & T_3 & 0\\
s_3 & 1-T_3 & 1-T_3 & 1\\
\Gamma & T_3 & T_3 & 1
\end{array}\right),\ \left(\begin{array}{cccc}
1 & s_1 & s_2 & s_3\\
s_1 & T_3 & 0 & 0\\
s_2 & 0 & T_3 & 0\\
s_3 & 1-T_3 & 1-T_3 & 1\\
\Gamma & T_3 & T_3 & 1
\end{array}\right),\ \mathrm{True}\right\}$$

```
{t1 = Xm3,1 // Γ // qΔ[1, 1, 2], t2 = (ϵ[2] Xm3,1 // Γ) ** (ϵ[1] Xm3,2 // Γ), t1 == t2}
```

$$\left\{\left(\begin{array}{cccc} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Gamma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{array}\right), \left(\begin{array}{cccc} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Gamma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{array}\right), \text{True}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right],\ \gamma 0\ //\ \mathrm{dm}[a, b, c]\right\}$$

$$\left\{\left(\begin{array}{cccc} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Gamma & \sigma_a & \sigma_b & \sigma_S \end{array}\right), \left(\begin{array}{ccc} -\left(-1+\beta\right)\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta} \\ \Gamma & \sigma_a\sigma_b & \sigma_S \end{array}\right)\right\}$$

```
γ0 // dm[a, b, c] // qΔ[c, c1, c2]
```

$$\left(\begin{array}{ccc} -\left(-1+\beta\right)\omega & s_{c1} & \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)\ (-1+T_{c1} T_{c2})} & \frac{(-1+T_{c1})\ T_{c2}}{(-} \\ s_{c2} & \frac{(-1+T_{c2})\ (-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)\ (-1+T_{c1} T_{c2})} & \frac{\gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_{c2}+}{(-} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Gamma & \sigma_a\sigma_b & \end{array}\right.$$

```
γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] // dm[a2, b2, c2]
```

$$\left(\begin{array}{ccc} -\left(-1+\beta\right)\omega & s_{c1} & \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)\ (-1+T_{c1} T_{c2})} & \frac{(-1+T_{c1})\ T_{c2}}{(-} \\ s_{c2} & \frac{(-1+T_{c2})\ (-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)\ (-1+T_{c1} T_{c2})} & \frac{\gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_{c2}+}{(-} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Gamma & \sigma_a\sigma_b & \end{array}\right.$$

```
(γ0 // dm[a, b, c] // qΔ[c, c1, c2]) ==
  (γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] // dm[a2, b2, c2]) // Simplify
```

```
True
```

---

dS tests for Γ

```
{Xp[1, 2] // Γ, Xm[1, 2] // Γ // dS[1], Xm[1, 2] // Γ // dS[2]}
```

$$\left\{\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right)\right\}$$

```
{Xm[1, 2] // Γ, Xp[1, 2] // Γ // dS[1], Xp[1, 2] // Γ // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
Xp[1, 2] // Γ // dS[1] // dS[2]
```

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right],\right.$$
$$\left.\ t1 = \gamma 0\ //\ dm[a, b, c]\ //\ dS[c],\ t2 = \gamma 0\ //\ dS[a]\ //\ dS[b]\ //\ dm[b, a, c],\ t1 == t2\right\}$$

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\,\Xi+\beta\gamma\,\Xi-\alpha\delta\,\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b} & s_c & s_S \\ s_c & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\,\Xi+\beta\gamma\,\Xi-\alpha\delta\,\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_S\,\sigma_S,\ \{t_a, t_S\}.\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}.\{h_a, h_S\}\right];$$

```
FullSimplify[{1, 1}.dS[a][γ0][A], And @@ Thread[{1, 1}.γ0[A] == {1, 1}]]
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{1-\phi}{\alpha}\right\}$$

```
{1, 1}.dS[a][γ0][A] // Simplify
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{\theta+\alpha\,\Xi-\theta\,\phi}{\alpha}\right\}$$

```
And @@ Thread[{1, 1}.γ0[A] == {1, 1}]
```

$$\alpha+\phi == 1\ \&\&\ \theta+\Xi == 1$$

```
Clear[α, θ, ϕ, Ξ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_S\,\sigma_S,\ \{t_a, t_S\}.\begin{pmatrix}\alpha & \theta\\ \phi & \Xi\end{pmatrix}.\{h_a, h_S\}\right];$$

```
γ0 // dS[a] // dS[a]
```

$$\left(\begin{array}{ccc}\omega & s_a & s_S\\ s_a & \alpha & \theta\\ s_S & \phi & \Xi\\ \Sigma & \sigma_a & \sigma_S\end{array}\right)$$

```
Clear[α, θ, ϕ, Ξ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_S\,\sigma_S,\ \{t_a, t_S\}.\begin{pmatrix}\alpha[T_a] & \theta[T_a]\\ \phi[T_a] & \Xi[T_a]\end{pmatrix}.\{h_a, h_S\}\right];$$

```
{(γ0 // dη[a]) (ϵ[a] // Γ), γ0 // qΔ[a, b, c] // dS[c] // dm[b, c, a],
 γ0 // qΔ[a, b, c] // dS[c] // dm[c, b, a],
 γ0 // qΔ[a, b, c] // dS[b] // dm[b, c, a], γ0 // qΔ[a, b, c] // dS[b] // dm[c, b, a]}
```

$$\left\{\left(\begin{array}{ccc}\omega & s_a & s_S\\ s_a & 1 & 0\\ s_S & 0 & \Xi[1]\\ \Sigma & 1 & \sigma_S\end{array}\right),\ \left(\begin{array}{ccc}\frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S\\ s_a & 1 & \frac{\theta[1]}{\alpha[1]}\\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]}\\ \Sigma & 1 & \sigma_S\end{array}\right),\right.$$

$$\left.\left(\begin{array}{ccc}\omega & s_a & s_S\\ s_a & 1 & \theta[1]\\ s_S & 0 & \Xi[1]\\ \Sigma & 1 & \sigma_S\end{array}\right),\ \left(\begin{array}{ccc}\omega & s_a & s_S\\ s_a & 1 & \theta[1]\\ s_S & 0 & \Xi[1]\\ \Sigma & 1 & \sigma_S\end{array}\right),\ \left(\begin{array}{ccc}\frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S\\ s_a & 1 & \frac{\theta[1]}{\alpha[1]}\\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]}\\ \Sigma & 1 & \sigma_S\end{array}\right)\right\}$$

```
{Xp[S, a] // Γ, Xp[S, a] // Γ // qΔ[a, b, c] // dS[c] // dm[c, b, a]}
```

$$\left\{\left(\begin{array}{ccc}1 & s_a & s_S\\ s_a & T_S & 0\\ s_S & 1-T_S & 1\\ \Sigma & T_S & 1\end{array}\right),\ \left(\begin{array}{ccc}1 & s_a & s_S\\ s_a & 1 & 0\\ s_S & 0 & 1\\ \Sigma & 1 & 1\end{array}\right)\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.{{α, θ}, {ϕ, Ξ}}.{h_a, h_S}];
{t1 = γ0 // qΔ[a, b, c] // dS[b] // dS[c],
 t2 = γ0 // dS[a] // qΔ[a, c, b], Simplify[t1 == t2]}
```

$$\left\{\left(\begin{array}{cccc} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{array}\right),\right.$$

$$\left.\left(\begin{array}{cccc} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{array}\right), \text{True}\right\}$$

---

dA tests for Γ

```
{Xp[1, 2] // Γ, (Xm[1, 2] // Γ) /. T_1 → 1/T_1, Xm[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{array}\right)\right\}$$

```
{Xm[1, 2] // Γ, (Xp[1, 2] // Γ) /. T_1 → 1/T_1, Xp[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{array}\right), \left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{array}\right)\right\}$$

$$\mathtt{Clear}[\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu,\omega];$$

$$\left\{\gamma0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_b\,\sigma_b+h_S\,\sigma_S,\ \{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right],\right.$$
$$\left.\mathtt{t1}=\gamma0\ \mathtt{//}\ \mathtt{dm}[a,b,c]\ \mathtt{//}\ \mathtt{dA}[c],\ \mathtt{t2}=\gamma0\ \mathtt{//}\ \mathtt{dA}[a]\ \mathtt{//}\ \mathtt{dA}[b]\ \mathtt{//}\ \mathtt{dm}[b,a,c],\ \mathtt{t1}==\mathtt{t2}\right\}$$

$$\left\{\begin{pmatrix}\omega&s_a&s_b&s_S\\s_a&\alpha&\beta&\theta\\s_b&\gamma&\delta&\epsilon\\s_S&\phi&\psi&\Xi\\\Sigma&\sigma_a&\sigma_b&\sigma_S\end{pmatrix},\ \begin{pmatrix}-\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b}&s_c&s_S\\s_c&\frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta}\\s_S&-\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta}\\\Sigma&\frac{1}{\sigma_a\,\sigma_b}&\sigma_S\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}-\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b}&s_c&s_S\\s_c&\frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta}\\s_S&-\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta}\\\Sigma&\frac{1}{\sigma_a\,\sigma_b}&\sigma_S\end{pmatrix},\ \mathrm{True}\right\}$$

$$\mathtt{Clear}[\alpha,\theta,\phi,\Xi,\omega];$$
$$\gamma0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,t_S\}.\begin{pmatrix}\alpha&\theta\\\phi&\Xi\end{pmatrix}.\{h_a,h_S\}\right];$$
$$\{\mathtt{t1}=\gamma0\ \mathtt{//}\ \mathtt{q}\Delta[a,b,c]\ \mathtt{//}\ \mathtt{dA}[b]\ \mathtt{//}\ \mathtt{dA}[c],$$
$$\ \mathtt{t2}=\gamma0\ \mathtt{//}\ \mathtt{dA}[a]\ \mathtt{//}\ \mathtt{q}\Delta[a,b,c],\ \mathtt{Simplify}[\mathtt{t1}==\mathtt{t2}]\}$$

$$\left\{\begin{pmatrix}\frac{\alpha\,\omega}{\sigma_a}&s_b&s_c&s_S\\s_b&\frac{-\alpha+\alpha T_c-T_c\sigma_a+T_bT_c\sigma_a}{\alpha(-1+T_bT_c)\sigma_a}&-\frac{(-1+T_b)T_c(\alpha-\sigma_a)}{\alpha(-1+T_bT_c)\sigma_a}&\frac{\theta(-1+T_b)T_c}{\alpha(-1+T_bT_c)}\\s_c&-\frac{(-1+T_c)(\alpha-\sigma_a)}{\alpha(-1+T_bT_c)\sigma_a}&\frac{-\alpha T_c+\alpha T_bT_c-\sigma_a+T_c\sigma_a}{\alpha(-1+T_bT_c)\sigma_a}&\frac{\theta(-1+T_c)}{\alpha(-1+T_bT_c)}\\s_S&-\frac{\phi}{\alpha}&-\frac{\phi}{\alpha}&\frac{\alpha\Xi-\theta\phi}{\alpha}\\\Sigma&\frac{1}{\sigma_a}&\frac{1}{\sigma_a}&\sigma_S\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\frac{\alpha\,\omega}{\sigma_a}&s_b&s_c&s_S\\s_b&\frac{-\alpha+\alpha T_c-T_c\sigma_a+T_bT_c\sigma_a}{\alpha(-1+T_bT_c)\sigma_a}&-\frac{(-1+T_b)T_c(\alpha-\sigma_a)}{\alpha(-1+T_bT_c)\sigma_a}&\frac{\theta(-1+T_b)T_c}{\alpha(-1+T_bT_c)}\\s_c&-\frac{(-1+T_c)(\alpha-\sigma_a)}{\alpha(-1+T_bT_c)\sigma_a}&\frac{-\alpha T_c+\alpha T_bT_c-\sigma_a+T_c\sigma_a}{\alpha(-1+T_bT_c)\sigma_a}&\frac{\theta(-1+T_c)}{\alpha(-1+T_bT_c)}\\s_S&-\frac{\phi}{\alpha}&-\frac{\phi}{\alpha}&\frac{\alpha\Xi-\theta\phi}{\alpha}\\\Sigma&\frac{1}{\sigma_a}&\frac{1}{\sigma_a}&\sigma_S\end{pmatrix},\ \mathrm{True}\right\}$$

$$n=4;\ \gamma0=\Gamma\left[\omega,\ \sum_{a=0}^{n}h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n}t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix}\omega&s_1&s_2&s_3&s_4\\s_1&\alpha_{11}&\alpha_{12}&\alpha_{13}&\alpha_{14}\\s_2&\alpha_{21}&\alpha_{22}&\alpha_{23}&\alpha_{24}\\s_3&\alpha_{31}&\alpha_{32}&\alpha_{33}&\alpha_{34}\\s_4&\alpha_{41}&\alpha_{42}&\alpha_{43}&\alpha_{44}\\\Sigma&\sigma_1&\sigma_2&\sigma_3&\sigma_4\end{pmatrix}$$

```
γ0 // dA[1] // dA[2] // dA[3] // dA[4]
```

$$\left(\frac{\omega\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}-\alpha_{11}\,\alpha}{}\right.$$

```
(γ0 // dA[1] // dA[2] // dA[3] // dA[4]) ** γ0
```

$$\left(\frac{\omega^2\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}-\alpha_{11}}{}\right.$$

$$\begin{aligned}(&\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41} - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41} - \alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41} + \alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41} + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41} - \alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41} -\\ &\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42} + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42} + \alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42} - \alpha_{11}\,\alpha_{24}\,\alpha_{33}\,\alpha_{42} - \alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42} +\\ &\alpha_{11}\,\alpha_{23}\,\alpha_{34}\,\alpha_{42} + \alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43} - \alpha_{12}\,\alpha_{24}\,\alpha_{31}\,\alpha_{43} - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43} + \alpha_{11}\,\alpha_{24}\,\alpha_{32}\,\alpha_{43} +\\ &\alpha_{12}\,\alpha_{21}\,\alpha_{34}\,\alpha_{43} - \alpha_{11}\,\alpha_{22}\,\alpha_{34}\,\alpha_{43} - \alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44} + \alpha_{12}\,\alpha_{23}\,\alpha_{31}\,\alpha_{44} + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44} -\\ &\alpha_{11}\,\alpha_{23}\,\alpha_{32}\,\alpha_{44} - \alpha_{12}\,\alpha_{21}\,\alpha_{33}\,\alpha_{44} + \alpha_{11}\,\alpha_{22}\,\alpha_{33}\,\alpha_{44}) == \mathrm{Det}\left[\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}\right]\end{aligned}$$

```
True
```

Θ tests

```
αSimplify[expr_] := expr // FullSimplify;
```

```
Θ[1, 2] // A
```

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1+e^{\frac{c_2}{2}}\ c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}}\ c_2+e^{\frac{c_2}{2}}\ c_2}{c_1^2+c_1\ c_2} & \frac{-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1+c_2} \\ t[2] & \frac{-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1+c_2} & \frac{e^{\frac{c_1}{2}}\ c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}}\ c_1-c_2+e^{\frac{c_1}{2}}\ c_2}{c_1\ c_2+c_2^2} \\ A & e^{\frac{c_2}{2}} & e^{\frac{c_1}{2}} \end{pmatrix}$$

```
(V // A) ** (Θ[1, 2] // A)
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \frac{-\sqrt{2}\,e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\,e^{\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\,e^{\frac{c_2}{2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \\ t[2] & \frac{\sqrt{2}\,e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} \end{pmatrix}$$

```
(Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1])
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \frac{\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1-e^{\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1-e^{c_1+c_2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1+e^{c_1+\frac{3c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1-e^{\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2+e^{c_1+\frac{3c_2}{2}}}{-\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2+e^{c_1+c_2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2-} \\ t[2] & \frac{\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}-e^{c_1+c_2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}+\sqrt{2}\,e^{c_1+c_2}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}}{-\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1+e^{c_1+c_2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1-} \end{pmatrix}$$

```
(V // A) ** (Θ[1, 2] // A) == (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
```

```
True
```

```
{t1 = Θ[1, 2] // A // Γ, t2 = Θi[1, 2] // A // Γ, t1 ** t2}
```

$$\left\{\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & \frac{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]} & -\frac{\mathrm{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]} \\ s_2 & -\frac{\mathrm{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]} & \frac{\mathrm{Log}[T_2]+\mathrm{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]} \\ \Sigma & \sqrt{T_2} & \sqrt{T_1} \end{array},\right.$$

$$\left.\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & \frac{\mathrm{Log}[T_2]+\mathrm{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} & \frac{\mathrm{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} \\ s_2 & \frac{\mathrm{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} & \frac{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\sqrt{T_1}\sqrt{T_2}} \\ \Sigma & \frac{1}{\sqrt{T_2}} & \frac{1}{\sqrt{T_1}} \end{array}, \begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & 1 & 0 \\ s_2 & 0 & 1 \\ \Sigma & 1 & 1 \end{array}\right\}$$

```
(V // A) ** (Θi[1, 2] // A) ==
  (Xm[2, 1] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // FullSimplify
```

```
True
```

```
(Θ[1, 2] // A) == (Vi // A) ** (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
```

```
Simplify::time : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the
    value of TimeConstraint option may improve the result of simplification. ≫

Simplify::time : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the
    value of TimeConstraint option may improve the result of simplification. ≫

Simplify::time : Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the
    value of TimeConstraint option may improve the result of simplification. ≫

General::stop : Further output of Simplify::time will be suppressed during this calculation. ≫

$Aborted
```

```
(Xp[2, 1] // A) == (V // A // dσ[1 → 2, 2 → 1]) ** (Θ[1, 2] // A) ** (Vi // A) // Simplify
```

```
True
```

```
Θ[1, 2] // A // Γ
```

$$\left(\begin{array}{ccc} 1 & s_1 & s_2 \\ s_1 & \frac{\text{Log}[T_1]+\text{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]} & -\frac{\text{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]} \\ s_2 & -\frac{\text{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]} & \frac{\text{Log}[T_2]+\text{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]} \\ \Gamma & \sqrt{T_2} & \sqrt{T_1} \end{array}\right)$$

Unitarity

```
MatrixForm[
 M = Simplify[(Θ[1, 2] // A // Γ)@A /. {Log[T_i_] :→ 2 b_i, √T_i_ :→ ⅇ^(b_i)}]
]
```

$$\left(\begin{array}{cc} \frac{b_1+e^{b_1+b_2} b_2}{b_1+b_2} & -\frac{\left(-1+e^{b_1+b_2}\right) b_1}{b_1+b_2} \\ -\frac{\left(-1+e^{b_1+b_2}\right) b_2}{b_1+b_2} & \frac{e^{b_1+b_2} b_1+b_2}{b_1+b_2} \end{array}\right)$$

```
Limit[Limit[M, b₁ → 0], b₂ → 0]
```

```
{{1, 0}, {0, 1}}
```

```
MatrixExp[( b  -b )] // Simplify // MatrixForm
          (-a   a )
```

$$\left(\begin{array}{cc} \frac{a+b\, e^{a+b}}{a+b} & \frac{b-b\, e^{a+b}}{a+b} \\ \frac{a-a\, e^{a+b}}{a+b} & \frac{b+a\, e^{a+b}}{a+b} \end{array}\right)$$

```
DD = DiagonalMatrix[{b₁, b₂}];
Inverse[DD].M.DD.Transpose[M /. b_i_ :→ -b_i] // FullSimplify // MatrixForm
```

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$$

Γb-Calculus

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma\theta = \Gamma b\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{h_a, h_b, h_S\}\right],\right.$$
$$\left.\gamma\theta\ //\ \Gamma\ //\ \mathrm{dm}[a, b, c]\ //\ \Gamma b\right\}$$

$$\left\{\left(\begin{array}{cccc}\omega & s_a & s_b & s_S\\ s_a & \alpha & \beta & \theta\\ s_b & \gamma & \delta & \epsilon\\ s_S & \phi & \psi & \Xi\\ \Sigma & \sigma_a & \sigma_b & \sigma_S\end{array}\right),\ \left(\begin{array}{ccc}-\beta+\omega & s_c & s_S\\ s_c & \frac{-\beta\,\gamma+\alpha\,\delta+\gamma\,\omega}{\omega} & \frac{-\beta\,\epsilon+\delta\,\theta+\epsilon\,\omega}{\omega}\\ s_S & \frac{-\beta\,\phi+\alpha\,\psi+\phi\,\omega}{\omega} & \frac{-\beta\,\Xi+\theta\,\psi+\Xi\,\omega}{\omega}\\ \Sigma & \sigma_a\,\sigma_b & \sigma_S\end{array}\right)\right\}$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &]
```

$$\left(\begin{array}{cc}\frac{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])^{1/4}\,(-1+T_1)^{1/4}\,(-1+T_2)^{1/4}}{\mathrm{Log}[T_1]^{1/4}\,\mathrm{Log}[T_2]^{1/4}\,(-1+T_1\,T_2)^{1/4}} & \\ s_1 & \mathrm{Log}[T_1]^{1/4}\left(\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\,\sqrt{-1+T_2}-\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\,T_1\,\sqrt{-\right.\\ s_2 & \\ \Sigma & \end{array}\right.$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &] // ΓbCollect[
  Assuming[c₁ > 0 && c₂ > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a_ :> ⅇ^c_a]]] &]
```

$$\left(\begin{array}{cc}\frac{((c_1+c_2)\,(-1+T_1))^{1/4}\,(-1+T_2)^{1/4}}{(c_1\,c_2\,(-1+T_1\,T_2))^{1/4}} & \\ s_1 & \sqrt{c_1\,(c_1+c_2)\,(-1+T_2)}-T_1\,\sqrt{c_1\,(c_1+c_2)\,(-1+T_2)}-T_1\,\sqrt{c_1\,(c_1+c_2)\,(-1+T_2)}\,T_2+T_1^2\,\sqrt{c_1\,(c_1+c_2)\,(-1+T_2)}\,T_2 \\ & (-1+T_1)^{3} \\ s_2 & \frac{c_2^{1/4}\,(-1+T_2)^{1/4}\,(-}{(c_1\,(} \\ \Sigma & \end{array}\right.$$

```
V // A // Γ // ΓCollect[FullSimplify[PowerExpand[#]] &] //
 ΓCollect[Assuming[c₁ > 0 && c₂ > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a_ :> ⅇ^c_a]]] &]
```

$$\left(\begin{array}{ccc}\frac{((c_1+c_2)\,(-1+T_1))^{1/4}\,(-1+T_2)^{1/4}}{(c_1\,c_2\,(-1+T_1\,T_2))^{1/4}} & s_1 & s_2\\ s_1 & \frac{c_1\,\sqrt{c_1\,c_2\,(-1+T_2)}+c_2\,\sqrt{c_1\,c_2\,(-1+T_2)}+c_1\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1\,T_2}}{(c_1+c_2)\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1\,T_2}} & \frac{-\sqrt{c_1\,c_2\,(c_1+c_2)\,(-1+T_1)}\,\sqrt{T}}{(c_1+c_2)\,\sqrt{-1+T}}\\ s_2 & \frac{-c_1\,\sqrt{c_1\,c_2\,(-1+T_2)}-c_2\,\sqrt{c_1\,c_2\,(-1+T_2)}+c_2\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1\,T_2}}{(c_1+c_2)\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1\,T_2}} & \frac{\sqrt{c_1\,c_2\,(c_1+c_2)\,(-1+T_1)}\,\sqrt{T_1}}{(c_1+c_2)\,\sqrt{-1+T}}\\ \Sigma & 1 & \sqrt{T}\end{array}\right.$$

$$\mathtt{V\ //\ A\ //\ \Gamma\ //\ \Gamma Collect[FullSimplify[PowerExpand[\#]]\ \&]\ //\ \Gamma Collect[}$$
$$\mathtt{Assuming[d_1>0\ \&\&\ d_2>0,\ FullSimplify[\#\ /.\ Log[x\_]\ :\!\to Log[x\ /.\ T_{a\_}:\!\to e^{d_a{}^2}]]]\ \&]}$$

$$\left(\begin{array}{ccc}
\frac{\left(\left(d_1^2+d_2^2\right)\left(-1+T_1\right)\right)^{1/4}\left(-1+T_2\right)^{1/4}}{\left(d_1^2\,d_2^2\left(-1+T_1\,T_2\right)\right)^{1/4}} & s_1 & s_2 \\
s_1 & \frac{d_1\left(d_1^2\,d_2\sqrt{-1+T_2}+d_2^3\sqrt{-1+T_2}+d_1\sqrt{\frac{-1+T_1}{T_1}}\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\,T_2\right)}\right)}{\left(d_1^2+d_2^2\right)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\,T_2\right)}} & \frac{d_1\left(-d_2\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\right)}\sqrt{T_1}\ -\right.}{\left(d_1^2+d_2^2\right)\sqrt{-1+T_2}} \\
s_2 & -\frac{d_2\left(d_1^3\sqrt{-1+T_2}+d_1\,d_2^2\sqrt{-1+T_2}-d_2\sqrt{\frac{-1+T_1}{T_1}}\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\,T_2\right)}\right)}{\left(d_1^2+d_2^2\right)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\,T_2\right)}} & \frac{d_2\left(d_1\sqrt{\left(d_1^2+d_2^2\right)\left(-1+T_1\right)}\sqrt{T_1}\ T\right.}{\left(d_1^2+d_2^2\right)\sqrt{-1+T_2}} \\
\Sigma & 1 & \sqrt{T_1}
\end{array}\right.$$

Γ1-Calculus

$$\mathtt{Clear[\alpha,\ \beta,\ \gamma,\ \delta,\ \theta,\ \epsilon,\ \phi,\ \psi,\ \Xi,\ \mu,\ \omega];}$$
$$\mathtt{\{\gamma 0 = \Gamma 1\left[\omega,\ h_a\,\sigma_a+h_b\,\sigma_b+h_S\,\sigma_S,\ \{t_a,\ t_b,\ t_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{h_a,\ h_b,\ h_S\}\right],\ \gamma 0\ //\ dm[a,\ b,\ c]\}}$$

$$\left\{\left(\begin{array}{cccc}
\omega & s_a & s_b & s_S \\
s_a & \alpha & \beta & \theta \\
s_b & \gamma & \delta & \epsilon \\
s_S & \phi & \psi & \Xi \\
\Gamma 1 & \sigma_a & \sigma_b & \sigma_S
\end{array}\right),\ \left(\begin{array}{ccc}
-\left(-1+\beta\right)\omega & s_c & s_S \\
s_c & \frac{-\gamma+\beta\,\gamma-\alpha\,\delta}{-1+\beta} & \frac{-\epsilon+\beta\,\epsilon-\delta\,\theta}{-1+\beta} \\
s_S & \frac{-\phi+\beta\,\phi-\alpha\,\psi}{-1+\beta} & \frac{-\Xi+\beta\,\Xi-\theta\,\psi}{-1+\beta} \\
\Gamma 1 & \sigma_a\,\sigma_b & \sigma_S
\end{array}\right)\right\}$$

$$\mathtt{Clear[\alpha,\ \theta,\ \phi,\ \Xi,\ \omega];}$$
$$\mathtt{\{\gamma 0 = \Gamma 1\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,\ t_S\}.\begin{pmatrix}\alpha & \theta\\ \phi & \Xi\end{pmatrix}.\{h_a,\ h_S\}\right],\ \gamma 0\ //\ q\Delta[a,\ b,\ c],}$$
$$\mathtt{\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,\ t_S\}.\begin{pmatrix}\alpha & \theta\\ \phi & \Xi\end{pmatrix}.\{h_a,\ h_S\}\right]\ //\ q\Delta[a,\ b,\ c]\}}$$

$$\left\{\left(\begin{array}{ccc}
\omega & s_a & s_S \\
s_a & \alpha & \theta \\
s_S & \phi & \Xi \\
\Gamma 1 & \sigma_a & \sigma_S
\end{array}\right),\ \left(\begin{array}{cccc}
\omega & s_b & s_c & s_S \\
s_b & \frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\left(-1+T_c\right)T_c\left(\alpha-\sigma_a\right)}{-1+T_b\,T_c} & \theta\,T_c \\
s_c & \frac{\left(-1+T_b\right)\left(\alpha-\sigma_a\right)}{-1+T_b\,T_c} & \frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{-1+T_b\,T_c} & \theta \\
s_S & \frac{\phi\left(-1+T_b\right)}{-1+T_b\,T_c} & \frac{\phi\left(-1+T_c\right)}{-1+T_b\,T_c} & \Xi \\
\Gamma 1 & \sigma_a & \sigma_a & \sigma_S
\end{array}\right),\right.$$

$$\left.\left(\begin{array}{cccc}
\omega & s_b & s_c & s_S \\
s_b & \frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\left(-1+T_b\right)T_c\left(\alpha-\sigma_a\right)}{-1+T_b\,T_c} & \frac{\theta\left(-1+T_b\right)T_c}{-1+T_b\,T_c} \\
s_c & \frac{\left(-1+T_c\right)\left(\alpha-\sigma_a\right)}{-1+T_b\,T_c} & \frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\theta\left(-1+T_c\right)}{-1+T_b\,T_c} \\
s_S & \phi & \phi & \Xi \\
\Gamma & \sigma_a & \sigma_a & \sigma_S
\end{array}\right)\right\}$$

```
{Xp[1, 2] // Γ, Xp[1, 2] // Γ1}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Gamma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_2 \\ s_2 & 0 & T_1 \\ \Gamma 1 & 1 & T_1 \end{pmatrix}\right\}$$

R3

$$\texttt{\{}Xm_{51}\,Xm_{62}\,Xp_{34}\texttt{ // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],}$$
$$Xp_{61}\,Xm_{24}\,Xm_{35}\texttt{ // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]\}}$$

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_1}{T_2} & \frac{-1+T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_1}{T_2} & \frac{-1+T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

```
V // A // Γ1 // Γ1Collect[FullSimplify[PowerExpand[#]] &]
```

$$\begin{pmatrix} \frac{(\text{Log}[T_1]+\text{Log}[T_2])^{1/4}\,(-1+T_1)^{1/4}\,(-1+T_2)^{1/4}}{\text{Log}[T_1]^{1/4}\,\text{Log}[T_2]^{1/4}\,(-1+T_1 T_2)^{1/4}} & s_1 \\ s_1 & \frac{\sqrt{\text{Log}[T_1]}\,\left(\text{Log}[T_1]\,\sqrt{\text{Log}[T_2]}\,\sqrt{-1+T_2}+\text{Log}[T_2]^{3/2}\,\sqrt{-1+T_2}+\sqrt{\text{Log}[T_1]}\,\sqrt{\text{Log}[T_1]+\text{Log}}\right.}{(\text{Log}[T_1]+\text{Log}[T_2])^{3/2}\,\sqrt{\frac{-1+T_1}{T_1}}\,\sqrt{-1+T_1 T_2}} \\ s_2 & -\frac{\sqrt{\text{Log}[T_2]}\,\left(\sqrt{\text{Log}[T_1]}\,\sqrt{\text{Log}[T_1]+\text{Log}[T_2]}\,\sqrt{-1+T_1}\,\sqrt{T_1}\,\sqrt{-1+T_2}+\sqrt{\text{Log}[T_2]}\,\sqrt{-1+T_1 T_2}\right.}{(\text{Log}[T_1]+\text{Log}[T_2])\,(-1+T_2)\,\sqrt{-1+T_1 T_2}} \\ \Sigma & 1 \end{pmatrix}$$

$$\texttt{V // A // Γ1 // Γ1Collect[FullSimplify[PowerExpand[\#]] \&] // Γ1Collect}\Big[$$
$$\texttt{Assuming}\Big[c_1 > 0\ \&\&\ c_2 > 0,\ \texttt{FullSimplify}\Big[\#\ \texttt{/.}\ \texttt{Log}[x\_] :\to \texttt{Log}\Big[x\ \texttt{/.}\ T_{a\_} :\to e^{c_a}\Big]\Big]\Big]\ \&\Big]$$

$$\begin{pmatrix} \frac{((c_1+c_2)\,(-1+T_1))^{1/4}\,(-1+T_2)^{1/4}}{(c_1\,c_2\,(-1+T_1 T_2))^{1/4}} & s_1 & \\ s_1 & \frac{c_1\,\sqrt{c_1\,c_2\,(-1+T_2)}+c_2\,\sqrt{c_1\,c_2\,(-1+T_2)}+c_1\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1 T_2}}{(c_1+c_2)\,\sqrt{\frac{(c_1+c_2)\,(-1+T_1)}{T_1}}\,\sqrt{-1+T_1 T_2}} & -\frac{\sqrt{c_1\,(-1+T_2)}\,\left(c_2^{3/2}\,\sqrt{-1+T}\right.}{} \\ s_2 & \frac{-\sqrt{c_1\,c_2\,(c_1+c_2)\,(-1+T_1)}\,\sqrt{T_1}\,\sqrt{-1+T_2}-c_2\,\sqrt{-1+T_1 T_2}+c_2\,T_1\,\sqrt{-1+T_1 T_2}}{(c_1+c_2)\,(-1+T_2)\,\sqrt{-1+T_1 T_2}} & \frac{\sqrt{c_1\,c_2}}{} \\ \Sigma & 1 & \end{pmatrix}$$