

Cheat Sheet Meta-Calculi

σ calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $tm_w^{uv} = (T_u, T_v \rightarrow T_w)$, $hm_z^{xy}(\sigma) = (\sigma \setminus \{x, y\}) \cup (z \rightarrow \sigma_x \sigma_y)$, $tha^{ux} = I$, $tr_c(\sigma) = \sigma \setminus c$, $R_{ux}^\pm \mapsto T_u^{\pm 1}$

α -calculus.

Constraints. • $\sum_u c_u A_{ux} = \sigma_x - 1$. • At $c_* = 0$, $\omega = 1$ and $A_{ux} = \frac{\partial \sigma_x}{\partial c_u} \Big|_{c_*=0}$.

$$\frac{\omega_1}{T_1} \Big| \frac{H_1}{A_1} * \frac{\omega_2}{T_2} \Big| \frac{H_2}{A_2} \stackrel{\alpha}{=} \frac{\omega_1 \omega_2}{T_1 T_2} \Big| \frac{H_1}{A_1} \quad \frac{H_2}{0 \quad A_2}$$

$$\frac{\omega}{u} \Big| \frac{H}{\alpha} \xrightarrow{\alpha::tm_w^{uv}} \frac{\omega}{w} \Big| \frac{H}{\alpha + \beta} \quad \frac{\omega}{T} \Big| \frac{x \quad y \quad H}{\alpha \quad \beta \quad \Xi} \xrightarrow{hm_z^{xy}} \frac{\omega}{T} \Big| \frac{z \quad H}{\alpha + \sigma_x \beta \quad \Xi}$$

$$\frac{\omega}{u} \Big| \frac{x \quad H}{\alpha \quad \theta} \xrightarrow{\alpha::tha^{ux}} \frac{v\omega}{u} \Big| \frac{x \quad H}{\sigma_x \alpha / v \quad \sigma_x \theta / v} \quad \frac{\omega}{a} \Big| \frac{S}{\alpha \quad \theta} \xrightarrow{\alpha::tr_a} \frac{\mu\omega}{S} \Big| \frac{S}{\Xi - c_a \phi \theta / \mu} \quad \rho_{ux}^\pm \stackrel{\alpha}{=} \frac{1}{u} \Big| \frac{x}{e^{\pm c_u} - 1}$$

β -calculus.

Constraints. • Sum of column x is $\sigma_x - 1$. • At $T_* = 1$, $\omega = 1$ and $A = 0$.

$$\frac{\omega_1}{T_1} \Big| \frac{H_1}{A_1} * \frac{\omega_2}{T_2} \Big| \frac{H_2}{A_2} \stackrel{\beta}{=} \frac{\omega_1 \omega_2}{T_1 T_2} \Big| \frac{H_1}{A_1} \quad \frac{H_2}{0 \quad A_2}$$

$$\frac{\omega}{u} \Big| \frac{H}{\alpha} \xrightarrow{\beta::tm_w^{uv}} \frac{\omega}{w} \Big| \frac{H}{\alpha + \beta} \quad \frac{\omega}{T} \Big| \frac{x \quad y \quad H}{\alpha \quad \beta \quad \Xi} \xrightarrow{hm_z^{xy}} \frac{\omega}{T} \Big| \frac{z \quad H}{\alpha + \sigma_x \beta \quad \Xi}$$

$$\frac{\omega}{u} \Big| \frac{x \quad H}{\alpha \quad \theta} \xrightarrow{\beta::tha^{ux}} \frac{v\omega}{u} \Big| \frac{x \quad H}{\sigma_x \alpha / v \quad \sigma_x \theta / v} \quad \rho_{ux}^\pm \stackrel{\beta}{=} \frac{1}{u} \Big| \frac{x}{T_u^{\pm 1} - 1}$$

Gassner calculus Γ .

Preserves $C_1 := [\text{col sum} = 1] (\Leftrightarrow \text{OC})$ and $\checkmark C_2 := [\forall a, b, (T_a - 1) \mid (A_{ab} - \delta_{ab} \sigma_b)]$

• Except under tr_c , at $T_* = 1$, $\omega = 1$ and $A = I$.

$$\frac{\omega}{a} \Big| \frac{a \quad b \quad S}{\alpha \quad \beta \quad \theta} \xrightarrow{m_c^{ab}} \frac{\mu\omega}{c} \Big| \frac{c \quad S}{\gamma + \alpha\delta/\mu \quad \epsilon + \delta\theta/\mu} \quad \frac{\omega}{c} \Big| \frac{c \quad S}{\alpha \quad \theta} \xrightarrow{\Gamma::tr_c} \frac{\mu\omega}{S} \Big| \frac{S}{\Xi + \psi\theta/\mu} \quad R_{ab}^\pm \stackrel{\Gamma}{=} \frac{1}{a} \Big| \frac{a \quad b}{1 \quad 1 - T_a^{\pm 1}} \quad \frac{1}{b} \Big| \frac{1 - T_a^{\pm 1}}{T_a^{\pm 1}}$$

$$\frac{\omega}{a} \Big| \frac{a \quad S}{\alpha \quad \theta} \xrightarrow{q\Delta_{bc}^a} \left(\frac{\omega}{b} \Big| \frac{b \quad c \quad S}{(\sigma_a - \alpha T_a - v T_c)/\mu \quad (T_b - 1)T_c v/\mu \quad (T_b - 1)T_c \theta/\mu} \right)_{T_a \mapsto T_b T_c}$$

Satisfies: $\checkmark R_{13}^+ // q\Delta_{12}^1 = R_{23}^+ \# R_{13}^+$
 $\checkmark R_{13}^- // q\Delta_{12}^1 = R_{13}^- \# R_{23}^-$
 $\checkmark q\Delta_{a_1 a_2}^a // q\Delta_{b_1 b_2}^b // m_{c_1}^{a_1 b_1} // m_{c_2}^{a_2 b_2} = m_c^{ab} // q\Delta_{c_1 c_2}^c$

$$\frac{\omega}{a} \Big| \frac{a \quad S}{\alpha \quad \theta} \xrightarrow{dS^a} \left(\frac{\alpha\omega/\sigma_a}{a} \Big| \frac{a \quad S}{1/\alpha \quad \theta/\alpha} \right)_{T_a \rightarrow T_a^{-1}}$$

Satisfies: $\checkmark R_{12}^+ // dS^{1 \text{ or } 2} = R_{12}^+$ • $\checkmark dm_c^{ab} // dS^c = dS^a // dS^b // dm_c^{ba}$
 $\checkmark dS^a // dS^a = I$ • $\checkmark q\Delta_{bc}^a // dS^b // dS^c = dS^a // q\Delta_{cb}^a$
 \checkmark Assuming C_2 , $d\eta^a // d\epsilon_a = q\Delta_{bc}^a // dS^c // dm_a^{bc}$ (also 3 variants).

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative.

The MVA mod units: $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ \checkmark

Bureau. On $b \in uB_n$, $Bu: \sigma_i^{\pm 1} \mapsto U_i^{\pm 1}$.

Unitarity. With $U = Bu(b)$, $\bar{U} \Omega_n U^T = \Omega_n$.

Thm. $\Gamma(b) = \frac{1}{s_1} \Big| \frac{s_{b(1)} \quad s_{b(2)} \quad \dots}{Bu(b)^T} \cdot U_i = \begin{pmatrix} I_i & & \\ & 1-t & t \\ & & 1 & 0 \\ & & & & I_{n-i-1} \end{pmatrix}, U_i^{-1} = \begin{pmatrix} I_i & & \\ & 0 & 1 \\ & \bar{t} & 1-\bar{t} \\ & & & & I_{n-i-1} \end{pmatrix}, \Omega_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1-t & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & \dots & 1 \end{pmatrix}$

Some matrices: $\begin{pmatrix} 1-t_i & 1 \\ t_i & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 \\ 1 & 1 & \frac{1}{1-t_3} \end{pmatrix}, \begin{pmatrix} 1-t_j & 1 \\ t_j & 0 \end{pmatrix}, \begin{pmatrix} -\frac{t_1-1}{t_1} & 0 & 0 \\ \frac{(t_1-1)(t_2-1)}{t_1} & -\frac{t_2-1}{t_2} & 0 \\ \frac{(t_1-1)(t_3-1)}{t_1} & \frac{(t_2-1)(t_3-1)}{t_2} & -\frac{t_3-1}{t_3} \end{pmatrix}$

To do. • Full verification program. • R1? • Precise relation with Bureau/Gassner. • Concordance. • Unitarity. • Planarity. • A depth-mirror property for u-objects. • Mutations? • Link relations? • Behaviour of A/MVA under mirror/strand reversal?

β -better calculus.

Constraints. • Sum of column x is $(\sigma_x - 1)\omega$. • $\omega^{k-1} \mid \Lambda^k A$. • At $T_* = 1$, $\omega = 1$ and $A = 0$.

$$\frac{\omega_1}{T_1} \left| \begin{array}{c|c} H_1 & \\ \hline A_1 & \end{array} \right| * \frac{\omega_2}{T_2} \left| \begin{array}{c|c} H_2 & \\ \hline A_2 & \end{array} \right| \stackrel{\beta_b}{=} \frac{\omega_1 \omega_2}{T_1 T_2} \left| \begin{array}{c|cc} H_1 & H_2 & \\ \hline \omega_2 A_1 & 0 & \\ T_2 & 0 & \omega_1 A_2 \end{array} \right|$$

$$\frac{\omega}{u} \left| \begin{array}{c|c} H & \\ \hline \alpha & \end{array} \right| \xrightarrow{\frac{tm_w^{uv}}{\beta_b}} \left(\frac{\omega}{w} \left| \begin{array}{c|c} H & \\ \hline \alpha + \beta & \end{array} \right| \right)_{T_u, T_v \rightarrow T_w} \quad \rho_{ux}^\pm = \frac{1}{\beta_b} \left| \begin{array}{c|c} 1 & x \\ \hline u & T_u^{\pm 1} - 1 \end{array} \right|$$

$$\frac{\omega}{T} \left| \begin{array}{c|cc} x & y & H \\ \hline \alpha & \beta & \gamma \end{array} \right| \xrightarrow{\frac{hm_z^{xy}}{\beta_b}} \frac{\omega}{T} \left| \begin{array}{c|c} z & H \\ \hline \alpha + \sigma_x \beta & \gamma \end{array} \right| \quad \frac{\omega}{u} \left| \begin{array}{c|cc} x & H & \\ \hline \alpha & \beta & \end{array} \right| \xrightarrow{\frac{tha^{ux}}{\beta_b}} \frac{\omega + \alpha}{u} \left| \begin{array}{c|cc} x & H & \\ \hline \sigma_x \alpha & \sigma_x \beta & \\ T & \gamma & \delta + \frac{\alpha \delta - \gamma \beta}{\omega} \end{array} \right| =: \left| \begin{array}{c|c} \cdot & - \\ \hline \left(\begin{array}{cc} \sigma_x & 0 \\ 0 & 1 \end{array} \right) & A^{ux} \end{array} \right|$$

$$\frac{\omega}{a} \left| \begin{array}{c|cc} a & b & S \\ \hline \alpha & \beta & \theta \\ b & \gamma & \delta \\ S & \phi & \psi \end{array} \right| \xrightarrow{\frac{m_c^{ab}}{\beta_b} \checkmark} \left(\frac{\omega + \beta}{c} \left| \begin{array}{c|c} c & S \\ \hline \gamma + \sigma_a \delta + \sigma_b (\alpha + \sigma_a \beta) + \frac{\beta \gamma - \alpha \delta}{\omega} & \epsilon + \sigma_b \theta + \frac{\beta \epsilon - \delta \theta}{\omega} \\ \phi + \sigma_a \psi + \frac{\beta \phi - \alpha \psi}{\omega} & \Xi + \frac{\beta \Xi - \psi \theta}{\omega} \end{array} \right| \right)_{T_a, T_b \rightarrow T_c}$$

The MVA (mod units): n -component $L \mapsto (\sigma, \omega, A) \mapsto \omega^{2-n} \det'(A - \omega \text{diag}((\sigma_i - 1)) / (1 - T'))$ ✓

Note. $A^{ux} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta + \frac{\alpha \delta - \gamma \beta}{\omega} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \frac{(\omega + \alpha) \delta - \gamma \beta}{\omega} \end{pmatrix} = \frac{1}{\omega} \left[(\omega + \alpha) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} - \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} \right] = \frac{1}{\omega} [(\omega + a_{ux})A - a_{*x} a_{u*}]$.

Claim. $\omega^{k-1} \mid \Lambda^k A$ and $\omega^k \mid \Lambda^{k+1} A$ implies $(\omega + \alpha)^{k-1} \mid \Lambda^k A^{ux}$, with $\alpha = a_{ux}$.

Proof. With $\bar{u} \in T^k$ and $\bar{x} \in H^k$, ω^k divides $\left| \begin{array}{c|c} \omega & 0 \\ 0 & a_{\bar{u}\bar{x}} \end{array} \right|$ and $\left| \begin{array}{cc} a_{u\bar{x}} & a_{u\bar{x}} \\ a_{\bar{u}x} & a_{\bar{u}\bar{x}} \end{array} \right|$ and hence their sum, $\left| \begin{array}{cc} \omega + \alpha & a_{u\bar{x}} \\ a_{\bar{u}x} & a_{\bar{u}\bar{x}} \end{array} \right| =$

$$(\omega + \alpha) \left| \begin{array}{c|c} 1 & 0 \\ 0 & a_{\bar{u}\bar{x}} - \frac{1}{\omega + \alpha} a_{\bar{u}x} a_{u\bar{x}} \end{array} \right| = \frac{1}{(\omega + \alpha)^{k-1}} |(\omega + \alpha) a_{\bar{u}\bar{x}} - a_{\bar{u}x} a_{u\bar{x}}|. \text{ So } \frac{1}{(\omega + \alpha)^{k-1}} \left| \frac{1}{\omega} [(\omega + \alpha) a_{\bar{u}\bar{x}} - a_{\bar{u}x} a_{u\bar{x}}] \right| \text{ is integral. } \square$$

That is, with $A_{\bar{u};\bar{x}}$ denoting minors, if $\omega^{k-1} \mu_{\bar{u};\bar{x}} = A_{\bar{u};\bar{x}}$ and $\omega^k \mu_{u\bar{u};x\bar{x}} = A_{u\bar{u};x\bar{x}}$, then $(\omega + \alpha)^{k-1} (\mu_{\bar{u};\bar{x}} + \mu_{u\bar{u};x\bar{x}}) = A_{\bar{u};\bar{x}}^{ux}$.

Λ -calculus. $\Lambda(T; H) = R(T) \otimes (\Lambda(T) \otimes \Lambda(H))_{\pm}$, with $R(T)$ Laurent polynomials in $\{T_u\}_{u \in T}$. $\lambda_1 * \lambda_2 = \lambda_1 (\wedge \otimes \wedge) \lambda_2$
 $tm_w^{uv} : u, v \rightarrow w, T_u, T_v \rightarrow T_w$ $hm_z^{xy} : x \rightarrow z, y \rightarrow \sigma_x z$ $tha^{ux} : \lambda \mapsto (1 + i_u \otimes i_x) \lambda // (u \rightarrow \sigma_x u)$ $\rho_{ux}^\pm = 1 + (T_u^{\pm 1} - 1) u x$

Relations. • $\rho_{ux}^+ \rho_{vy}^- // tm_w^{uv} // hm_z^{xy} = t \epsilon_w h \epsilon_z$. • $\rho_{ux}^{s_1} \rho_{vy}^{s_2} \rho_{wz}^{s_2} // tm_v^{vw} // hm_x^{xy} // tha^{ux} = \rho_{vx}^{s_2} \rho_{wz}^{s_2} \rho_{uy}^{s_1} // tm_v^{vw} // hm_x^{xy}$.