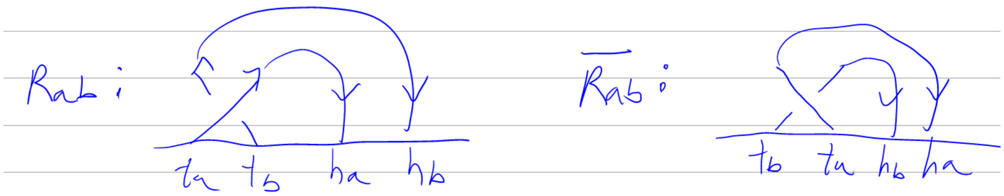
Pensieve header: Unitarity for Γ-calculus.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\MetaCalculi"];
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```




In[∘]:=
```
R[a_,b_] := Γ@<|ς → {t[a], t[b], h[a], h[b]}, ω → 1, σ → h[a] + h[b] T[a], λ → {t[a], t[b]}.(T[a], 1 - T[a]; 0, 1).{h[a], h[b]}|>;
R̄[a_,b_] := Γ@<|ς → {t[b], t[a], h[b], h[a]}, ω → 1, σ → h[a] + h[b] T[a]^-1, λ → {t[a], t[b]}.(T[a]^-1, 1 - T[a]^-1; 0, 1).{h[a], h[b]}|>;
```

$$R_{a\_,b\_} := \Gamma@\langle|\varsigma \to \{t_a, t_b, h_a, h_b\},\ \omega \to 1,\ \sigma \to h_a + h_b T_a,\ \lambda \to \{t_a, t_b\}.\begin{pmatrix} T_a & 1-T_a \\ 0 & 1 \end{pmatrix}.\{h_a, h_b\}|\rangle;$$

$$\bar{R}_{a\_,b\_} := \Gamma@\langle|\varsigma \to \{t_b, t_a, h_b, h_a\},\ \omega \to 1,\ \sigma \to h_a + h_b T_a^{-1},\ \lambda \to \{t_a, t_b\}.\begin{pmatrix} T_a^{-1} & 1-T_a^{-1} \\ 0 & 1 \end{pmatrix}.\{h_a, h_b\}|\rangle;$$

In[∘]:= $\bar{R}_{1,2}$

Out[∘]=

$$\Gamma\left[\left\langle\left|\varsigma \to \{t_2, t_1, h_2, h_1\},\ \omega \to 1,\ \sigma \to h_1 + \frac{h_2}{T_1},\ \lambda \to h_2\left(t_2 + t_1\left(1 - \frac{1}{T_1}\right)\right) + \frac{h_1 t_1}{T_1}\right|\right\rangle\right]$$

In[∘]:=
```
Γ[α_][ϰ_] := α[ϰ];
Γ[α_][S] := Union@Cases[α[ς], t[a_] :→ a, ∞];
Γ[α_][n] := Length[Γ[α][S]];
Γ[α_][Σ] := (∂[h[#]] α[σ]) & /@ Γ[α][S];
Γ[α_][A] := Outer[Factor[∂[#1,#2] Γ[α][λ]] &, Cases[α[ς], t_], Cases[α[ς], h_]];
```

In[∘]:=
```
Table[{γ@ς, γ@ω, γ@σ, γ@λ, γ@S, γ@Σ, γ@A // MatrixForm}, {γ, {R[1,2], R̄[1,2]}}] // Transpose //
 MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \{t_1, t_2, h_1, h_2\} & \{t_2, t_1, h_2, h_1\} \\ 1 & 1 \\ h_1 + h_2 T_1 & h_1 + \frac{h_2}{T_1} \\ h_2\,(t_2 + t_1\,(1 - T_1)) + h_1 t_1 T_1 & h_2\left(t_2 + t_1\left(1 - \frac{1}{T_1}\right)\right) + \frac{h_1 t_1}{T_1} \\ \{1, 2\} & \{1, 2\} \\ \{1, T_1\} & \left\{1, \frac{1}{T_1}\right\} \\ \begin{pmatrix} T_1 & 1 - T_1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ \frac{-1+T_1}{T_1} & \frac{1}{T_1} \end{pmatrix} \end{pmatrix}$$

In[∘]:=
```
ΓCollect[γ_Γ] := Γ[<|ς → γ@ς, ω → Factor@γ@ω, σ → Expand@γ@σ,
   λ → Total[CoefficientRules[γ@λ, γ@ς] /. (ps_ → c_) :→ Factor[c] Times @@ ((γ@ς)^ps)]|>]
```

In[•]:=
```
ΓCollect[R̄1,2]
```
Out[•]=

$$\Gamma\left[\left\langle\left|\varsigma \to \{t_2, t_1, h_2, h_1\}, \omega \to 1, \sigma \to h_1 + \frac{h_2}{T_1}, \lambda \to h_2 t_2 + \frac{h_1 t_1}{T_1} + \frac{h_2 t_1 (-1 + T_1)}{T_1}\right|\right\rangle\right]$$

In[•]:=
```
ConservativeQ[γ_Γ] := Simplify[Expand[γ@λ /. h_ → 1] == Sum[ta, {a, γ@S}]]
```

In[•]:=
```
ConservativeQ[R1,2]
```
Out[•]=

True

In[•]:=
```
T* ^:= T^-1;
vpnς_[c_, d_] := Expand@Module[{e, f}, (* Vortex Pairing North *)
    Expand[c (d /. {T → T*, ti_ :→ ti*, hi_ :→ hi*})] /.
     {ti_* ti_ → 0, hi_* hi_ → T - T*, (f : t | h)j_* (e : t | h)i_ :→
       If[Position[ς, ei][[1, 1]] < Position[ς, fj][[1, 1]], T - 1, 1 - T*]}
   ];
vpsς_[c_, d_] :=
  Expand@Module[{e, f}, (* Vortex Pairing South (rel. to north, T→T* and t↔h) *)
    Expand[c (d /. {T → T*, ti_ :→ ti*, hi_ :→ hi*})] /.
     {ti_* ti_ → T* - T, hi_* hi_ → 0, (f : t | h)j_* (e : t | h)i_ :→
       If[Position[ς, ei][[1, 1]] < Position[ς, fj][[1, 1]], T* - 1, 1 - T]}
   ];
UnitaryQ[γ_Γ] := Module[{vs},
   vs = Table[-ti + ∂ti γ@λ /. T_ → T, {i, γ@S}];
   And @@ Flatten@Table[
      Simplify[vpnγ@ς[vs[[i]], vs[[j]]] == 0 == vpsγ@ς[vs[[i]], vs[[j]]]], {i, γ@n}, {j, γ@n}]
  ];
```

In[•]:=
```
UnitaryQ /@ {R1,2, R̄1,2}
```
Out[•]=

{True, True}

In[∘]:=
```
ΓForm[γ_Γ] := Module[{M},
   M = γ[A] // Transpose;
   PrependTo[M, t# & /@ γ[S]];
   M = Join[
     {Prepend[h# & /@ γ[S], γ[ω]]},
     Transpose[M],
     {Prepend[γ[Σ], If[TrueQ[ConservativeQ@γ ∧ UnitaryQ@γ], 🟩, 🟥]]}
    ];
   Column[{γ[ς], MatrixForm[M]}]
  ];
ΓForm[else_] := else /. γ_Γ :→ ΓForm[γ];
Format[γ_Γ, StandardForm] := ΓForm[γ];
```

In[∘]:= $\{R_{1,2}, \overline{R}_{1,2}\}$

Out[∘]=

$$\left\{ \begin{matrix}\{t_1, t_2, h_1, h_2\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ 🟩 & 1 & T_1 \end{pmatrix}\end{matrix},\ \begin{matrix}\{t_2, t_1, h_2, h_1\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ 🟩 & 1 & \frac{1}{T_1} \end{pmatrix}\end{matrix} \right\}$$

In[∘]:=
```
dm_{i_, j_→k_}[γ_Γ] := Module[{a, b, c, d, θ, ϵ, ϕ, ψ, Ξ, μ},
  ( a b θ )   ( ∂_{t_i,h_i} γ@λ  ∂_{t_i,h_j} γ@λ  ∂_{t_i} γ@λ )
  ( c d ϵ ) = ( ∂_{t_j,h_i} γ@λ  ∂_{t_j,h_j} γ@λ  ∂_{t_j} γ@λ ) /. (t | h)_{i|j} → 0;
  ( ϕ ψ Ξ )   (   ∂_{h_i} γ@λ      ∂_{h_j} γ@λ       γ@λ     )
  ΓCollect[Γ[<|
     ς → DeleteCases[γ@ς, h_i | t_j] /. {t_i → t_k, h_j → h_k},
     ω → (μ = 1 - c) γ@ω,
     σ → h_k (∂_{h_i} σ) (∂_{h_j} σ) + (σ /. h_{i|j} → 0),
     λ → {t_k, 1}.( b + a d / μ   θ + a ϵ / μ ).{h_k, 1}
                  ( ψ + d ϕ / μ   Ξ + ϵ ϕ / μ )
     |>] /. {T_i → T_k, T_j → T_k}]
  ];
```

In[∘]:= $\{R_{1,2}\ //\ dm_{1,2\to1},\ R_{1,2}\ //\ dm_{2,1\to1},\ \overline{R}_{1,2}\ //\ dm_{1,2\to1},\ \overline{R}_{1,2}\ //\ dm_{2,1\to1}\}$

Out[∘]=

$$\left\{ \begin{matrix}\{t_1, h_1\}\\ \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ 🟩 & 0 \end{pmatrix}\end{matrix},\ \begin{matrix}\{t_1, h_1\}\\ \begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \\ 🟩 & 0 \end{pmatrix}\end{matrix},\ \begin{matrix}\{t_1, h_1\}\\ \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ 🟩 & 0 \end{pmatrix}\end{matrix},\ \begin{matrix}\{t_1, h_1\}\\ \begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \\ 🟩 & 0 \end{pmatrix}\end{matrix} \right\}$$

In[∘]:= **With**[{n = 3},

$$\gamma 0 = \Gamma @ \langle| \varsigma \to \text{Flatten@Table}[\{t_a, h_a\}, \{a, n\}],\ \omega \to \omega,\ \sigma \to \sum_{a=1}^{n} h_a\,\sigma_a,\ \lambda \to \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \alpha_{ab} |\rangle\Big]$$

$\gamma 0$ // $dm_{1,2\to 1}$

Out[∘]=

$\{t_1, h_1, t_2, h_2, t_3, h_3\}$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 \\ t_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ t_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ t_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \\ \blacksquare & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

Out[∘]=

$\{t_1, h_1, t_3, h_3\}$

$$\begin{pmatrix} -\omega\,(-1+\alpha_{2,1}) & h_1 & h_3 \\ t_1 & \frac{-\alpha_{1,2}+\alpha_{1,2}\,\alpha_{2,1}-\alpha_{1,1}\,\alpha_{2,2}}{-1+\alpha_{2,1}} & \frac{-\alpha_{1,3}+\alpha_{1,3}\,\alpha_{2,1}-\alpha_{1,1}\,\alpha_{2,3}}{-1+\alpha_{2,1}} \\ t_3 & \frac{-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}}{-1+\alpha_{2,1}} & \frac{-\alpha_{2,3}\,\alpha_{3,1}-\alpha_{3,3}+\alpha_{2,1}\,\alpha_{3,3}}{-1+\alpha_{2,1}} \\ \blacksquare & 0 & 0 \end{pmatrix}$$

In[∘]:= **With**[{n = 4},

$$\gamma 0 = \Gamma @ \langle| \varsigma \to \text{Flatten@Table}[\{t_a, h_a\}, \{a, n\}],\ \omega \to \omega,\ \sigma \to \sum_{a=1}^{n} h_a\,\sigma_a,\ \lambda \to \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \alpha_{ab} |\rangle\Big]$$

Out[∘]=

$\{t_1, h_1, t_2, h_2, t_3, h_3, t_4, h_4\}$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_4 \\ t_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ t_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ t_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ t_4 & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \blacksquare & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

In[∘]:= **lhs** = $\gamma 0$ // $dm_{1,2\to 1}$ // $dm_{1,3\to 1}$ // **ΓCollect**

Out[∘]=

$\{t_1, h_1, t_4, h_4\}$

$$\begin{pmatrix} \omega\,(1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}) & h_1 \\ t_1 & \frac{\alpha_{1,3}-\alpha_{1,3}\,\alpha_{2,1}+\alpha_{1,1}\,\alpha_{2,3}-\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1}+\alpha_{1,2}\,\alpha_{2,3}\,\alpha_{3,1}-\alpha_{1,3}\,\alpha_{3,2}+\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2}-\alpha_{1,1}\,\alpha_{2,3}\,\alpha_{3,2}+}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ t_4 & \frac{\alpha_{2,3}\,\alpha_{4,1}-\alpha_{2,3}\,\alpha_{3,2}\,\alpha_{4,1}+\alpha_{2,2}\,\alpha_{3,3}\,\alpha_{4,1}+\alpha_{2,3}\,\alpha_{3,1}\,\alpha_{4,2}+\alpha_{3,3}\,\alpha_{4,2}-\alpha_{2,1}\,\alpha_{3,3}\,\alpha_{4,2}+\alpha_{4,3}-\alpha_{2,1}\,\alpha_{4,3}-}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ \blacksquare & 0 \end{pmatrix}$$

In[∘]:=
```
rhs = γ0 // dm_{2,3→2} // dm_{1,2→1} // ΓCollect
```

Out[∘]=

$\{t_1, h_1, t_4, h_4\}$

$$\begin{pmatrix} \omega\ (1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}) & h_1 \\ t_1 & \frac{\alpha_{1,3}-\alpha_{1,3}\,\alpha_{2,1}+\alpha_{1,1}\,\alpha_{2,3}-\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1}+\alpha_{1,2}\,\alpha_{2,3}\,\alpha_{3,1}-\alpha_{1,3}\,\alpha_{3,2}+\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2}-\alpha_{1,1}\,\alpha_{2,3}\,\alpha_{3,2}+c}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ t_4 & \frac{\alpha_{2,3}\,\alpha_{4,1}-\alpha_{2,3}\,\alpha_{3,2}\,\alpha_{4,1}+\alpha_{2,2}\,\alpha_{3,3}\,\alpha_{4,1}+\alpha_{2,3}\,\alpha_{3,1}\,\alpha_{4,2}+\alpha_{3,3}\,\alpha_{4,2}-\alpha_{2,1}\,\alpha_{3,3}\,\alpha_{4,2}+\alpha_{4,3}-\alpha_{2,1}\,\alpha_{4,3}-c}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ \blacksquare & 0 \end{pmatrix}$$

In[∘]:=
```
lhs@λ == rhs@λ
```

Out[∘]=

True

In[∘]:=
```
Γ /: RotateLeft[Γ[α_], n___] := Γ@ReplacePart[α, Key@ς → RotateLeft[α@ς, n]]
```

In[∘]:=
```
{Table[RotateLeft[R_{1,2}, k], {k, 0, 3}], Table[RotateLeft[R̄_{1,2}, k], {k, 0, 3}]}
```

Out[∘]=

$$\left\{\left\{ \begin{matrix}\{t_1, t_2, h_1, h_2\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}\end{matrix}, \begin{matrix}\{t_2, h_1, h_2, t_1\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 0 & 1 \\ t_2 & T_1 & 1-T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}\end{matrix}, \begin{matrix}\{h_1, h_2, t_1, t_2\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}\end{matrix}, \begin{matrix}\{h_2, t_1, t_2, h_1\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1-T_1 & T_1 \\ t_2 & 1 & 0 \\ \blacksquare & 1 & T_1 \end{pmatrix}\end{matrix} \right\},\right.$$

$$\left.\left\{ \begin{matrix}\{t_2, t_1, h_2, h_1\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\end{matrix}, \begin{matrix}\{t_1, h_2, h_1, t_2\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ t_2 & 1 & 0 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\end{matrix}, \begin{matrix}\{h_2, h_1, t_2, t_1\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\end{matrix}, \begin{matrix}\{h_1, t_2, t_1, h_2\}\\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 0 & 1 \\ t_2 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\end{matrix} \right\}\right\}$$

In[∘]:=
```
Γ /: Insert[γ1_Γ, γ2_Γ, k_] := Γ@<|
    ς → Flatten[Insert[γ1@ς, γ2@ς, k]],
    ω → γ1@ω γ2@ω,
    σ → γ1@σ + γ2@σ,
    λ → γ1@λ + γ2@λ
    |>
Γ /: Insert[γ2_Γ, k_][γ1_Γ] := Insert[γ1, γ2, k]
```

In[∘]:=
```
R_{1,2} // Insert[R_{3,4}, 2]
```

Out[∘]=

$\{t_1, t_3, t_4, h_3, h_4, t_2, h_1, h_2\}$

$$\begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & 0 & 0 & T_1 & 1-T_1 \\ t_2 & T_3 & 1-T_3 & 0 & 0 \\ t_3 & 0 & 1 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 \end{pmatrix}$$

In[ ]:= ComposeList[{Insert[$R_{3,4}$, 5], Insert[$R_{5,6}$, 9], $dm_{2,3\to 2}$, $dm_{1,4\to 1}$, $dm_{1,5\to 1}$, $dm_{2,6\to 2}$, $dm_{2,1\to 2}$}, $R_{1,2}$]

Out[∘]=

$$\left\{ \overset{\{t_1, t_2, h_1, h_2\}}{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}}, \overset{\{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4\}}{\begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & T_1 & 1-T_1 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1-T_3 \\ t_4 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 \end{pmatrix}}, \right.$$

$$\overset{\{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4, t_5, t_6, h_5, h_6\}}{\begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ t_1 & T_1 & 1-T_1 & 0 & 0 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1-T_3 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 & 1 & T_5 \end{pmatrix}},$$

$$\overset{\{t_1, t_2, h_1, t_4, h_2, h_4, t_5, t_6, h_5, h_6\}}{\begin{pmatrix} 1 & h_1 & h_2 & h_4 & h_5 & h_6 \\ t_1 & T_1 & -((-1+T_1)\,T_2) & (-1+T_1)\,(-1+T_2) & 0 & 0 \\ t_2 & 0 & T_2 & 1-T_2 & 0 & 0 \\ t_4 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 & 0 \end{pmatrix}},$$

$$\overset{\{t_1, t_2, h_2, h_1, t_5, t_6, h_5, h_6\}}{\begin{pmatrix} 1 & h_1 & h_2 & h_5 & h_6 \\ t_1 & -((-1+T_1)\,T_2) & 1-T_2+T_1\,T_2 & 0 & 0 \\ t_2 & T_2 & 1-T_2 & 0 & 0 \\ t_5 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}},$$

$$\overset{\{t_1, t_2, h_2, t_6, h_1, h_6\}}{\begin{pmatrix} 1 & h_1 & h_2 & h_6 \\ t_1 & -((-1+T_1)\,T_2) & T_1\,(1-T_2+T_1\,T_2) & -((-1+T_1)\,(1-T_2+T_1\,T_2)) \\ t_2 & T_2 & -T_1\,(-1+T_2) & (-1+T_1)\,(-1+T_2) \\ t_6 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}},$$

$$\left. \overset{\{t_1, t_2, h_1, h_2\}}{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1\,(1-T_2+T_1\,T_2) & -((-1+T_1)\,(1+T_1\,T_2)) \\ t_2 & -T_1\,(-1+T_2) & 1-T_1+T_1\,T_2 \\ \blacksquare & 0 & 0 \end{pmatrix}}, \overset{\{t_2, h_2\}}{\begin{pmatrix} T_2\,\left(1-T_2+T_2^2\right) & h_2 \\ t_2 & 1 \\ \blacksquare & 0 \end{pmatrix}} \right\}$$

In[∘]:= {Insert[RotateLeft[$\overline{R}_{4,11}$, 1], 4], $dm_{1,4\to1}$, $dm_{11,12\to11}$, , Insert[$R_{10,15}$, 5]}

Out[∘]=

{Insert[ {$t_4$, $h_{11}$, $h_4$, $t_{11}$} $\begin{pmatrix} 1 & h_4 & h_{11} \\ t_4 & \frac{-1+T_4}{T_4} & \frac{1}{T_4} \\ t_{11} & 1 & 0 \\ \blacksquare & 1 & \frac{1}{T_4} \end{pmatrix}$ , 4], $dm_{1,4\to1}$, $dm_{11,12\to11}$, Null, Insert[ {$t_{10}$, $t_{15}$, $h_{10}$, $h_{15}$} $\begin{pmatrix} 1 & h_{10} & h_{15} \\ t_{10} & T_{10} & 1-T_{10} \\ t_{15} & 0 & 1 \\ \blacksquare & 1 & T_{10} \end{pmatrix}$ , 5]}

In[∘]:= Column@Reverse@
ComposeList[{Insert[RotateLeft[$\overline{R}_{2,7}$, 1], 4], $dm_{1,2\to1}$, Insert[RotateLeft[$R_{6,13}$, 1], 7],
$dm_{12,13\to12}$, $dm_{6,7\to6}$, Insert[$\overline{R}_{8,3}$, 5], $dm_{1,3\to1}$, $dm_{6,8\to6}$, Insert[$R_{14,9}$, 6], $dm_{12,14\to12}$,
$dm_{6,9\to6}$, Insert[RotateLeft[$\overline{R}_{4,11}$, 1], 4], $dm_{1,4\to1}$, $dm_{11,12\to11}$, Insert[$R_{10,15}$, 6],
$dm_{6,10\to6}$, $dm_{11,15\to11}$, $dm_{6,11\to6}$, Insert[$R_{16,5}$, 4], $dm_{6,16\to6}$, $dm_{1,5\to1}$, $dm_{1,6\to1}$
},
$\overline{R}_{12,1}$]

Out[∘]=

{$t_1$, $h_1$}

$$\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix}$$

{$t_1$, $h_6$, $h_1$, $t_6$}

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2} & h_1 \\ t_1 & \frac{(-1+T_1)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} \\ t_6 & -\frac{T_6(1-2T_1+T_1^2-2T_6+5T_1T_6-3T_1^2T_6+2T_6^2-6T_1T_6^2+2T_1^2T_6^2-2T_6^3+5T_1T_6^3-T_1^2T_6^3+T_6^4-3T_1T_6^4+T_1T_6^5}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} \\ \blacksquare & 0 \end{pmatrix}$$

{$t_1$, $h_1$, $t_5$, $h_6$, $h_5$, $t_6$}

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2} & h_1 & \\ t_1 & \frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} & \frac{(-1+T_1)T_6(1-}{-1+2T_1-T_1^2+T_6-4} \\ t_5 & 0 & \\ t_6 & \frac{(-1+T_6)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} & -\frac{T_6(1-2T_1+T_1^2-2T_6+5T_1T_6-3T_1^2T_6+2}{-1+2T_1-T_1^2+T_6-} \\ \blacksquare & 0 & \end{pmatrix}$$

{$t_1$, $h_1$, $h_6$, $t_{16}$, $t_5$, $h_{16}$, $h_5$, $t_6$}

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2} & h_1 & \\ t_1 & \frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} & \frac{(-1+T_1)(1-T_1-}{-1+2T_1-T_1^2+T_6-4T_1} \\ t_5 & 0 & \\ t_6 & 0 & \\ t_{16} & \frac{(-1+T_6)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3} & \frac{-1+2T_1-T_1^2+2T_6-5T_1T_6+3T_1^2T_6-2T_6^2+}{-1+2T_1-T_1^2+T_6-4T_1} \\ \blacksquare & 0 & \end{pmatrix}$$

$\{t_1, h_1, h_6, t_6\}$

| $\frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2}$ | $h_1$ | |
|---|---|---|
| $t_1$ | $\frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$ | $\frac{(-1+T_1)(1-T_1-\ldots}{-1+2T_1-T_1^2+T_6-4T_1\ldots}$ |
| $t_6$ | $\frac{(-1+T_6)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$ | $\frac{-1+2T_1-T_1^2+2T_6-5T_1T_6+3T_1^2T_6-2T_6^2+\ldots}{-1+2T_1-T_1^2+T_6-4T_1\ldots}$ |
| ■ | 0 | |

$\{t_1, h_1, t_{11}, h_6, h_{11}, t_6\}$

| $\frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}}$ | $h_1$ | |
|---|---|---|
| $t_1$ | $\frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | |
| $t_6$ | $\frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | |
| $t_{11}$ | $\frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{T_6(-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+2T_{11}-4\ldots}{\ldots}$ |
| ■ | 0 | |

$\{t_1, h_1, t_{11}, h_{11}, t_{15}, h_6, h_{15}, t_6\}$

| $\frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}}$ | $h_1$ | $h_6$ |
|---|---|---|
| $t_1$ | $\frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{(-1+T_1)(1-T_1+T_1T_6\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_6$ | $\frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{T_1^2T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_{11}$ | 0 | 0 |
| $t_{15}$ | $\frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $-\frac{(-1+T_6)T_{11}(-1+2T_1-T_1^2+T_{11}-2T_1\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| ■ | 0 | 0 |

$\{t_1, h_1, t_{11}, h_{11}, h_6, t_{10}, t_{15}, h_{10}, h_{15}, t_6\}$

| $\frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}}$ | $h_1$ | $h_6$ |
|---|---|---|
| $t_1$ | $\frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{(-1+T_1)(1-T_1+T_1T_6\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_6$ | $\frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{T_1^2T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_{10}$ | 0 | 0 |
| $t_{11}$ | 0 | 0 |
| $t_{15}$ | $\frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $-\frac{(-1+T_6)T_{11}(-1+2T_1-T_1^2+T_{11}-2T_1\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| ■ | 0 | 0 |

$\{t_1, h_1, t_{11}, h_{11}, h_6, t_6\}$

| $\frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}}$ | $h_1$ | $h_6$ |
|---|---|---|
| $t_1$ | $\frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{(-1+T_1)(1-T_1+T_1T_6\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_6$ | $\frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $\frac{T_1^2T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| $t_{11}$ | $\frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}$ | $-\frac{(-1+T_6)T_{11}(-1+2T_1-T_1^2+T_{11}-2T_1\ldots}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1\ldots}$ |
| ■ | 0 | 0 |

$\{t_1, t_{12}, h_{11}, h_1, t_{11}, h_{12}, h_6, t_6\}$

| 1 | $h_1$ | $h_6$ | $h_{11}$ | $h_{12}$ |
|---|---|---|---|---|
| $t_1$ | $\frac{(-1+T_1)(1-T_1+T_1 T_6)}{T_1^2 T_6}$ | $\frac{1-T_1+T_1 T_6}{T_1^2 T_6}$ | $0$ | $\frac{-1+T_1}{T_1 T_6}$ |
| $t_6$ | $\frac{(-1+T_1)(1-T_1+T_1 T_6)(-1+T_{12})}{T_1^2 T_6 T_{12}}$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1^2 T_6 T_{12}}$ | $1$ | $-\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ |
| $t_{11}$ | $1$ | $0$ | $0$ | $0$ |
| $t_{12}$ | $\frac{(-1+T_1)(-1+T_6)}{T_1}$ | $\frac{-1+T_6}{T_1}$ | $-((-1+T_6) T_{12})$ | $2-T_6-T_{12}+T_6 T_{12}$ |
| ■ | $0$ | $0$ | $0$ | $0$ |

$\{t_1, t_{12}, h_1, t_4, h_{11}, h_4, t_{11}, h_{12}, h_6, t_6\}$

| 1 | $h_1$ | $h_4$ | $h_6$ | $h_{11}$ | $h_{12}$ |
|---|---|---|---|---|---|
| $t_1$ | $\frac{1-T_1+T_1 T_6}{T_1 T_6}$ | $0$ | $0$ | $0$ | $\frac{-1+T_1}{T_1 T_6}$ |
| $t_4$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $0$ | $0$ | $1$ | $-\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ |
| $t_6$ | $0$ | $\frac{-1+T_4}{T_4}$ | $\frac{1}{T_4}$ | $0$ | $0$ |
| $t_{11}$ | $0$ | $1$ | $0$ | $0$ | $0$ |
| $t_{12}$ | $-1+T_6$ | $0$ | $0$ | $-((-1+T_6) T_{12})$ | $2-T_6-T_{12}+T_6 T_{12}$ |
| ■ | $0$ | $1$ | $0$ | $\frac{1}{T_4}$ | $0$ |

$\{t_1, t_{12}, h_1, h_{12}, h_6, t_6\}$

| 1 | $h_1$ | $h_6$ | $h_{12}$ |
|---|---|---|---|
| $t_1$ | $\frac{1-T_1+T_1 T_6}{T_1 T_6}$ | $0$ | $\frac{-1+T_1}{T_1 T_6}$ |
| $t_6$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $1$ | $-\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ |
| $t_{12}$ | $-1+T_6$ | $-((-1+T_6) T_{12})$ | $2-T_6-T_{12}+T_6 T_{12}$ |
| ■ | $0$ | $0$ | $0$ |

$\{t_1, t_{12}, h_1, h_6, t_9, h_{12}, h_9, t_6\}$

| 1 | $h_1$ | $h_6$ | $h_9$ | $h_{12}$ |
|---|---|---|---|---|
| $t_1$ | $\frac{1-T_1+T_1 T_6}{T_1 T_6}$ | $\frac{-1+T_1}{T_1 T_6}$ | $0$ | $0$ |
| $t_6$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $\frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $1$ | $-\frac{-1+T_{12}}{T_{12}}$ |
| $t_9$ | $0$ | $0$ | $0$ | $1$ |
| $t_{12}$ | $-1+T_6$ | $1$ | $-((-1+T_6) T_{12})$ | $(-1+T_6)(-1+T_{12})$ |
| ■ | $0$ | $0$ | $0$ | $0$ |

$\{t_1, t_{12}, h_1, h_6, h_{12}, t_{14}, t_9, h_{14}, h_9, t_6\}$

| 1 | $h_1$ | $h_6$ | $h_9$ | $h_{12}$ | $h_{14}$ |
|---|---|---|---|---|---|
| $t_1$ | $\frac{1-T_1+T_1 T_6}{T_1 T_6}$ | $\frac{-1+T_1}{T_1 T_6}$ | $0$ | $0$ | $0$ |
| $t_6$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $\frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $\frac{1}{T_{12}}$ | $0$ | $0$ |
| $t_9$ | $0$ | $0$ | $0$ | $T_{14}$ | $1-T_{14}$ |
| $t_{12}$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $t_{14}$ | $-1+T_6$ | $1$ | $1-T_6$ | $0$ | $0$ |
| ■ | $0$ | $0$ | $T_{14}$ | $0$ | $1$ |

$\{t_1, t_{12}, h_1, h_6, h_{12}, t_6\}$

| 1 | $h_1$ | $h_6$ | $h_{12}$ |
|---|---|---|---|
| $t_1$ | $\frac{1-T_1+T_1 T_6}{T_1 T_6}$ | $\frac{-1+T_1}{T_1 T_6}$ | $0$ |
| $t_6$ | $\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $\frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}}$ | $\frac{1}{T_{12}}$ |
| $t_{12}$ | $-1+T_6$ | $1$ | $1-T_6$ |
| ■ | $0$ | $0$ | $0$ |

$\{t_1, t_{12}, h_6, t_8, h_1, h_8, h_{12}, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & \frac{1}{T_{12}} \\ t_8 & 0 & \frac{-1+T_8}{T_8} & \frac{1}{T_8} & 0 \\ t_{12} & T_6 & 0 & 0 & 1-T_6 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_6, h_1, t_3, t_8, h_3, h_8, h_{12}, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_3 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 & 0 \\ t_3 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & 0 & \frac{1}{T_{12}} \\ t_6 & 0 & 0 & 1 & 0 & 0 \\ t_8 & 0 & 0 & \frac{-1+T_8}{T_8} & \frac{1}{T_8} & 0 \\ t_{12} & T_6 & 0 & 0 & 0 & 1-T_6 \\ \blacksquare & 0 & \frac{1}{T_8} & 0 & 1 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_6, h_1, h_{12}, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} \\ t_{12} & T_6 & 0 & 1-T_6 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_7, h_1, t_7, h_6, h_{12}, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & \frac{1}{T_{12}} \\ t_7 & 1 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & T_6 & 1-T_6 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}, t_{13}, h_6, h_{13}, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} & h_{13} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} & 0 & 0 \\ t_7 & 1 & 0 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & 0 & 0 & 1 \\ t_{13} & 0 & 0 & 0 & T_6 & 1-T_6 \\ \blacksquare & 0 & 1 & 0 & 0 & T_6 \end{pmatrix}$$

$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}\}$

$$\begin{pmatrix} 1 & h_1 & h_7 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 \\ t_7 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} \\ t_{12} & 1 & 0 & 0 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_1, t_2, h_7, h_2, t_7, h_{12}\}$

| 1 | $h_1$ | $h_2$ | $h_7$ | $h_{12}$ |
|---|---|---|---|---|
| $t_1$ | 1 | 0 | 0 | 0 |
| $t_2$ | $\frac{-1+T_{12}}{T_{12}}$ | 0 | 0 | $\frac{1}{T_{12}}$ |
| $t_7$ | 0 | $\frac{-1+T_2}{T_2}$ | $\frac{1}{T_2}$ | 0 |
| $t_{12}$ | 0 | 1 | 0 | 0 |
| ■ | $\frac{1}{T_{12}}$ | 1 | $\frac{1}{T_2}$ | 1 |

$\{t_1, t_{12}, h_1, h_{12}\}$

| 1 | $h_1$ | $h_{12}$ |
|---|---|---|
| $t_1$ | 1 | 0 |
| $t_{12}$ | $\frac{-1+T_{12}}{T_{12}}$ | $\frac{1}{T_{12}}$ |
| ■ | $\frac{1}{T_{12}}$ | 1 |

In[∘]:=
```
seq = Factor[
  Reverse@ComposeList[{Insert[RotateLeft[R̄_{2,7}, 1], 4], dm_{1,2→1}, Insert[RotateLeft[R_{6,13}, 1],
      7], dm_{12,13→12}, dm_{6,7→6}, Insert[R̄_{8,3}, 5], dm_{1,3→1}, dm_{6,8→6}, Insert[R_{14,9}, 6], dm_{12,14→12},
     dm_{6,9→6}, Insert[RotateLeft[R̄_{4,11}, 1], 4], dm_{1,4→1}, dm_{11,12→11}, Insert[R_{10,15}, 6],
     dm_{6,10→6}, dm_{11,15→11}, dm_{6,11→6}, Insert[R_{16,5}, 4], dm_{6,16→6}, dm_{1,5→1}, dm_{1,6→1}
    },
    R̄_{12,1}] /. T_ → T
 ];
```

In[∘]:=
```
MatrixForm@Transpose[
  Factor[{Det[#@A], (#@ω), Det[#@A] (#@ω), Det[#@A]^2 (#@ω), Det[#@A]^-2 (#@ω),
     Det[#@A] (#@ω)^2, Det[#@A] (#@ω)^-2}] & /@ seq]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}
1 & \frac{T^2\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)}{-1+3T-5T^2+5T^3-2T^4+T^5} & \frac{T^2\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\
-\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3} & \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} & \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\
-\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3} & \frac{-1+2T-5T^2+5T^3-3T^4+T^5}{T^2} & \frac{-1+2T-5T^2+5T^3-3T^4+T^5}{T^2} \\
-\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3} & \frac{\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)^2}{-1+3T-5T^2+5T^3-2T^4+T^5} & \frac{\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)^2}{-1+3T-5T^2+5T^3-2T^4+T^5} \\
-\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3} & \frac{\left(-1+3T-5T^2+5T^3-2T^4+T^5\right)^3}{T^8\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)^2} & \frac{\left(-1+3T-5T^2+5T^3-2T^4+T^5\right)^3}{T^8\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)^2} \\
\frac{\left(1-4T+8T^2-11T^3+8T^4-4T^5+T^6\right)^2}{T^6} & \frac{\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)\left(-1+3T-5T^2+5T^3-2T^4+T^5\right)}{T^6} & \frac{\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)\left(-1+3T-5T^2+5T^3-2T\right.}{T^6} \\
\frac{T^6}{\left(1-4T+8T^2-11T^3+8T^4-4T^5+T^6\right)^2} & \frac{T^{10}\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)}{\left(-1+3T-5T^2+5T^3-2T^4+T^5\right)^3} & \frac{T^{10}\left(-1+2T-5T^2+5T^3-3T^4+T^5\right)}{\left(-1+3T-5T^2+5T^3-2T^4+T^5\right)^3}
\end{pmatrix}$$