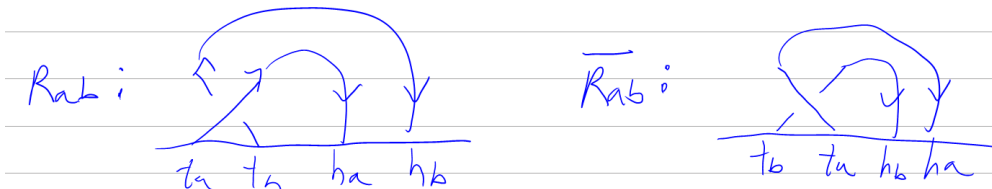


Pensieve header: Unitarity for Γ -calculus.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\MetaCalculi"];
Once[<< KnotTheory`]
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
 Read more at <http://katlas.org/wiki/KnotTheory>.



```
In[*]:= R_{a,b} := \Gamma @ < | \varsigma \to \{t_a, t_b, h_a, h_b\}, \omega \to 1, \sigma \to h_a + h_b T_a, \lambda \to \{t_a, t_b\} \cdot \begin{pmatrix} T_a & 1 - T_a \\ 0 & 1 \end{pmatrix} \cdot \{h_a, h_b\} | >;
\bar{R}_{a,b} := \Gamma @ < | \varsigma \to \{t_b, t_a, h_b, h_a\}, \omega \to 1, \sigma \to h_a + h_b T_a^{-1}, \lambda \to \{t_a, t_b\} \cdot \begin{pmatrix} T_a^{-1} & 1 - T_a^{-1} \\ 0 & 1 \end{pmatrix} \cdot \{h_a, h_b\} | >;
```

```
In[*]:= \bar{R}_{1,2}
```

Out[*]=

$$\Gamma \left[\left\langle \left| \varsigma \to \{t_2, t_1, h_2, h_1\}, \omega \to 1, \sigma \to h_1 + \frac{h_2}{T_1}, \lambda \to h_2 \left(t_2 + t_1 \left(1 - \frac{1}{T_1} \right) \right) + \frac{h_1 t_1}{T_1} \right| \right\rangle \right]$$

```
In[*]:= \Gamma[\alpha_][x_] := \alpha[x];
\Gamma[\alpha_][S] := Union@Cases[\alpha[\varsigma], t_a_ -> a, \infty];
\Gamma[\alpha_][n] := Length[\Gamma[\alpha_][S]];
\Gamma[\alpha_][\Sigma] := (\partial_{h_#} \alpha[\sigma]) & /@ \Gamma[\alpha_][S];
\Gamma[\alpha_][A] := Outer[Factor[\partial_{#1, #2} \Gamma[\alpha_][\lambda]] &, Cases[\alpha[\varsigma], t_], Cases[\alpha[\varsigma], h_]]];
```

```
In[*]:= Table[\{\gamma @ \varsigma, \gamma @ \omega, \gamma @ \sigma, \gamma @ \lambda, \gamma @ S, \gamma @ \Sigma, \gamma @ A // MatrixForm\}, \{\gamma, \{R_{1,2}, \bar{R}_{1,2}\}\} // Transpose // MatrixForm
```

Out[*]//MatrixForm=

$$\begin{pmatrix} \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ 1 \end{matrix} & \begin{matrix} \{t_2, t_1, h_2, h_1\} \\ 1 \end{matrix} \\ \begin{matrix} h_1 + h_2 T_1 \\ h_2 (t_2 + t_1 (1 - T_1)) + h_1 t_1 T_1 \end{matrix} & \begin{matrix} h_1 + \frac{h_2}{T_1} \\ h_2 \left(t_2 + t_1 \left(1 - \frac{1}{T_1} \right) \right) + \frac{h_1 t_1}{T_1} \end{matrix} \\ \{1, 2\} & \{1, 2\} \\ \{1, T_1\} & \left\{1, \frac{1}{T_1}\right\} \\ \begin{pmatrix} T_1 & 1 - T_1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -\frac{1+T_1}{T_1} & \frac{1}{T_1} \end{pmatrix} \end{pmatrix}$$

```
In[*]:= \GammaCollect[\gamma_I] := \Gamma @ < | \varsigma \to \gamma @ \varsigma, \omega \to Factor @ \gamma @ \omega, \sigma \to Expand @ \gamma @ \sigma,
\lambda \to Total[CoefficientRules[\gamma @ \lambda, \gamma @ \varsigma] /. (ps_ -> c_) -> Factor[c] Times @@ (\gamma @ \varsigma)^{ps}] | >
```

```
In[*]:= RCollect[ $\bar{R}_{1,2}$ ]
```

```
Out[*]=
```

$$\Gamma \left[\left\langle \left| \varsigma \rightarrow \{t_2, t_1, h_2, h_1\}, \omega \rightarrow 1, \sigma \rightarrow h_1 + \frac{h_2}{T_1}, \lambda \rightarrow h_2 t_2 + \frac{h_1 t_1}{T_1} + \frac{h_2 t_1 (-1 + T_1)}{T_1} \right| \right\rangle \right]$$

```
In[*]:= ConservativeQ[ $\gamma_{-I}$ ] := Simplify[Expand[ $\gamma @ \lambda /. h_ \rightarrow 1$ ] == Sum[ $t_a$ , { $a$ ,  $\gamma @ S$ }]]
```

```
In[*]:= ConservativeQ[ $R_{1,2}$ ]
```

```
Out[*]=
```

```
True
```

```
In[*]:=  $T^* \wedge := T^{-1}$ ;
```

```
 $\langle c_ , d_ \rangle_{c_} := Expand@Module [{ $e$ ,  $f$ },  
  Expand [ $c (d /. \{T \rightarrow T^*, t_{i_} \rightarrow t_{i_*}, h_{i_} \rightarrow h_{i_*}\})$ ] /.  
    { $t_{i_*} t_{i_} \rightarrow 0$ ,  $h_{i_*} h_{i_} \rightarrow T - T^*$ , ( $f : t | h$ ) $_{j_}^* (e : t | h)_{i_} \rightarrow$   
      If[Position[ $c$ ,  $e_i$ ][[1, 1]] < Position[ $f_j$ ][[1, 1]],  $T - 1$ ,  $1 - T^*$ ]}  
  ];$ 
```

```
UnitaryQ[ $\gamma_{-I}$ ] := Module [{ $vs$ },  
   $vs = Table[-t_i + \partial_{t_i} \gamma @ \lambda /. T_ \rightarrow T$ , { $i$ ,  $\gamma @ S$ }];  
  And@@Flatten@Table[Simplify[ $\langle vs[[i]$ ,  $vs[[j]] \rangle_{\gamma @ c} = 0$ ], { $i$ ,  $\gamma @ n$ }, { $j$ ,  $\gamma @ n$ }]  
  ];
```

```
In[*]:= UnitaryQ /@ { $R_{1,2}$ ,  $\bar{R}_{1,2}$ }
```

```
Out[*]=
```

```
{True, True}
```

```
In[*]:= IForm[ $\gamma_{-I}$ ] := Module [{ $M$ },  
   $M = \gamma[A]$  // Transpose;  
  PrependTo[ $M$ ,  $t_{\#}$  & /@  $\gamma[S]$ ];  
   $M = Join$   
    {Prepend[ $h_{\#}$  & /@  $\gamma[S]$ ,  $\gamma[\omega]$ ]},  
    Transpose[ $M$ ],  
    {Prepend[ $\gamma[\Sigma]$ , If[TrueQ[ConservativeQ@ $\gamma \wedge$  UnitaryQ@ $\gamma$ ],  $\blacksquare$ ,  $\blacksquare$ ]]}  
  ];  
  Column[{ $\gamma[\varsigma]$ , MatrixForm[ $M$ ]}]  
  ];  
IForm[ $else_$ ] :=  $else$  /.  $\gamma_{-I} \rightarrow$  IForm[ $\gamma$ ];  
Format[ $\gamma_{-I}$ , StandardForm] := IForm[ $\gamma$ ];
```

In[*]:= {R_{1,2}, $\bar{R}_{1,2}$ }

Out[*]=

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1 - T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix} \end{matrix} , \begin{matrix} \{t_2, t_1, h_2, h_1\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix} \end{matrix} \right\}$$

In[*]:= `dmi_j→k_[γ , T] := Module[{a, b, c, d, θ , ϵ , ϕ , ψ , Ξ , μ },`

$$\begin{pmatrix} a & b & \theta \\ c & d & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_i, h_i} \gamma @ \lambda & \partial_{t_i, h_j} \gamma @ \lambda & \partial_{t_i} \gamma @ \lambda \\ \partial_{t_j, h_i} \gamma @ \lambda & \partial_{t_j, h_j} \gamma @ \lambda & \partial_{t_j} \gamma @ \lambda \\ \partial_{h_i} \gamma @ \lambda & \partial_{h_j} \gamma @ \lambda & \gamma @ \lambda \end{pmatrix} /. (t | h)_{i|j} \rightarrow \theta;$$

`rCollect[Γ [<|`

`c → DeleteCases[γ @c, hi | tj] /. {ti → tk, hj → hk},`

`ω → ($\mu = 1 - c$) γ @ ω ,`

`σ → hk ($\partial_{h_i} \sigma$) ($\partial_{h_j} \sigma$) + (σ /. hi|j → θ),`

`λ → {tk, 1} . (b + a d / μ θ + a ϵ / μ) . {hk, 1}`

`|>] /. {Ti → Tk, Tj → Tk}`

`];`

In[*]:= {R_{1,2} // dm_{1,2→1}, R_{1,2} // dm_{2,1→1}, $\bar{R}_{1,2}$ // dm_{1,2→1}, $\bar{R}_{1,2}$ // dm_{2,1→1}}

Out[*]=

$$\left\{ \begin{matrix} \{t_1, h_1\} \\ \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} , \begin{matrix} \{t_1, h_1\} \\ \begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} , \begin{matrix} \{t_1, h_1\} \\ \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} , \begin{matrix} \{t_1, h_1\} \\ \begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} \right\}$$

In[]:= With[{n = 3},

$$\gamma\theta = \Gamma@<|\zeta \rightarrow \text{Flatten@Table}[\{t_a, h_a\}, \{a, n\}], \omega \rightarrow \omega, \sigma \rightarrow \sum_{a=1}^n h_a \sigma_a, \lambda \rightarrow \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{ab} |>$$

$\gamma\theta$ // dm_{1,2→1}

Out[]:=

$$\{t_1, h_1, t_2, h_2, t_3, h_3\}$$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 \\ t_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ t_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ t_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \\ \blacksquare & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

Out[]:=

$$\{t_1, h_1, t_3, h_3\}$$

$$\begin{pmatrix} -\omega & (-1 + \alpha_{2,1}) & h_1 & h_3 \\ t_1 & \frac{-\alpha_{1,2} + \alpha_{1,2} \alpha_{2,1} - \alpha_{1,1} \alpha_{2,2}}{-1 + \alpha_{2,1}} & \frac{-\alpha_{1,3} + \alpha_{1,3} \alpha_{2,1} - \alpha_{1,1} \alpha_{2,3}}{-1 + \alpha_{2,1}} \\ t_3 & \frac{-\alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}}{-1 + \alpha_{2,1}} & \frac{-\alpha_{2,3} \alpha_{3,1} - \alpha_{3,3} + \alpha_{2,1} \alpha_{3,3}}{-1 + \alpha_{2,1}} \\ \blacksquare & 0 & 0 \end{pmatrix}$$

In[]:= With[{n = 4},

$$\gamma\theta = \Gamma@<|\zeta \rightarrow \text{Flatten@Table}[\{t_a, h_a\}, \{a, n\}], \omega \rightarrow \omega, \sigma \rightarrow \sum_{a=1}^n h_a \sigma_a, \lambda \rightarrow \sum_{a=1}^n \sum_{b=1}^n t_a h_b \alpha_{ab} |>$$

Out[]:=

$$\{t_1, h_1, t_2, h_2, t_3, h_3, t_4, h_4\}$$

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_4 \\ t_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ t_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ t_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ t_4 & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \blacksquare & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

In[]:= lhs = $\gamma\theta$ // dm_{1,2→1} // dm_{1,3→1} // Γ Collect

Out[]:=

$$\{t_1, h_1, t_4, h_4\}$$

$$\begin{pmatrix} \omega & (1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}) & h_1 \\ t_1 & \frac{\alpha_{1,3} - \alpha_{1,3} \alpha_{2,1} + \alpha_{1,1} \alpha_{2,3} - \alpha_{1,3} \alpha_{2,2} \alpha_{3,1} + \alpha_{1,2} \alpha_{2,3} \alpha_{3,1} - \alpha_{1,3} \alpha_{3,2} + \alpha_{1,3} \alpha_{2,1} \alpha_{3,2} - \alpha_{1,1} \alpha_{2,3} \alpha_{3,2} + \alpha_{1,2} \alpha_{3,2}}{1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}} \\ t_4 & \frac{\alpha_{2,3} \alpha_{4,1} - \alpha_{2,3} \alpha_{3,2} \alpha_{4,1} + \alpha_{2,2} \alpha_{3,3} \alpha_{4,1} + \alpha_{2,3} \alpha_{3,1} \alpha_{4,2} + \alpha_{3,3} \alpha_{4,2} - \alpha_{2,1} \alpha_{3,3} \alpha_{4,2} + \alpha_{4,3} - \alpha_{2,1} \alpha_{4,3} - \alpha_{2,2} \alpha_{3,3} \alpha_{4,3}}{1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}} \\ \blacksquare & 0 \end{pmatrix}$$

In[*]:= rhs = γ_0 // dm_{2,3→2} // dm_{1,2→1} // Γ Collect

Out[*]=

$$\begin{pmatrix} \{t_1, h_1, t_4, h_4\} \\ \omega (1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}) \\ t_1 \\ t_4 \\ \blacksquare \end{pmatrix} \begin{matrix} h_1 \\ \frac{\alpha_{1,3} - \alpha_{1,3} \alpha_{2,1} + \alpha_{1,1} \alpha_{2,3} - \alpha_{1,3} \alpha_{2,2} \alpha_{3,1} + \alpha_{1,2} \alpha_{2,3} \alpha_{3,1} - \alpha_{1,3} \alpha_{3,2} + \alpha_{1,1} \alpha_{2,1} \alpha_{3,2} - \alpha_{1,1} \alpha_{2,3} \alpha_{3,2} + \alpha_{1,1} \alpha_{2,3} \alpha_{3,2}}{1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}} \\ \frac{\alpha_{2,3} \alpha_{4,1} - \alpha_{2,3} \alpha_{3,2} \alpha_{4,1} + \alpha_{2,2} \alpha_{3,3} \alpha_{4,1} + \alpha_{2,3} \alpha_{4,1} + \alpha_{2,3} \alpha_{3,1} \alpha_{4,2} + \alpha_{3,3} \alpha_{4,2} - \alpha_{2,1} \alpha_{3,3} \alpha_{4,2} + \alpha_{4,3} - \alpha_{2,1} \alpha_{4,3} - \alpha_{2,1} \alpha_{4,3}}{1 - \alpha_{2,1} - \alpha_{2,2} \alpha_{3,1} - \alpha_{3,2} + \alpha_{2,1} \alpha_{3,2}} \\ 0 \end{matrix}$$

In[*]:= lhs@ λ == rhs@ λ

Out[*]=

True

In[*]:= Γ /: RotateLeft[Γ [α], n___] := Γ @ReplacePart[α , Key@ ζ → RotateLeft[α @ ζ , n]]

In[*]:= {Table[RotateLeft[R_{1,2}, k], {k, 0, 3}], Table[RotateLeft[R̄_{1,2}, k], {k, 0, 3}]}

Out[*]=

$$\left\{ \left\{ \begin{pmatrix} \{t_1, t_2, h_1, h_2\} \\ 1 & h_1 & h_2 \\ t_1 & T_1 & 1 - T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} \{t_2, h_1, h_2, t_1\} \\ 1 & h_1 & h_2 \\ t_1 & 0 & 1 \\ t_2 & T_1 & 1 - T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} \{h_1, h_2, t_1, t_2\} \\ 1 & h_1 & h_2 \\ t_1 & T_1 & 1 - T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} \{h_2, t_1, t_2, h_1\} \\ 1 & h_1 & h_2 \\ t_1 & 1 - T_1 & T_1 \\ t_2 & 1 & 0 \\ \blacksquare & 1 & T_1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \{t_2, t_1, h_2, h_1\} \\ 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} \{t_1, h_2, h_1, t_2\} \\ 1 & h_1 & h_2 \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ t_2 & 1 & 0 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} \{h_2, h_1, t_2, t_1\} \\ 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} \{h_1, t_2, t_1, h_2\} \\ 1 & h_1 & h_2 \\ t_1 & 0 & 1 \\ t_2 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix} \right\} \right\}$$

In[*]:= Γ /: Insert[γ_1 _ Γ , γ_2 _ Γ , k_] := Γ @<|
 ζ → Flatten[Insert[γ_1 @ ζ , γ_2 @ ζ , k]],
 ω → γ_1 @ ω γ_2 @ ω ,
 σ → γ_1 @ σ + γ_2 @ σ ,
 λ → γ_1 @ λ + γ_2 @ λ
 |>
 Γ /: Insert[γ_2 _ Γ , k_] [γ_1 _ Γ] := Insert[γ_1 , γ_2 , k]

In[*]:= R_{1,2} // Insert[R_{3,4}, 2]

Out[*]=

$$\begin{pmatrix} \{t_1, t_3, t_4, h_3, h_4, t_2, h_1, h_2\} \\ 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & 0 & 0 & T_1 & 1 - T_1 \\ t_2 & T_3 & 1 - T_3 & 0 & 0 \\ t_3 & 0 & 1 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 \end{pmatrix}$$

```
In[*]:= ComposeList[{Insert[R3,4, 5], Insert[R5,6, 9], dm2,3→2, dm1,4→1, dm1,5→1, dm2,6→2, dm2,1→2},  
R1,2]
```

Out[]=

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1 - T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix} \end{matrix} \right\}, \left\{ \begin{matrix} \{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & T_1 & 1 - T_1 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1 - T_3 \\ t_4 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 \end{pmatrix} \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ t_1 & T_1 & 1 - T_1 & 0 & 0 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1 - T_3 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & 0 & T_5 & 1 - T_5 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 & 1 & T_5 \end{pmatrix} \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, t_4, h_2, h_4, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_4 & h_5 & h_6 \\ t_1 & T_1 & -((-1 + T_1) T_2) & (-1 + T_1) (-1 + T_2) & 0 & 0 \\ t_2 & 0 & T_2 & 1 - T_2 & 0 & 0 \\ t_4 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & T_5 & 1 - T_5 \\ t_6 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \{t_1, t_2, h_2, h_1, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_5 & h_6 \\ t_1 & -((-1 + T_1) T_2) & 1 - T_2 + T_1 T_2 & 0 & 0 \\ t_2 & T_2 & 1 - T_2 & 0 & 0 \\ t_5 & 0 & 0 & T_5 & 1 - T_5 \\ t_6 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \{t_1, t_2, h_2, t_6, h_1, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_6 \\ t_1 & -((-1 + T_1) T_2) & T_1 (1 - T_2 + T_1 T_2) & -((-1 + T_1) (1 - T_2 + T_1 T_2)) \\ t_2 & T_2 & -T_1 (-1 + T_2) & (-1 + T_1) (-1 + T_2) \\ t_6 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix} \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 (1 - T_2 + T_1 T_2) & -((-1 + T_1) (1 + T_1 T_2)) \\ t_2 & -T_1 (-1 + T_2) & 1 - T_1 + T_1 T_2 \\ \blacksquare & 0 & 0 \end{pmatrix} \end{matrix} \right\}, \left\{ \begin{matrix} \{t_2, h_2\} \\ \begin{pmatrix} T_2 (1 - T_2 + T_2^2) & h_2 \\ t_2 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} \right\}$$

In[*]:= {Insert[RotateLeft[$\bar{R}_{4,11}$, 1], 4], dm_{1,4→1}, dm_{11,12→11}, , Insert[R_{10,15}, 5]}

Out[*]=

$$\left\{ \text{Insert} \left[\begin{pmatrix} t_4 & h_{11} & h_4 & t_{11} \\ 1 & h_4 & h_{11} \\ t_4 & \frac{-1+T_4}{T_4} & \frac{1}{T_4} \\ t_{11} & 1 & \emptyset \\ \blacksquare & 1 & \frac{1}{T_4} \end{pmatrix}, 4 \right], dm_{1,4 \rightarrow 1}, dm_{11,12 \rightarrow 11}, \text{Null}, \text{Insert} \left[\begin{pmatrix} t_{10} & t_{15} & h_{10} & h_{15} \\ 1 & h_{10} & h_{15} \\ t_{10} & T_{10} & 1 - T_{10} \\ t_{15} & \emptyset & 1 \\ \blacksquare & 1 & T_{10} \end{pmatrix}, 5 \right] \right\}$$

In[*]:= Column@Reverse@

ComposeList[{Insert[RotateLeft[$\bar{R}_{2,7}$, 1], 4], dm_{1,2→1}, Insert[RotateLeft[R_{6,13}, 1], 7],
 dm_{12,13→12}, dm_{6,7→6}, Insert[$\bar{R}_{8,3}$, 5], dm_{1,3→1}, dm_{6,8→6}, Insert[R_{14,9}, 6], dm_{12,14→12},
 dm_{6,9→6}, Insert[RotateLeft[$\bar{R}_{4,11}$, 1], 4], dm_{1,4→1}, dm_{11,12→11}, Insert[R_{10,15}, 6],
 dm_{6,10→6}, dm_{11,15→11}, dm_{6,11→6}, Insert[R_{16,5}, 4], dm_{6,16→6}, dm_{1,5→1}, dm_{1,6→1}},
 $\bar{R}_{12,1}$]

Out[*]=

$$\begin{pmatrix} t_1 & h_1 \\ -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^2} & h_1 \\ t_1 & 1 \\ \blacksquare & \emptyset \end{pmatrix}$$

$$\begin{pmatrix} t_1 & h_6 & h_1 & t_6 \\ \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3}{T_1^2T_6^2} & h_1 \\ t_1 & \frac{(-1+T_1)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ t_6 & -\frac{T_6(1-2T_1+T_1^2-2T_6+5T_1T_6-3T_1^2T_6+2T_1^2T_6^2-6T_1T_6^2+2T_1^2T_6^2-2T_6^3+5T_1T_6^3-T_1^3T_6^3+T_6^4-3T_1T_6^4+T_1T_6^5)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ \blacksquare & \emptyset \end{pmatrix}$$

$$\begin{pmatrix} t_1 & h_1 & t_5 & h_6 & h_5 & t_6 \\ \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3}{T_1^2T_6^2} & h_1 \\ t_1 & \frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} & \frac{(-1+T_1)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ t_5 & \emptyset & \emptyset \\ t_6 & \frac{(-1+T_6)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} & -\frac{T_6(1-2T_1+T_1^2-2T_6+5T_1T_6-3T_1^2T_6+2T_1^2T_6^2-6T_1T_6^2+2T_1^2T_6^2-2T_6^3+5T_1T_6^3-T_1^3T_6^3+T_6^4-3T_1T_6^4+T_1T_6^5)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ \blacksquare & \emptyset & \emptyset \end{pmatrix}$$

$$\begin{pmatrix} t_1 & h_1 & h_6 & t_{16} & t_5 & h_{16} & h_5 & t_6 \\ \frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3}{T_1^2T_6^2} & h_1 \\ t_1 & \frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} & \frac{(-1+T_1)(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ t_5 & \emptyset & \emptyset \\ t_6 & \frac{(-1+T_6)T_6(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} & -\frac{1-2T_1-T_1^2+2T_6-5T_1T_6+3T_1^2T_6-2T_6^2+T_1T_6^2-2T_6^3+5T_1T_6^3-T_1^3T_6^3+T_6^4-3T_1T_6^4+T_1T_6^5}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^3T_6^3} \\ \blacksquare & \emptyset & \emptyset \end{pmatrix}$$

$\left\{ \begin{array}{l} \mathbf{t_1, h_1, h_6, t_6} \\ \frac{-1+2 T_1 - T_1^2 + T_6 - 4 T_1 T_6 + 3 T_1^2 T_6 + 2 T_1 T_6^2 - 2 T_1^2 T_6^2 + T_1^3 T_6^3}{T_1^2 T_6^2} \\ \mathbf{t_1} \\ \mathbf{t_6} \\ \blacksquare \end{array} \right.$	$\begin{array}{l} \mathbf{h_1} \\ \frac{T_6^2 (1 - T_1 + T_1 T_6)}{-1+2 T_1 - T_1^2 + T_6 - 4 T_1 T_6 + 3 T_1^2 T_6 + 2 T_1 T_6^2 - 2 T_1^2 T_6^2 + T_1^3 T_6^3} \\ \frac{(-1+T_6) T_6 (1 - T_1 - T_6 + 3 T_1 T_6 + T_6^2 - 2 T_1 T_6^2 + T_1 T_6^3)}{-1+2 T_1 - T_1^2 + T_6 - 4 T_1 T_6 + 3 T_1^2 T_6 + 2 T_1 T_6^2 - 2 T_1^2 T_6^2 + T_1^3 T_6^3} \\ \mathbf{0} \end{array}$	$\frac{(-1+T_1) (1 - T_1 - T_6)}{-1+2 T_1 - T_1^2 + T_6 - 4 T_1 T_6 + 3 T_1^2 T_6 + 2 T_1 T_6^2 - 2 T_1^2 T_6^2 + T_1^3 T_6^3}$
$\left\{ \begin{array}{l} \mathbf{t_1, h_1, t_{11}, h_6, h_{11}, t_6} \\ \frac{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}{T_1^2 T_6 T_{11}} \\ \mathbf{t_1} \\ \mathbf{t_6} \\ \mathbf{t_{11}} \\ \blacksquare \end{array} \right.$	$\begin{array}{l} \mathbf{h_1} \\ \frac{(1 - T_1 + T_1 T_6) T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{(1 - T_1 + T_1 T_6) (-1 + T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{T_1 (-1 + T_6) T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$	$\frac{T_6 (-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + 2 T_{11} - 4 T_1 T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}$
$\left\{ \begin{array}{l} \mathbf{t_1, h_1, t_{11}, h_{11}, t_{15}, h_6, h_{15}, t_6} \\ \frac{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}{T_1^2 T_6 T_{11}} \\ \mathbf{t_1} \\ \mathbf{t_6} \\ \mathbf{t_{11}} \\ \mathbf{t_{15}} \\ \blacksquare \end{array} \right.$	$\begin{array}{l} \mathbf{h_1} \\ \frac{(1 - T_1 + T_1 T_6) T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{(1 - T_1 + T_1 T_6) (-1 + T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \\ \frac{T_1 (-1 + T_6) T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$	$\begin{array}{l} \mathbf{h_6} \\ \frac{(-1+T_1) (1 - T_1 + T_1 T_6)}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{T_1^2 T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \\ \frac{(-1+T_6) T_{11} (-1+2 T_1 - T_1^2 + T_{11} - 2 T_1 T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$
$\left\{ \begin{array}{l} \mathbf{t_1, h_1, t_{11}, h_{11}, h_6, t_{10}, t_{15}, h_{10}, h_{15}, t_6} \\ \frac{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}{T_1^2 T_6 T_{11}} \\ \mathbf{t_1} \\ \mathbf{t_6} \\ \mathbf{t_{10}} \\ \mathbf{t_{11}} \\ \mathbf{t_{15}} \\ \blacksquare \end{array} \right.$	$\begin{array}{l} \mathbf{h_1} \\ \frac{(1 - T_1 + T_1 T_6) T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{(1 - T_1 + T_1 T_6) (-1 + T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{T_1 (-1 + T_6) T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$	$\begin{array}{l} \mathbf{h_6} \\ \frac{(-1+T_1) (1 - T_1 + T_1 T_6)}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{T_1^2 T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{(-1+T_6) T_{11} (-1+2 T_1 - T_1^2 + T_{11} - 2 T_1 T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$
$\left\{ \begin{array}{l} \mathbf{t_1, h_1, t_{11}, h_{11}, h_6, t_6} \\ \frac{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}{T_1^2 T_6 T_{11}} \\ \mathbf{t_1} \\ \mathbf{t_6} \\ \mathbf{t_{11}} \\ \blacksquare \end{array} \right.$	$\begin{array}{l} \mathbf{h_1} \\ \frac{(1 - T_1 + T_1 T_6) T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{(1 - T_1 + T_1 T_6) (-1 + T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{T_1 (-1 + T_6) T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$	$\begin{array}{l} \mathbf{h_6} \\ \frac{(-1+T_1) (1 - T_1 + T_1 T_6)}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{T_1^2 T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \frac{(-1+T_6) T_{11} (-1+2 T_1 - T_1^2 + T_{11} - 2 T_1 T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} \\ \mathbf{0} \end{array}$

$$\left\{ t_1, t_{12}, h_{11}, h_1, t_{11}, h_{12}, h_6, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{11} & h_{12} \\ t_1 & \frac{(-1+T_1)(1-T_1+T_1 T_6)}{T_1^2 T_6} & \frac{1-T_1+T_1 T_6}{T_1^2 T_6} & 0 & \frac{-1+T_1}{T_1 T_6} \\ t_6 & \frac{(-1+T_1)(1-T_1+T_1 T_6)(-1+T_{12})}{T_1^2 T_6 T_{12}} & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1^2 T_6 T_{12}} & 1 & -\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} \\ t_{11} & 1 & 0 & 0 & 0 \\ t_{12} & \frac{(-1+T_1)(-1+T_6)}{T_1} & \frac{-1+T_6}{T_1} & -((-1 + T_6) T_{12}) & 2 - T_6 - T_{12} + T_6 T_{12} \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{ t_1, t_{12}, h_1, t_4, h_{11}, h_4, t_{11}, h_{12}, h_6, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_4 & h_6 & h_{11} & h_{12} \\ t_1 & \frac{1-T_1+T_1 T_6}{T_1 T_6} & 0 & 0 & 0 & \frac{-1+T_1}{T_1 T_6} \\ t_4 & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} & 0 & 0 & 1 & -\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} \\ t_6 & 0 & \frac{-1+T_4}{T_4} & \frac{1}{T_4} & 0 & 0 \\ t_{11} & 0 & 1 & 0 & 0 & 0 \\ t_{12} & -1 + T_6 & 0 & 0 & -((-1 + T_6) T_{12}) & 2 - T_6 - T_{12} + T_6 T_{12} \\ \blacksquare & 0 & 1 & 0 & \frac{1}{T_4} & 0 \end{pmatrix}$$

$$\left\{ t_1, t_{12}, h_1, h_{12}, h_6, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{1-T_1+T_1 T_6}{T_1 T_6} & 0 & \frac{-1+T_1}{T_1 T_6} \\ t_6 & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} & 1 & -\frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} \\ t_{12} & -1 + T_6 & -((-1 + T_6) T_{12}) & 2 - T_6 - T_{12} + T_6 T_{12} \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{ t_1, t_{12}, h_1, h_6, t_9, h_{12}, h_9, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_9 & h_{12} \\ t_1 & \frac{1-T_1+T_1 T_6}{T_1 T_6} & \frac{-1+T_1}{T_1 T_6} & 0 & 0 \\ t_6 & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}} & 1 & -\frac{-1+T_{12}}{T_{12}} \\ t_9 & 0 & 0 & 0 & 1 \\ t_{12} & -1 + T_6 & 1 & -((-1 + T_6) T_{12}) & (-1 + T_6)(-1 + T_{12}) \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{ t_1, t_{12}, h_1, h_6, h_{12}, t_{14}, t_9, h_{14}, h_9, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_9 & h_{12} & h_{14} \\ t_1 & \frac{1-T_1+T_1 T_6}{T_1 T_6} & \frac{-1+T_1}{T_1 T_6} & 0 & 0 & 0 \\ t_6 & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}} & \frac{1}{T_{12}} & 0 & 0 \\ t_9 & 0 & 0 & 0 & T_{14} & 1 - T_{14} \\ t_{12} & 0 & 0 & 0 & 0 & 1 \\ t_{14} & -1 + T_6 & 1 & 1 - T_6 & 0 & 0 \\ \blacksquare & 0 & 0 & T_{14} & 0 & 1 \end{pmatrix}$$

$$\left\{ t_1, t_{12}, h_1, h_6, h_{12}, t_6 \right\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{1-T_1+T_1 T_6}{T_1 T_6} & \frac{-1+T_1}{T_1 T_6} & 0 \\ t_6 & \frac{(1-T_1+T_1 T_6)(-1+T_{12})}{T_1 T_6 T_{12}} & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_6 T_{12}} & \frac{1}{T_{12}} \\ t_{12} & -1 + T_6 & 1 & 1 - T_6 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

$$\{t_1, t_{12}, h_6, t_8, h_1, h_8, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & \frac{1}{T_{12}} \\ t_8 & 0 & \frac{-1+T_8}{T_8} & \frac{1}{T_8} & 0 \\ t_{12} & T_6 & 0 & 0 & 1 - T_6 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\{t_1, t_{12}, h_6, h_1, t_3, t_8, h_3, h_8, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_3 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 & 0 \\ t_3 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & 0 & \frac{1}{T_{12}} \\ t_6 & 0 & 0 & 1 & 0 & 0 \\ t_8 & 0 & 0 & \frac{-1+T_8}{T_8} & \frac{1}{T_8} & 0 \\ t_{12} & T_6 & 0 & 0 & 0 & 1 - T_6 \\ \blacksquare & 0 & \frac{1}{T_8} & 0 & 1 & 0 \end{pmatrix}$$

$$\{t_1, t_{12}, h_6, h_1, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} \\ t_{12} & T_6 & 0 & 1 - T_6 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

$$\{t_1, t_{12}, h_7, h_1, t_7, h_6, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & 0 & \frac{1}{T_{12}} \\ t_7 & 1 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & T_6 & 1 - T_6 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}, t_{13}, h_6, h_{13}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} & h_{13} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 & 0 & 0 \\ t_6 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} & 0 & 0 \\ t_7 & 1 & 0 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & 0 & 0 & 1 \\ t_{13} & 0 & 0 & 0 & T_6 & 1 - T_6 \\ \blacksquare & 0 & 1 & 0 & 0 & T_6 \end{pmatrix}$$

$$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}\}$$

$$\begin{pmatrix} 1 & h_1 & h_7 & h_{12} \\ t_1 & \frac{-1+T_1}{T_1} & \frac{1}{T_1} & 0 \\ t_7 & \frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}} & \frac{-1+T_{12}}{T_1 T_{12}} & \frac{1}{T_{12}} \\ t_{12} & 1 & 0 & 0 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}$$

{t₁, t₁₂, h₁, t₂, h₇, h₂, t₇, h₁₂}

$$\begin{pmatrix} 1 & h_1 & h_2 & h_7 & h_{12} \\ t_1 & 1 & 0 & 0 & 0 \\ t_2 & \frac{-1+T_{12}}{T_{12}} & 0 & 0 & \frac{1}{T_{12}} \\ t_7 & 0 & \frac{-1+T_2}{T_2} & \frac{1}{T_2} & 0 \\ t_{12} & 0 & 1 & 0 & 0 \\ \blacksquare & \frac{1}{T_{12}} & 1 & \frac{1}{T_2} & 1 \end{pmatrix}$$

{t₁, t₁₂, h₁, h₁₂}

$$\begin{pmatrix} 1 & h_1 & h_{12} \\ t_1 & 1 & 0 \\ t_{12} & \frac{-1+T_{12}}{T_{12}} & \frac{1}{T_{12}} \\ \blacksquare & \frac{1}{T_{12}} & 1 \end{pmatrix}$$

```
In[*]:= γ0 = R̄12,1 // Insert[RotateLeft[R̄2,7, 1], 4] // dm1,2→1 // Insert[RotateLeft[R6,13, 1], 7] //
dm12,13→12 // dm6,7→6 // Insert[R̄8,3, 5] // dm1,3→1 // dm6,8→6 // Insert[R14,9, 6] //
dm12,14→12 // dm6,9→6 // Insert[RotateLeft[R̄4,11, 1], 4] // dm1,4→1 // dm11,12→11
```

Out[*]=

$$\begin{pmatrix} t_1 & h_1 & t_{11} & h_{11} & h_6 & t_6 \\ \frac{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}}{T_1^2 T_6 T_{11}} & h_1 & & & h_6 & \\ t_1 & \frac{(1-T_1+T_1 T_6) T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} & & & \frac{(-1+T_1) (1-T_1+T_1 T_6)}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11}} & \\ t_6 & \frac{(1-T_1+T_1 T_6) (-1+T_{11})}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} & & & \frac{T_1^2 T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11}} & \\ t_{11} & \frac{T_1 (-1+T_6) T_6 T_{11}}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11} + T_1^2 T_{11} + T_1 T_6 T_{11}} & & & \frac{(-1+T_6) T_{11} (-1+2 T_1 - T_1^2 + T_{11} - 2 T_1)}{-1+2 T_1 - T_1^2 - T_1 T_6 + T_1^2 T_6 + T_{11} - 2 T_1 T_{11}} & \\ \blacksquare & 0 & & & 0 & \end{pmatrix}$$

```
In[*]:= Module[{vs},
vs = Table[-τ + ∂τγ0@λ /. T_ -> T, {τ, Cases[γ0@c, t_]}];
MatrixForm@Table[Simplify[⟨vs[[i]], vs[[j]]⟩γ0@c, {i, γ0@n}, {j, γ0@n}]
]
```

Out[*]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
In[*]:= seq = Factor[
Reverse@ComposeList[{{Insert[RotateLeft[R̄2,7, 1], 4], dm1,2→1, Insert[RotateLeft[R6,13, 1], 7], dm12,13→12, dm6,7→6, Insert[R̄8,3, 5], dm1,3→1, dm6,8→6, Insert[R14,9, 6], dm12,14→12, dm6,9→6, Insert[RotateLeft[R̄4,11, 1], 4], dm1,4→1, dm11,12→11, Insert[R10,15, 6], dm6,10→6, dm11,15→11, dm6,11→6, Insert[R16,5, 4], dm6,16→6, dm1,5→1, dm1,6→1},
R̄12,1] /. T_ -> T
]
```

Out[*]=

$$\{t_1, h_1\} \left\{ \begin{pmatrix} -\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^3} h_1 \\ t_1 \\ \blacksquare \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

<http://drorbn.net/AcademicPensieve/Projects/MetaCalculi/Archive/#MathematicaNotebooks>

$$\left(\begin{array}{c} \{t_1, h_6, h_1, t_6\} \\ \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\ t_1 \\ t_6 \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{(-1+T)T(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ -\frac{T(1-4T+8T^2-11T^3+8T^4-4T^5+T^6)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \end{array} \begin{array}{c} h_6 \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \frac{(-1+T)(1-T+T^2)^3}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \end{array} \right), \left(\begin{array}{c} \{t_1, h_1, t_5, h_6, h_5, t_6\} \\ \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\ t_1 \\ t_5 \\ t_6 \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{(-1+T)T(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \frac{(-1+T)T(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \end{array} \begin{array}{c} h_6 \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ T \end{array} \begin{array}{c} h_5 \\ \frac{(-1+T)(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \\ T \end{array} \begin{array}{c} h_6 \\ \emptyset \\ T \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_{16} \\ \emptyset \\ 1-T \\ 1 \\ \emptyset \\ 1 \end{array} \right), \left(\begin{array}{c} \{t_1, h_1, h_6, t_6\} \\ \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\ t_1 \\ t_6 \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{T^2(1-T+T^2)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_6 \\ \frac{(-1+T)(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \end{array} \right),$$

$$\left(\begin{array}{c} \{t_1, h_1, h_6, t_{16}, t_5, h_{16}, h_5, t_6\} \\ \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\ t_1 \\ t_5 \\ t_6 \\ t_{16} \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{T^2(1-T+T^2)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_5 \\ \frac{(-1+T)(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \\ T \end{array} \begin{array}{c} h_6 \\ \emptyset \\ T \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_{16} \\ \emptyset \\ 1-T \\ 1 \\ \emptyset \\ 1 \end{array} \right),$$

$$\left(\begin{array}{c} \{t_1, h_1, h_6, t_6\} \\ \frac{-1+3T-5T^2+5T^3-2T^4+T^5}{T^4} \\ t_1 \\ t_6 \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{T^2(1-T+T^2)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_6 \\ \frac{(-1+T)(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \end{array} \right),$$

$$\left(\begin{array}{c} \{t_1, h_1, t_{11}, h_6, h_{11}, t_6\} \\ \frac{-1+3T-4T^2+3T^3}{T^4} \\ t_1 \\ t_6 \\ t_{11} \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{T(1-T+T^2)}{-1+3T-4T^2+3T^3} \\ \frac{(-1+T)(1-T+T^2)}{-1+3T-4T^2+3T^3} \\ \frac{(-1+T)T^3}{-1+3T-4T^2+3T^3} \\ \emptyset \end{array} \begin{array}{c} h_6 \\ -\frac{(-1+T)T(-1+2T-3T^2+T^3)}{-1+3T-4T^2+3T^3} \\ -\frac{(-1+T)T^2(1-T+T^2)}{-1+3T-4T^2+3T^3} \\ \frac{T(-1+4T-8T^2+10T^3-6T^4+2T^5)}{-1+3T-4T^2+3T^3} \\ \emptyset \end{array} \begin{array}{c} h_{11} \\ \frac{(-1+T)(1-2T+4T^2-3T^3+T^4)}{-1+3T-4T^2+3T^3} \\ \frac{T(1-3T+4T^2-2T^3+T^4)}{-1+3T-4T^2+3T^3} \\ -\frac{(-1+T)(-1+3T-5T^2+7T^3-4T^4+2T^5)}{-1+3T-4T^2+3T^3} \\ \emptyset \end{array} \right), \left(\begin{array}{c} \{t_1, h_1, t_{11}, h_1\} \\ \frac{-1+3T-4T^2+3T^3}{T^4} \\ t_1 \\ t_6 \\ t_{11} \\ t_{15} \\ \blacksquare \end{array} \begin{array}{c} h_1 \\ \frac{(-1+T)T(1-2T+4T^2-2T^3+T^4)}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_6 \\ -\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{-1+3T-5T^2+5T^3-2T^4+T^5} \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{array} \begin{array}{c} h_{11} \\ \frac{(-1+T)(1-2T+4T^2-3T^3+T^4)}{-1+3T-4T^2+3T^3} \\ \frac{T(1-3T+4T^2-2T^3+T^4)}{-1+3T-4T^2+3T^3} \\ -\frac{(-1+T)(-1+3T-5T^2+7T^3-4T^4+2T^5)}{-1+3T-4T^2+3T^3} \\ \emptyset \end{array} \right),$$

$$\{t_1, h_1, t_{11}, h_{11}, h_6, t_{10}, t_{15}, h_{10}, h_{15}, t_6\}$$

$$\begin{pmatrix} \frac{-1+3T-4T^2+3T^3}{T^4} & h_1 & h_6 & h_{10} & h_{11} & h_{15} \\ t_1 & \frac{T(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{(-1+T)T(1-T+T^2)}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)(-1+2T-3T^2+T^3)}{-1+3T-4T^2+3T^3} & 0 & 0 \\ t_6 & \frac{(-1+T)(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{T^4}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)T(1-T+T^2)}{-1+3T-4T^2+3T^3} & 0 & 0 \\ t_{10} & 0 & 0 & 0 & T & 1-T \\ t_{11} & 0 & 0 & 0 & 0 & 1 \\ t_{15} & \frac{(-1+T)T^3}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)T(-1+2T)(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{-1+4T-8T^2+10T^3-6T^4+2T^5}{-1+3T-4T^2+3T^3} & 0 & 0 \\ \blacksquare & 0 & 0 & 1 & 0 & T \end{pmatrix},$$

$$\{t_1, h_1, t_{11}, h_{11}, h_6, t_6\}$$

$$\begin{pmatrix} \frac{-1+3T-4T^2+3T^3}{T^4} & h_1 & h_6 & h_{11} \\ t_1 & \frac{T(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{(-1+T)T(1-T+T^2)}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)(-1+2T-3T^2+T^3)}{-1+3T-4T^2+3T^3} \\ t_6 & \frac{(-1+T)(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{T^4}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)T(1-T+T^2)}{-1+3T-4T^2+3T^3} \\ t_{11} & \frac{(-1+T)T^3}{-1+3T-4T^2+3T^3} & -\frac{(-1+T)T(-1+2T)(1-T+T^2)}{-1+3T-4T^2+3T^3} & \frac{-1+4T-8T^2+10T^3-6T^4+2T^5}{-1+3T-4T^2+3T^3} \\ \blacksquare & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_{11}, h_1, t_{11}, h_{12}, h_6, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{11} & h_{12} \\ t_1 & \frac{(-1+T)(1-T+T^2)}{T^3} & \frac{1-T+T^2}{T^3} & 0 & \frac{-1+T}{T^2} \\ t_6 & \frac{(-1+T)^2(1-T+T^2)}{T^4} & \frac{(-1+T)(1-T+T^2)}{T^4} & 1 & -\frac{(-1+T)(1-T+T^2)}{T^3} \\ t_{11} & 1 & 0 & 0 & 0 \\ t_{12} & \frac{(-1+T)^2}{T} & \frac{-1+T}{T} & -((-1+T)T) & 2-2T+T^2 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_1, t_4, h_{11}, h_4, t_{11}, h_{12}, h_6, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_4 & h_6 & h_{11} & h_{12} \\ t_1 & \frac{1-T+T^2}{T^2} & 0 & 0 & 0 & \frac{-1+T}{T^2} \\ t_4 & \frac{(-1+T)(1-T+T^2)}{T^3} & 0 & 0 & 1 & -\frac{(-1+T)(1-T+T^2)}{T^3} \\ t_6 & 0 & \frac{-1+T}{T} & \frac{1}{T} & 0 & 0 \\ t_{11} & 0 & 1 & 0 & 0 & 0 \\ t_{12} & -1+T & 0 & 0 & -((-1+T)T) & 2-2T+T^2 \\ \blacksquare & 0 & 1 & 0 & \frac{1}{T} & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_1, h_{12}, h_6, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{1-T+T^2}{T^2} & 0 & \frac{-1+T}{T^2} \\ t_6 & \frac{(-1+T)(1-T+T^2)}{T^3} & 1 & -\frac{(-1+T)(1-T+T^2)}{T^3} \\ t_{12} & -1+T & -((-1+T)T) & 2-2T+T^2 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_1, h_6, t_9, h_{12}, h_9, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_9 & h_{12} \\ t_1 & \frac{1-T+T^2}{T^2} & \frac{-1+T}{T^2} & 0 & 0 \\ t_6 & \frac{(-1+T)(1-T+T^2)}{T^3} & \frac{(-1+T)^2}{T^3} & 1 & -\frac{-1+T}{T} \\ t_9 & 0 & 0 & 0 & 1 \\ t_{12} & -1+T & 1 & -((-1+T)T) & (-1+T)^2 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_1, h_6, h_{12}, t_{14}, t_9, h_{14}, h_9, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_9 & h_{12} & h_{14} \\ t_1 & \frac{1-T+T^2}{T^2} & \frac{-1+T}{T^2} & 0 & 0 & 0 \\ t_6 & \frac{(-1+T)(1-T+T^2)}{T^3} & \frac{(-1+T)^2}{T^3} & \frac{1}{T} & 0 & 0 \\ t_9 & 0 & 0 & 0 & T & 1-T \\ t_{12} & 0 & 0 & 0 & 0 & 1 \\ t_{14} & -1+T & 1 & 1-T & 0 & 0 \\ \blacksquare & 0 & 0 & T & 0 & 1 \end{pmatrix}, \quad \{t_1, t_{12}, h_1, h_6, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{1-T+T^2}{T^2} & \frac{-1+T}{T^2} & 0 \\ t_6 & \frac{(-1+T)(1-T+T^2)}{T^3} & \frac{(-1+T)^2}{T^3} & \frac{1}{T} \\ t_{12} & -1+T & 1 & 1-T \\ \blacksquare & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_6, t_8, h_1, h_8, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 & 0 \\ t_6 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & 0 & \frac{1}{T} \\ t_8 & 0 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_{12} & T & 0 & 0 & 1-T \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \{t_1, t_{12}, h_6, h_1, t_3, t_8, h_3, h_8, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_3 & h_6 & h_8 & h_{12} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 & 0 & 0 \\ t_3 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & 0 & 0 & \frac{1}{T} \\ t_6 & 0 & 0 & 1 & 0 & 0 \\ t_8 & 0 & 0 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_{12} & T & 0 & 0 & 0 & 1-T \\ \blacksquare & 0 & \frac{1}{T} & 0 & 1 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_6, h_1, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{12} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_6 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & \frac{1}{T} \\ t_{12} & T & 0 & 1-T \\ \blacksquare & 0 & 0 & 0 \end{pmatrix}, \quad \{t_1, t_{12}, h_7, h_1, t_7, h_6, h_{12}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 & 0 \\ t_6 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & 0 & \frac{1}{T} \\ t_7 & 1 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & T & 1-T \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}, t_{13}, h_6, h_{13}, t_6\}$$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_7 & h_{12} & h_{13} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 & 0 & 0 \\ t_6 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & \frac{1}{T} & 0 & 0 \\ t_7 & 1 & 0 & 0 & 0 & 0 \\ t_{12} & 0 & 0 & 0 & 0 & 1 \\ t_{13} & 0 & 0 & 0 & T & 1-T \\ \blacksquare & 0 & 1 & 0 & 0 & T \end{pmatrix}, \quad \{t_1, t_{12}, h_7, h_1, t_7, h_{12}\}$$

$$\begin{pmatrix} 1 & h_1 & h_7 & h_{12} \\ t_1 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_7 & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T^2} & \frac{1}{T} \\ t_{12} & 1 & 0 & 0 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix},$$

$$\{t_1, t_{12}, h_1, t_2, h_7, h_2, t_7, h_{12}\}$$

$$\begin{pmatrix} 1 & h_1 & h_2 & h_7 & h_{12} \\ t_1 & 1 & 0 & 0 & 0 \\ t_2 & \frac{-1+T}{T} & 0 & 0 & \frac{1}{T} \\ t_7 & 0 & \frac{-1+T}{T} & \frac{1}{T} & 0 \\ t_{12} & 0 & 1 & 0 & 0 \\ \blacksquare & \frac{1}{T} & 1 & \frac{1}{T} & 1 \end{pmatrix}, \quad \{t_1, t_{12}, h_1, h_{12}\}$$

$$\begin{pmatrix} 1 & h_1 & h_{12} \\ t_1 & 1 & 0 \\ t_{12} & \frac{-1+T}{T} & \frac{1}{T} \\ \blacksquare & \frac{1}{T} & 1 \end{pmatrix}$$

In[*]:= Factor[Det[#@A]] & /@ seq

Out[*]=

$$\left\{ 1, \frac{T^2(-1+2T-5T^2+5T^3-3T^4+T^5)}{-1+3T-5T^2+5T^3-2T^4+T^5}, \frac{T^2(-1+2T-5T^2+5T^3-3T^4+T^5)}{-1+3T-5T^2+5T^3-2T^4+T^5}, \right.$$

$$-\frac{T^2(-1+2T-5T^2+5T^3-3T^4+T^5)}{-1+3T-5T^2+5T^3-2T^4+T^5}, -\frac{T(-1+2T-5T^2+5T^3-3T^4+T^5)}{-1+3T-5T^2+5T^3-2T^4+T^5},$$

$$-\frac{T^3}{-1+3T-4T^2+3T^3}, -\frac{T^3}{-1+3T-4T^2+3T^3}, -\frac{T^3}{-1+3T-4T^2+3T^3},$$

$$\left. \frac{T^2}{-1+3T-4T^2+3T^3}, \frac{1}{T^2}, -\frac{1}{T^2}, \frac{1}{T}, \frac{1}{T}, -\frac{1}{T}, -\frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T}, -\frac{1}{T}, -\frac{1}{T}, \frac{1}{T^2}, -\frac{1}{T^2}, \frac{1}{T} \right\}$$


```
In[*]:= MatrixForm@Transpose[
  Factor[{Det[#@A], (#@w), Det[#@A] (#@w), Det[#@A]^2 (#@w), Det[#@A]^-2 (#@w),
    Det[#@A] (#@w)^2, Det[#@A] (#@w)^-2}] & /@ seq]
```

Out[*]//MatrixForm=

$$\left(\begin{array}{ccc} 1 & \frac{T^2 (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)}{-1+3 T-5 T^2+5 T^3-2 T^4+T^5} & \frac{T^2 (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)}{-1+3 T-5 T^2+5 T^3-2 T^4+T^5} \\ -\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^3} & \frac{-1+3 T-5 T^2+5 T^3-2 T^4+T^5}{T^4} & \frac{-1+3 T-5 T^2+5 T^3-2 T^4+T^5}{T^4} \\ -\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^3} & \frac{-1+2 T-5 T^2+5 T^3-3 T^4+T^5}{T^2} & \frac{-1+2 T-5 T^2+5 T^3-3 T^4+T^5}{T^2} \\ -\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^3} & \frac{(-1+2 T-5 T^2+5 T^3-3 T^4+T^5)^2}{-1+3 T-5 T^2+5 T^3-2 T^4+T^5} & \frac{(-1+2 T-5 T^2+5 T^3-3 T^4+T^5)^2}{-1+3 T-5 T^2+5 T^3-2 T^4+T^5} \\ -\frac{1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6}{T^3} & \frac{(-1+3 T-5 T^2+5 T^3-2 T^4+T^5)^3}{T^8 (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)^2} & \frac{(-1+3 T-5 T^2+5 T^3-2 T^4+T^5)^3}{T^8 (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)^2} \\ \frac{(1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6)^2}{T^6} & \frac{(-1+2 T-5 T^2+5 T^3-3 T^4+T^5) (-1+3 T-5 T^2+5 T^3-2 T^4+T^5)}{T^6} & \frac{(-1+2 T-5 T^2+5 T^3-3 T^4+T^5) (-1+3 T-5 T^2+5 T^3-2 T^4+T^5)}{T^6} \\ \frac{(1-4 T+8 T^2-11 T^3+8 T^4-4 T^5+T^6)^2}{T^6} & \frac{T^{10} (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)}{(-1+3 T-5 T^2+5 T^3-2 T^4+T^5)^3} & \frac{T^{10} (-1+2 T-5 T^2+5 T^3-3 T^4+T^5)}{(-1+3 T-5 T^2+5 T^3-2 T^4+T^5)^3} \end{array} \right)$$

```
In[*]:= Factor[
  (Det[#@A] (#@w)^2) / (Det[#@A] (#@w)^2 /. T -> 1 / T) ] & /@ seq
```

Out[*]=

$$\left\{ 1, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^4}, -\frac{-1+3 T-4 T^2+3 T^3}{T^7 (-3+4 T-3 T^2+T^3)}, -\frac{-1+3 T-4 T^2+3 T^3}{T^7 (-3+4 T-3 T^2+T^3)}, -\frac{-1+3 T-4 T^2+3 T^3}{T^7 (-3+4 T-3 T^2+T^3)}, \right. \\ \left. -\frac{-1+3 T-4 T^2+3 T^3}{T^9 (-3+4 T-3 T^2+T^3)}, \frac{1}{T^4}, \frac{1}{T^4}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^4}, \frac{1}{T^4}, \frac{1}{T^4}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^4}, \frac{1}{T^4}, \frac{1}{T^2} \right\}$$

```
In[*]:= Factor[
  (Det[#@A] (#@w)) / (#@w /. T -> 1 / T) ] & /@ seq
```

Out[*]=

$$\left\{ 1, -\frac{1}{T}, -\frac{1}{T}, \frac{1}{T}, \frac{1}{T^2}, \frac{1}{T^2 (-3+4 T-3 T^2+T^3)}, \frac{1}{T^2 (-3+4 T-3 T^2+T^3)}, -\frac{1}{T^2 (-3+4 T-3 T^2+T^3)}, \right. \\ \left. -\frac{1}{T^3 (-3+4 T-3 T^2+T^3)}, \frac{1}{T^2}, -\frac{1}{T^2}, \frac{1}{T}, \frac{1}{T}, -\frac{1}{T}, -\frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T^2}, \frac{1}{T}, -\frac{1}{T}, -\frac{1}{T}, \frac{1}{T^2}, -\frac{1}{T^2}, \frac{1}{T} \right\}$$

$$\text{In[*]} := \text{Factor} \left[\frac{(\text{Det}[\# @ \mathbf{A}] (\# @ \omega /. \mathbf{T} \rightarrow \mathbf{1} / \mathbf{T}))}{(\# @ \omega)} \right] \& /@ \text{seq}$$

Out[*]=

$$\left\{ 1, -\frac{\mathbf{T}^5 (-1 + 2 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 3 \mathbf{T}^4 + \mathbf{T}^5)^2}{(-1 + 3 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 2 \mathbf{T}^4 + \mathbf{T}^5)^2}, -\frac{\mathbf{T}^5 (-1 + 2 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 3 \mathbf{T}^4 + \mathbf{T}^5)^2}{(-1 + 3 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 2 \mathbf{T}^4 + \mathbf{T}^5)^2}, \right.$$

$$\frac{\mathbf{T}^5 (-1 + 2 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 3 \mathbf{T}^4 + \mathbf{T}^5)^2}{(-1 + 3 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 2 \mathbf{T}^4 + \mathbf{T}^5)^2}, \frac{\mathbf{T}^4 (-1 + 2 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 3 \mathbf{T}^4 + \mathbf{T}^5)^2}{(-1 + 3 \mathbf{T} - 5 \mathbf{T}^2 + 5 \mathbf{T}^3 - 2 \mathbf{T}^4 + \mathbf{T}^5)^2},$$

$$\frac{\mathbf{T}^8 (-3 + 4 \mathbf{T} - 3 \mathbf{T}^2 + \mathbf{T}^3)}{(-1 + 3 \mathbf{T} - 4 \mathbf{T}^2 + 3 \mathbf{T}^3)^2}, \frac{\mathbf{T}^8 (-3 + 4 \mathbf{T} - 3 \mathbf{T}^2 + \mathbf{T}^3)}{(-1 + 3 \mathbf{T} - 4 \mathbf{T}^2 + 3 \mathbf{T}^3)^2}, -\frac{\mathbf{T}^8 (-3 + 4 \mathbf{T} - 3 \mathbf{T}^2 + \mathbf{T}^3)}{(-1 + 3 \mathbf{T} - 4 \mathbf{T}^2 + 3 \mathbf{T}^3)^2},$$

$$\left. -\frac{\mathbf{T}^7 (-3 + 4 \mathbf{T} - 3 \mathbf{T}^2 + \mathbf{T}^3)}{(-1 + 3 \mathbf{T} - 4 \mathbf{T}^2 + 3 \mathbf{T}^3)^2}, \frac{1}{\mathbf{T}^2}, -\frac{1}{\mathbf{T}^2}, \frac{1}{\mathbf{T}}, \frac{1}{\mathbf{T}}, -\frac{1}{\mathbf{T}}, -\frac{1}{\mathbf{T}^2}, \frac{1}{\mathbf{T}^2}, \frac{1}{\mathbf{T}^2}, \frac{1}{\mathbf{T}}, -\frac{1}{\mathbf{T}}, -\frac{1}{\mathbf{T}}, \frac{1}{\mathbf{T}^2}, -\frac{1}{\mathbf{T}^2}, \frac{1}{\mathbf{T}} \right\}$$