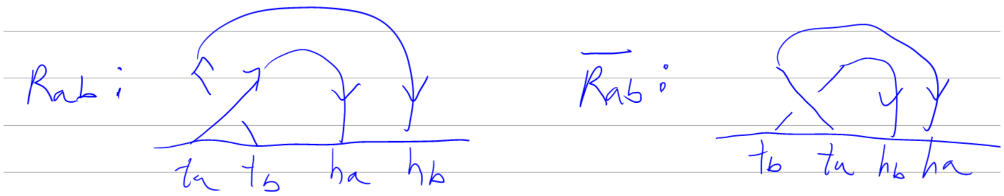
Pensieve header: Unitarity for Γ-calculus.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\MetaCalculi"];
Once[<< KnotTheory`]
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.



In[∘]:=
```
R[a_,b_] := Γ@<|ς → {t[a], t[b], h[a], h[b]}, ω → 1, σ → h[a] + h[b] T[a], λ → {t[a], t[b]}.({{T[a], 1 - T[a]}, {0, 1}}).{h[a], h[b]}|>;

R̄[a_,b_] := Γ@<|ς → {t[b], t[a], h[b], h[a]}, ω → 1, σ → h[a] + h[b] T[a]^-1, λ → {t[a], t[b]}.({{T[a]^-1, 1 - T[a]^-1}, {0, 1}}).{h[a], h[b]}|>;
```

In[∘]:= $\overline{R}_{1,2}$

Out[∘]=

$$\Gamma\left[\left\langle\left|\varsigma\to\{t_2, t_1, h_2, h_1\},\ \omega\to 1,\ \sigma\to h_1+\frac{h_2}{T_1},\ \lambda\to h_2\left(t_2+t_1\left(1-\frac{1}{T_1}\right)\right)+\frac{h_1 t_1}{T_1}\right|\right\rangle\right]$$

In[∘]:=
```
Γ[α_][ϰ_] := α[ϰ];
Γ[α_][S] := Union@Cases[α[ς], t[a_] :> a, ∞];
Γ[α_][n] := Length[Γ[α][S]];
Γ[α_][Σ] := (∂[h[#]] α[σ]) & /@ Γ[α][S];
Γ[α_][A] := Outer[Factor[∂[t[#1], h[#2]] Γ[α][λ]] &, Γ[α][S], Γ[α][S]];
```

In[∘]:=
```
{R̄[1,2][ς], R̄[1,2][ω], R̄[1,2][σ], R̄[1,2][λ], R̄[1,2][S], R̄[1,2][Σ], R̄[1,2][A] // MatrixForm} // Column
```

Out[∘]=

$\{t_2, t_1, h_2, h_1\}$

1

$h_1 + \frac{h_2}{T_1}$

$h_2\left(t_2 + t_1\left(1-\frac{1}{T_1}\right)\right) + \frac{h_1 t_1}{T_1}$

$\{1, 2\}$

$\left\{1, \frac{1}{T_1}\right\}$

$$\begin{pmatrix} \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ 0 & 1 \end{pmatrix}$$

In[∘]:=
```
ΓCollect[γ_Γ] := Γ[<|ς → γ@ς, ω → Factor@γ@ω, σ → Expand@γ@σ,
    λ → Total[CoefficientRules[γ@λ, γ@ς] /. (ps_ → c_) :> Factor[c] Times @@ ((γ@ς)^ps)]|>]
```

In[∘]:=
```
ΓCollect[R̄1,2]
```
Out[∘]=

$$\Gamma\left[\left\langle\left|\varsigma\to\{t_2, t_1, h_2, h_1\}, \omega\to 1, \sigma\to h_1+\frac{h_2}{T_1}, \lambda\to h_2\,t_2+\frac{h_1\,t_1}{T_1}+\frac{h_2\,t_1\,(-1+T_1)}{T_1}\right|\right\rangle\right]$$

In[∘]:=
```
ConservativeQ[γ_Γ] := Simplify[Expand[γ@λ /. h_ → 1] == Sum[t_a, {a, γ@S}]]
```

In[∘]:=
```
ConservativeQ[R1,2]
```
Out[∘]=

True

In[∘]:=
```
T* ^:= T^-1;
⟨c_, d_⟩_ς_ := Expand@Module[{e, f},
    Expand[c (d /. {T → T*, t_i_ :> t_i*, h_i_ :> h_i*})] /.
     {t_i_* t_i_ → 0, h_i_* h_i_ → T - T*, (f : t | h)_j_* (e : t | h)_i_ :>
       If[Position[ς, e_i][[1, 1]] < Position[ς, f_j][[1, 1]], T - 1, 1 - T*]}
   ];
UnitaryQ[γ_Γ] := Module[{vs},
   vs = Table[-t_i + ∂_t_i γ@λ /. T_ → T, {i, γ@S}];
   And @@ Flatten@Table[Simplify[⟨vs[[i]], vs[[j]]⟩_γ@ς == 0], {i, γ@n}, {j, γ@n}]
  ];
```

In[∘]:=
```
UnitaryQ /@ {R1,2, R̄1,2}
```
Out[∘]=

{True, True}

In[∘]:=
```
ΓForm[γ_Γ] := Module[{M},
   M = γ[A] // Transpose;
   PrependTo[M, t_# & /@ γ[S]];
   M = Join[
     {Prepend[h_# & /@ γ[S], γ[ω]]},
     Transpose[M],
     {Prepend[γ[Σ], If[TrueQ[ConservativeQ@γ ∧ UnitaryQ@γ], ■, ■]]}
    ];
   Column[{γ[ς], MatrixForm[M]}]
  ];
ΓForm[else_] := else /. γ_Γ :> ΓForm[γ];
Format[γ_Γ, StandardForm] := ΓForm[γ];
```

In[∘]:= $\{R_{1,2}, \overline{R}_{1,2}\}$

Out[∘]=

$$\left\{ \overset{\{t_1, t_2, h_1, h_2\}}{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}}, \overset{\{t_2, t_1, h_2, h_1\}}{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ t_2 & 0 & 1 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}} \right\}$$

In[∘]:=

$$\mathrm{dm}_{i\_,j\_\to k\_}[\gamma\_\Gamma] := \mathrm{Module}\Big[\{a, b, c, d, \theta, \epsilon, \phi, \psi, \Xi, \mu\},$$

$$\begin{pmatrix} a & b & \theta \\ c & d & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_i,h_i}\gamma@\lambda & \partial_{t_i,h_j}\gamma@\lambda & \partial_{t_i}\gamma@\lambda \\ \partial_{t_j,h_i}\gamma@\lambda & \partial_{t_j,h_j}\gamma@\lambda & \partial_{t_j}\gamma@\lambda \\ \partial_{h_i}\gamma@\lambda & \partial_{h_j}\gamma@\lambda & \gamma@\lambda \end{pmatrix} \;/.\; (t \mid h)_{i|j} \to 0;$$

$$\Gamma\mathrm{Collect}\Big[\Gamma\Big[\langle|$$

$$\varsigma \to \mathrm{DeleteCases}\big[\gamma@\varsigma, h_i \mid t_j\big] \;/.\; \{t_i \to t_k, h_j \to h_k\},$$

$$\omega \to (\mu = 1 - c)\ \gamma@\omega,$$

$$\sigma \to h_k\ (\partial_{h_i}\sigma)\ \big(\partial_{h_j}\sigma\big) + \big(\sigma \;/.\; h_{i|j} \to 0\big),$$

$$\lambda \to \{t_k, 1\}.\begin{pmatrix} b + a\,d/\mu & \theta + a\,\epsilon/\mu \\ \psi + d\,\phi/\mu & \Xi + \epsilon\,\phi/\mu \end{pmatrix}.\{h_k, 1\}$$

$$|\rangle\Big] \;/.\; \{T_i \to T_k, T_j \to T_k\}\Big]$$

$$\Big];$$

In[∘]:= $\{R_{1,2} \,//\, \mathrm{dm}_{1,2\to1}, R_{1,2} \,//\, \mathrm{dm}_{2,1\to1}, \overline{R}_{1,2} \,//\, \mathrm{dm}_{1,2\to1}, \overline{R}_{1,2} \,//\, \mathrm{dm}_{2,1\to1}\}$

Out[∘]=

$$\left\{ \overset{\{t_1, h_1\}}{\begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare\blacksquare & 0 \end{pmatrix}}, \overset{\{t_1, h_1\}}{\begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \\ \blacksquare\blacksquare & 0 \end{pmatrix}}, \overset{\{t_1, h_1\}}{\begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare\blacksquare & 0 \end{pmatrix}}, \overset{\{t_1, h_1\}}{\begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \\ \blacksquare\blacksquare & 0 \end{pmatrix}} \right\}$$

In[∘]:= $\mathrm{With}\Big[\{n = 4\},$

$$\gamma 0 = \Gamma@\langle|\varsigma \to \mathrm{Flatten}@\mathrm{Table}[\{t_a, h_a\}, \{a, n\}], \omega \to \omega, \sigma \to \sum_{a=1}^{n} h_a\,\sigma_a, \lambda \to \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{ab}\,|\rangle\Big]$$

Out[∘]=

$$\overset{\{t_1, h_1, t_2, h_2, t_3, h_3, t_4, h_4\}}{\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_4 \\ t_1 & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ t_2 & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ t_3 & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ t_4 & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \blacksquare\blacksquare & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}}$$

In[∘]:= **lhs = γ0 // $dm_{1,2\to 1}$ // $dm_{1,3\to 1}$ // ΓCollect**

Out[∘]=

{t_1, h_1, t_4, h_4}

$$\begin{pmatrix} \omega\ (1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}) & h_1 \\ t_1 & \frac{\alpha_{1,3}-\alpha_{1,3}\,\alpha_{2,1}+\alpha_{1,1}\,\alpha_{2,3}-\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1}+\alpha_{1,2}\,\alpha_{2,3}\,\alpha_{3,1}-\alpha_{1,3}\,\alpha_{3,2}+\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2}-\alpha_{1,1}\,\alpha_{2,3}\,\alpha_{3,2}+\cdots}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ t_4 & \frac{\alpha_{2,3}\,\alpha_{4,1}-\alpha_{2,3}\,\alpha_{3,2}\,\alpha_{4,1}+\alpha_{2,2}\,\alpha_{3,3}\,\alpha_{4,1}+\alpha_{2,3}\,\alpha_{3,1}\,\alpha_{4,2}+\alpha_{3,3}\,\alpha_{4,2}-\alpha_{2,1}\,\alpha_{3,3}\,\alpha_{4,2}+\alpha_{4,3}-\alpha_{2,1}\,\alpha_{4,3}-\cdots}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ \blacksquare & 0 \end{pmatrix}$$

In[∘]:= **rhs = γ0 // $dm_{2,3\to 2}$ // $dm_{1,2\to 1}$ // ΓCollect**

Out[∘]=

{t_1, h_1, t_4, h_4}

$$\begin{pmatrix} \omega\ (1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}) & h_1 \\ t_1 & \frac{\alpha_{1,3}-\alpha_{1,3}\,\alpha_{2,1}+\alpha_{1,1}\,\alpha_{2,3}-\alpha_{1,3}\,\alpha_{2,2}\,\alpha_{3,1}+\alpha_{1,2}\,\alpha_{2,3}\,\alpha_{3,1}-\alpha_{1,3}\,\alpha_{3,2}+\alpha_{1,3}\,\alpha_{2,1}\,\alpha_{3,2}-\alpha_{1,1}\,\alpha_{2,3}\,\alpha_{3,2}+\cdots}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ t_4 & \frac{\alpha_{2,3}\,\alpha_{4,1}-\alpha_{2,3}\,\alpha_{3,2}\,\alpha_{4,1}+\alpha_{2,2}\,\alpha_{3,3}\,\alpha_{4,1}+\alpha_{2,3}\,\alpha_{3,1}\,\alpha_{4,2}+\alpha_{3,3}\,\alpha_{4,2}-\alpha_{2,1}\,\alpha_{3,3}\,\alpha_{4,2}+\alpha_{4,3}-\alpha_{2,1}\,\alpha_{4,3}-\cdots}{1-\alpha_{2,1}-\alpha_{2,2}\,\alpha_{3,1}-\alpha_{3,2}+\alpha_{2,1}\,\alpha_{3,2}} \\ \blacksquare & 0 \end{pmatrix}$$

In[∘]:= **lhs@λ == rhs@λ**

Out[∘]=

True

In[∘]:=
```
Γ /: RotateLeft[Γ[α_], n___] := Γ@ReplacePart[α, Key@ς → RotateLeft[α@ς, n]]
```

In[∘]:= **{Table[RotateLeft[$R_{1,2}$, k], {k, 0, 3}], Table[RotateLeft[$\overline{R}_{1,2}$, k], {k, 0, 3}]}**

Out[∘]=

{t_1, t_2, h_1, h_2} {t_2, h_1, h_2, t_1} {h_1, h_2, t_1, t_2} {h_2, t_1, t_2, h_1}

$$\left\{\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix}\right\},\right.$$

{t_2, t_1, h_2, h_1} {t_1, h_2, h_1, t_2} {h_2, h_1, t_2, t_1} {h_1, t_2, t_1, h_2}

$$\left.\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ t_2 & 0 & 1 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ t_2 & 0 & 1 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ t_2 & 0 & 1 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & \frac{1}{T_1} & \frac{-1+T_1}{T_1} \\ t_2 & 0 & 1 \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\right\}\right\}$$

In[∘]:=
```
Γ /: Insert[γ1_Γ, γ2_Γ, k_] := Γ@<|
    ς → Flatten[Insert[γ1@ς, γ2@ς, k]],
    ω → γ1@ω γ2@ω,
    σ → γ1@σ + γ2@σ,
    λ → γ1@λ + γ2@λ
    |>
Γ /: Insert[γ2_Γ, k_][γ1_Γ] := Insert[γ1, γ2, k]
```

In[◦]:= $R_{1,2}$ // Insert[$R_{3,4}$, 2]

Out[◦]=

$\{t_1, t_3, t_4, h_3, h_4, t_2, h_1, h_2\}$

$$\begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & T_1 & 1 - T_1 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1 - T_3 \\ t_4 & 0 & 0 & 0 & 1 \\ & 1 & T_1 & 1 & T_3 \end{pmatrix}$$

In[◦]:= ComposeList[{Insert[$R_{3,4}$, 5], Insert[$R_{5,6}$, 9], $dm_{2,3\to 2}$, $dm_{1,4\to 1}$, $dm_{1,5\to 1}$, $dm_{2,6\to 2}$, $dm_{2,1\to 2}$}, $R_{1,2}$]

Out[ ]=

$$\left\{ \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1 & 1-T_1 \\ t_2 & 0 & 1 \\ \blacksquare & 1 & T_1 \end{pmatrix} \end{matrix}, \begin{matrix} \{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 \\ t_1 & T_1 & 1-T_1 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1-T_3 \\ t_4 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 \end{pmatrix} \end{matrix}, \right.$$

$$\begin{matrix} \{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ t_1 & T_1 & 1-T_1 & 0 & 0 & 0 & 0 \\ t_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ t_3 & 0 & 0 & T_3 & 1-T_3 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 1 & T_1 & 1 & T_3 & 1 & T_5 \end{pmatrix} \end{matrix},$$

$$\begin{matrix} \{t_1, t_2, h_1, t_4, h_2, h_4, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_4 & h_5 & h_6 \\ t_1 & T_1 & -((-1+T_1)\,T_2) & (-1+T_1)\,(-1+T_2) & 0 & 0 \\ t_2 & 0 & T_2 & 1-T_2 & 0 & 0 \\ t_4 & 0 & 0 & 1 & 0 & 0 \\ t_5 & 0 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

$$\begin{matrix} \{t_1, t_2, h_2, h_1, t_5, t_6, h_5, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_5 & h_6 \\ t_1 & 1-T_2+T_1 T_2 & -((-1+T_1)\,T_2) & 0 & 0 \\ t_2 & 1-T_2 & T_2 & 0 & 0 \\ t_5 & 0 & 0 & T_5 & 1-T_5 \\ t_6 & 0 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

$$\begin{matrix} \{t_1, t_2, h_2, t_6, h_1, h_6\} \\ \begin{pmatrix} 1 & h_1 & h_2 & h_6 \\ t_1 & T_1\,(1-T_2+T_1 T_2) & -((-1+T_1)\,T_2) & -((-1+T_1)\,(1-T_2+T_1 T_2)) \\ t_2 & -T_1\,(-1+T_2) & T_2 & (-1+T_1)\,(-1+T_2) \\ t_6 & 0 & 0 & 1 \\ \blacksquare & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

$$\left. \begin{matrix} \{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & T_1\,(1-T_2+T_1 T_2) & -((-1+T_1)\,(1+T_1 T_2)) \\ t_2 & -T_1\,(-1+T_2) & 1-T_1+T_1 T_2 \\ \blacksquare & 0 & 0 \end{pmatrix} \end{matrix}, \begin{matrix} \{t_2, h_2\} \\ \begin{pmatrix} T_2\,\left(1-T_2+T_2^2\right) & h_2 \\ t_2 & 1 \\ \blacksquare & 0 \end{pmatrix} \end{matrix} \right\}$$

$\{\text{Insert}[\text{RotateLeft}[\overline{R}_{4,11}, 1], 4], dm_{1,4\to1}, dm_{11,12\to11}, , \text{Insert}[R_{10,15}, 5]\}$

In[∘]:= **Column@Reverse@**
ComposeList$[\{\text{Insert}[\text{RotateLeft}[\overline{R}_{2,7}, 1], 4], dm_{1,2\to1}, \text{Insert}[\text{RotateLeft}[R_{6,13}, 1], 7],$
$dm_{12,13\to12}, dm_{6,7\to6}, \text{Insert}[\overline{R}_{8,3}, 5], dm_{1,3\to1}, dm_{6,8\to6}, \text{Insert}[R_{14,9}, 6], dm_{12,14\to12},$
$dm_{6,9\to6}, \text{Insert}[\text{RotateLeft}[\overline{R}_{4,11}, 1], 4], dm_{1,4\to1}, dm_{11,12\to11}, \text{Insert}[R_{10,15}, 6],$
$dm_{6,10\to6}, dm_{11,15\to11}, dm_{6,11\to6}, \text{Insert}[R_{16,5}, 4], dm_{6,16\to6}, dm_{1,5\to1}, dm_{1,6\to1}$
$\},$
$\overline{R}_{12,1}]$

Out[∘]=

$\{t_1, h_1\}$

$-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3}$	h_1
t_1	1
■	0

$\{t_1, h_6, h_1, t_6\}$

$\frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2}$	h_1
t_1	$-\frac{1-2T_1+T_1^2-2T_6+6T_1T_6-4T_1^2T_6+T_6^2-6T_1T_6^2+5T_1^2T_6^2-T_6^3+3T_1T_6^3-3T_1^2T_6^3-T_1T_6^4+T_1^2T_6^4}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$
t_6	$\frac{(-1+T_6)\left(1-2T_1+T_1^2-T_6+4T_1T_6-3T_1^2T_6+T_6^2-3T_1T_6^2+2T_1^2T_6^2-T_6^3+3T_1T_6^3-T_1^2T_6^3+T_6^4-2T_1T_6^4+T_1T_6^5\right)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$
■	0

$\{t_1, h_1, t_5, h_6, h_5, t_6\}$

$\frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2}$	h_1	
t_1	$\frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	$-\frac{(-1+T_1)(-1+T\ldots}{-1+2T_1-T_1^2+T_6\ldots}$
t_5	0	
t_6	$\frac{(-1+T_6)T_6\left(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3\right)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	$\frac{(-1+T_6)\left(1-2T_1+T_1^2-2T_6+5T_1T_6-3T_1^2\ldots\right.}{-1+2T_1-T_1^2+T_6\ldots}$
■	0	

$\{t_1, h_1, h_6, t_{16}, t_5, h_{16}, h_5, t_6\}$

$\frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2}$	h_1	h_5	
t_1	$\frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	0	$\frac{(-1+\ldots}{-1+2T_1-\ldots}$
t_5	0	1	
t_6	$\frac{(-1+T_6)T_6\left(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3\right)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	0	$\frac{-1+2T_1-T_1^2+2T_6-5T_1T_6+3\ldots}{-1+2T_1-\ldots}$
t_{16}	0	$1-T_{16}$	
■	0	T_{16}	

$\{t_1, h_1, h_6, t_6\}$

$\frac{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}{T_1^2T_6^2}$	h_1	
t_1	$\frac{T_6^2(1-T_1+T_1T_6)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	$\frac{(-1+T_1)(1-T_1-\ldots}{-1+2T_1-T_1^2+T_6-4T_1\ldots}$
t_6	$\frac{(-1+T_6)T_6\left(1-T_1-T_6+3T_1T_6+T_6^2-2T_1T_6^2+T_1T_6^3\right)}{-1+2T_1-T_1^2+T_6-4T_1T_6+3T_1^2T_6+2T_1T_6^2-2T_1^2T_6^2+T_1^2T_6^3}$	$\frac{-1+2T_1-T_1^2+2T_6-5T_1T_6+3T_1^2T_6-2T_6^2+\ldots}{-1+2T_1-T_1^2+T_6-4T_1\ldots}$
■	0	

$\{t_1, h_1, t_{11}, h_6, h_{11}, t_6\}$

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}} & h_1 & \\ t_1 & \frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_6 & \frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \frac{T_6\,(-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+2T_{11}-4\,\mathrm{T}}{} \\ t_{11} & \frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ \blacksquare & 0 & \end{pmatrix}$$

$\{t_1, h_1, t_{11}, h_{11}, t_{15}, h_6, h_{15}, t_6\}$

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}} & h_1 & \\ t_1 & \frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_6 & \frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \frac{T_6\,(-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+2T_{11}-4\,\mathrm{T}}{} \\ t_{11} & \frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_{15} & 0 & \\ \blacksquare & 0 & \end{pmatrix}$$

$\{t_1, h_1, t_{11}, h_{11}, h_6, t_{10}, t_{15}, h_{10}, h_{15}, t_6\}$

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}} & h_1 & \\ t_1 & \frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_6 & \frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+2T_{11}-4T_1T_1}{} \\ t_{10} & 0 & \\ t_{11} & \frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_{15} & 0 & \\ \blacksquare & 0 & \end{pmatrix}$$

$\{t_1, h_1, t_{11}, h_{11}, h_6, t_6\}$

$$\begin{pmatrix} \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}}{T_1^2T_6T_{11}} & h_1 & \\ t_1 & \frac{(1-T_1+T_1T_6)T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ t_6 & \frac{T_1(-1+T_6)T_6T_{11}}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \frac{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+2T_{11}-4T_1T_1}{} \\ t_{11} & \frac{(1-T_1+T_1T_6)(-1+T_{11})}{-1+2T_1-T_1^2-T_1T_6+T_1^2T_6+T_{11}-2T_1T_{11}+T_1^2T_{11}+T_1T_6T_{11}} & \\ \blacksquare & 0 & \end{pmatrix}$$

$\{t_1, t_{12}, h_{11}, h_1, t_{11}, h_{12}, h_6, t_6\}$

$$\begin{pmatrix} 1 & h_1 & h_6 & h_{11} & h_{12} \\ t_1 & \frac{1-T_1+T_1T_6}{T_1^2T_6} & \frac{-1+T_1}{T_1T_6} & \frac{(-1+T_1)(1-T_1+T_1T_6)}{T_1^2T_6} & 0 \\ t_6 & \frac{-1+T_6}{T_1} & 2-T_6-T_{12}+T_6T_{12} & \frac{(-1+T_1)(-1+T_6)}{T_1} & -((-1+T_6)T_{12}) \\ t_{11} & 0 & 0 & 1 & 0 \\ t_{12} & \frac{(1-T_1+T_1T_6)(-1+T_{12})}{T_1^2T_6T_{12}} & -\frac{(1-T_1+T_1T_6)(-1+T_{12})}{T_1T_6T_{12}} & \frac{(-1+T_1)(1-T_1+T_1T_6)(-1+T_{12})}{T_1^2T_6T_{12}} & 1 \\ \blacksquare & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{t_1, t_{12}, h_1, t_4, h_{11}, h_4, t_{11}, h_{12}, h_6, t_6\}$

1	h_1	h_4	h_6	h_{11}	h_{12}
t_1	$\frac{1-T_1+T_1 T_6}{T_1 T_6}$	0	$\frac{-1+T_1}{T_1 T_6}$	0	0
t_4	0	$\frac{1}{T_4}$	0	$\frac{-1+T_4}{T_4}$	0
t_6	$-1+T_6$	0	$2-T_6-T_{12}+T_6 T_{12}$	0	$-((-1+T_6)\,T_{12})$
t_{11}	0	0	0	1	0
t_{12}	$\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	0	$-\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	0	1
■	0	1	0	$\frac{1}{T_4}$	0

$\{t_1, t_{12}, h_1, h_{12}, h_6, t_6\}$

1	h_1	h_6	h_{12}
t_1	$\frac{1-T_1+T_1 T_6}{T_1 T_6}$	$\frac{-1+T_1}{T_1 T_6}$	0
t_6	$-1+T_6$	$2-T_6-T_{12}+T_6 T_{12}$	$-((-1+T_6)\,T_{12})$
t_{12}	$\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$-\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	1
■	0	0	0

$\{t_1, t_{12}, h_1, h_6, t_9, h_{12}, h_9, t_6\}$

1	h_1	h_6	h_9	h_{12}
t_1	$\frac{1-T_1+T_1 T_6}{T_1 T_6}$	$\frac{-1+T_1}{T_1 T_6}$	0	0
t_6	$-1+T_6$	1	$(-1+T_6)\,(-1+T_{12})$	$-((-1+T_6)\,T_{12})$
t_9	0	0	1	0
t_{12}	$\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$\frac{(-1+T_1)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$-\frac{-1+T_{12}}{T_{12}}$	1
■	0	0	0	0

$\{t_1, t_{12}, h_1, h_6, h_{12}, t_{14}, t_9, h_{14}, h_9, t_6\}$

1	h_1	h_6	h_9	h_{12}	h_{14}
t_1	$\frac{1-T_1+T_1 T_6}{T_1 T_6}$	$\frac{-1+T_1}{T_1 T_6}$	0	0	0
t_6	$-1+T_6$	1	0	$1-T_6$	0
t_9	0	0	1	0	0
t_{12}	$\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$\frac{(-1+T_1)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	0	$\frac{1}{T_{12}}$	0
t_{14}	0	0	$1-T_{14}$	0	T_{14}
■	0	0	T_{14}	0	1

$\{t_1, t_{12}, h_1, h_6, h_{12}, t_6\}$

1	h_1	h_6	h_{12}
t_1	$\frac{1-T_1+T_1 T_6}{T_1 T_6}$	$\frac{-1+T_1}{T_1 T_6}$	0
t_6	$-1+T_6$	1	$1-T_6$
t_{12}	$\frac{(1-T_1+T_1 T_6)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$\frac{(-1+T_1)\,(-1+T_{12})}{T_1 T_6 T_{12}}$	$\frac{1}{T_{12}}$
■	0	0	0

$\{t_1, t_{12}, h_6, t_8, h_1, h_8, h_{12}, t_6\}$

1	h_1	h_6	h_8	h_{12}
t_1	$\frac{1}{T_1}$	$\frac{-1+T_1}{T_1}$	0	0
t_6	0	T_6	0	$1-T_6$
t_8	$\frac{-1+T_8}{T_8}$	0	$\frac{1}{T_8}$	0
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	$\frac{(-1+T_1)\,(-1+T_{12})}{T_1 T_{12}}$	0	$\frac{1}{T_{12}}$
■	0	0	0	0

$\{t_1, t_{12}, h_6, h_1, t_3, t_8, h_3, h_8, h_{12}, t_6\}$

1	h_1	h_3	h_6	h_8	h_{12}
t_1	$\frac{1}{T_1}$	0	$\frac{-1+T_1}{T_1}$	0	0
t_3	0	1	0	0	0
t_6	0	0	T_6	0	$1-T_6$
t_8	0	$\frac{-1+T_8}{T_8}$	0	$\frac{1}{T_8}$	0
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	0	$\frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}}$	0	$\frac{1}{T_{12}}$
■	0	$\frac{1}{T_8}$	0	1	0

$\{t_1, t_{12}, h_6, h_1, h_{12}, t_6\}$

1	h_1	h_6	h_{12}
t_1	$\frac{1}{T_1}$	$\frac{-1+T_1}{T_1}$	0
t_6	0	T_6	$1-T_6$
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	$\frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}}$	$\frac{1}{T_{12}}$
■	0	0	0

$\{t_1, t_{12}, h_7, h_1, t_7, h_6, h_{12}, t_6\}$

1	h_1	h_6	h_7	h_{12}
t_1	$\frac{1}{T_1}$	0	$\frac{-1+T_1}{T_1}$	0
t_6	0	T_6	0	$1-T_6$
t_7	0	0	1	0
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	0	$\frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}}$	$\frac{1}{T_{12}}$
■	0	0	0	0

$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}, t_{13}, h_6, h_{13}, t_6\}$

1	h_1	h_6	h_7	h_{12}	h_{13}
t_1	$\frac{1}{T_1}$	0	$\frac{-1+T_1}{T_1}$	0	0
t_6	0	T_6	0	0	$1-T_6$
t_7	0	0	1	0	0
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	0	$\frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}}$	$\frac{1}{T_{12}}$	0
t_{13}	0	0	0	0	1
■	0	1	0	0	T_6

$\{t_1, t_{12}, h_7, h_1, t_7, h_{12}\}$

1	h_1	h_7	h_{12}
t_1	$\frac{1}{T_1}$	$\frac{-1+T_1}{T_1}$	0
t_7	0	1	0
t_{12}	$\frac{-1+T_{12}}{T_1 T_{12}}$	$\frac{(-1+T_1)(-1+T_{12})}{T_1 T_{12}}$	$\frac{1}{T_{12}}$
■	0	0	0

$\{t_1, t_{12}, h_1, t_2, h_7, h_2, t_7, h_{12}\}$

1	h_1	h_2	h_7	h_{12}
t_1	1	0	0	0
t_2	0	$\frac{1}{T_2}$	$\frac{-1+T_2}{T_2}$	0
t_7	0	0	1	0
t_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	0	$\frac{1}{T_{12}}$
■	$\frac{1}{T_{12}}$	1	$\frac{1}{T_2}$	1

$\{t_1, t_{12}, h_1, h_{12}\}$

$$\begin{pmatrix} 1 & h_1 & h_{12} \\ t_1 & 1 & 0 \\ t_{12} & \frac{-1+T_{12}}{T_{12}} & \frac{1}{T_{12}} \\ & \frac{1}{T_{12}} & 1 \end{pmatrix}$$