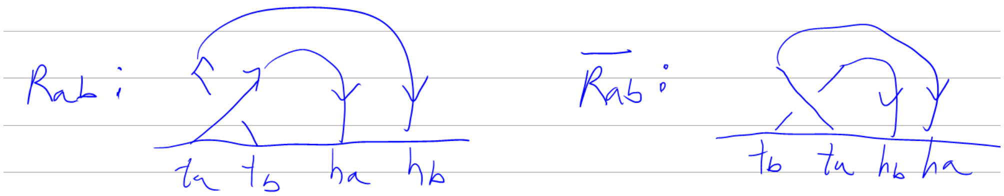
Pensieve header: Unitarity for Γ-calculus.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\MetaCalculi"];
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```



In[∘]:=
```
R_{a_,b_} := Γ@⟨|ς → {t_a, t_b, h_a, h_b}, ω → 1, σ → h_a + h_b T_a, λ → {t_a, t_b}.( (1, 1 - T_a), (0, T_a) ).{h_a, h_b}|⟩;
R̄_{a_,b_} := Γ@⟨|ς → {t_b, t_a, h_b, h_a}, ω → 1, σ → h_a + h_b T_a^-1, λ → {t_a, t_b}.( (1, 1 - T_a^-1), (0, T_a^-1) ).{h_a, h_b}|⟩;
```

In[∘]:= `R̄_{1,2}`

Out[∘]=

$$\Gamma\left[\left\langle\left|\varsigma\to\{t_2,t_1,h_2,h_1\},\ \omega\to 1,\ \sigma\to h_1+\frac{h_2}{T_1},\ \lambda\to h_1 t_1+h_2\left(t_1\left(1-\frac{1}{T_1}\right)+\frac{t_2}{T_1}\right)\right|\right\rangle\right]$$

In[∘]:=
```
Γ[α_][κ_] := α[κ];
Γ[α_][S] := Union@Cases[α[ς], t_a_ ⧴ a, ∞];
Γ[α_][n] := Length[Γ[α][S]];
Γ[α_][Σ] := (∂_{h_#} α[σ]) & /@ Γ[α][S];
Γ[α_][A] := Outer[Factor[∂_{t_#1 h_#2} Γ[α][λ]] &, Γ[α][S], Γ[α][S]];
```

In[∘]:=
```
{R̄_{1,2}[ς], R̄_{1,2}[ω], R̄_{1,2}[σ], R̄_{1,2}[λ], R̄_{1,2}[S], R̄_{1,2}[Σ], R̄_{1,2}[A] // MatrixForm} // Column
```

Out[∘]=

$\{t_2, t_1, h_2, h_1\}$

1

$h_1 + \frac{h_2}{T_1}$

$h_1 t_1 + h_2\left(t_1\left(1-\frac{1}{T_1}\right)+\frac{t_2}{T_1}\right)$

$\{1, 2\}$

$\left\{1, \frac{1}{T_1}\right\}$

$$\begin{pmatrix} 1 & \frac{-1+T_1}{T_1} \\ 0 & \frac{1}{T_1} \end{pmatrix}$$

In[∘]:=
```
ΓCollect[γ_Γ] := Γ[⟨|ς → γ@ς, ω → Factor@γ@ω, σ → Expand@γ@σ,
   λ → Total[CoefficientRules[γ@λ, γ@ς] /. (ps_ → c_) ⧴ Factor[c] Times @@ ((γ@ς)^ps)]|⟩]
```

In[∘]:=
```
ΓCollect[R̄1,2]
```
Out[∘]=

$$\Gamma\left[\left\langle\left|\varsigma \to \{t_2, t_1, h_2, h_1\},\ \omega \to 1,\ \sigma \to h_1 + \frac{h_2}{T_1},\ \lambda \to h_1 t_1 + \frac{h_2 t_2}{T_1} + \frac{h_2 t_1 (-1 + T_1)}{T_1}\right|\right\rangle\right]$$

```
T* ^:= T^-1;
⟨c_, d_⟩ς_ := Expand@Module[{e, f},
    Expand[c (d /. {T → T*, t_i_ :→ t_i*, h_i_ :→ h_i*})] /.
     {t_i_* t_i_ → 0, h_i_* h_i_ → T - T*, (f : t | h)_j_* (e : t | h)_i_ :→
       If[Position[ς, e_i][[1, 1]] < Position[ς, f_j][[1, 1]], T - 1, 1 - T*]}
   ];
UnitaryQ[γ_Γ] := Module[{vs},
  vs = Table[-h_i + ∂_{h_i} γ@λ /. T_ → T, {i, γ@S}];
  And @@ Flatten@Table[⟨vs[[i]], vs[[j]]⟩γ@ς == 0, {i, γ@n}, {j, γ@n}]
 ]
```

In[∘]:=
```
UnitaryQ /@ {R1,2, R̄1,2}
```
Out[∘]=
```
{True, True}
```

In[∘]:=
```
ΓForm[γ_Γ] := Module[{M},
   M = γ[A] // Transpose;
   PrependTo[M, t# & /@ γ[S]];
   M = Join[
     {Prepend[h# & /@ γ[S], γ[ω]]},
     Transpose[M],
     {Prepend[γ[Σ], If[TrueQ@UnitaryQ@γ, ■, ■]]}
    ];
   Column[{γ[ς], MatrixForm[M]}]
  ];
ΓForm[else_] := else /. γ_Γ :→ ΓForm[γ];
Format[γ_Γ, StandardForm] := ΓForm[γ];
```

In[∘]:=
```
{R1,2, R̄1,2}
```
Out[∘]=

$$\left\{ \begin{array}{c}\{t_1, t_2, h_1, h_2\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 1 - T_1 \\ t_2 & 0 & T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}\end{array},\ \begin{array}{c}\{t_2, t_1, h_2, h_1\} \\ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & \frac{-1+T_1}{T_1} \\ t_2 & 0 & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}\end{array} \right\}$$

In[∘]:=
```
dm_{i_,j_→k_}[γ_Γ] := Module[{a, b, c, d, θ, ϵ, ϕ, ψ, Ξ, μ},
  (a b θ; c d ϵ; ϕ ψ Ξ) = (∂_{t_i,h_i} γ@λ  ∂_{t_i,h_j} γ@λ  ∂_{t_i} γ@λ;
                          ∂_{t_j,h_i} γ@λ  ∂_{t_j,h_j} γ@λ  ∂_{t_j} γ@λ;
                          ∂_{h_i} γ@λ      ∂_{h_j} γ@λ      γ@λ) /. (t | h)_{i|j} → 0;
  ΓCollect[Γ[<|
      ς → DeleteCases[γ@ς, h_i | t_j] /. {t_i → t_k, h_j → h_k},
      ω → (μ = 1 - b) γ@ω,
      σ → h_k (∂_{h_i} σ) (∂_{h_j} σ) + (σ /. h_{i|j} → 0),
      λ → {t_k, 1}.(c + a d / μ   ϵ + d θ / μ;  ϕ + a ψ / μ   Ξ + θ ψ / μ).{h_k, 1}
    |>] /. {T_i → T_k, T_j → T_k}]
];
```

In[∘]:= $\{R_{1,2}$ // $dm_{1,2\to 1}$, $R_{1,2}$ // $dm_{2,1\to 1}$, $\overline{R}_{1,2}$ // $dm_{1,2\to 1}$, $\overline{R}_{1,2}$ // $dm_{2,1\to 1}\}$

Out[∘]=

$$\left\{ \begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix}_{\{t_1, h_1\}}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix}_{\{t_1, h_1\}}, \begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix}_{\{t_1, h_1\}}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \\ \blacksquare & 0 \end{pmatrix}_{\{t_1, h_1\}} \right\}$$

In[∘]:=
```
Γ /: RotateLeft[Γ[α_], n___] := Γ@ReplacePart[α, Key@ς → RotateLeft[α@ς, n]]
```

In[∘]:= $\{$Table[RotateLeft[$R_{1,2}$, k], {k, 0, 3}], Table[RotateLeft[$\overline{R}_{1,2}$, k], {k, 0, 3}]$\}$

Out[∘]=

$$\left\{\left\{ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 1-T_1 \\ t_2 & 0 & T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}_{\{t_1, t_2, h_1, h_2\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 1-T_1 \\ t_2 & 0 & T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}_{\{t_2, h_1, h_2, t_1\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 1-T_1 \\ t_2 & 0 & T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}_{\{h_1, h_2, t_1, t_2\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 1-T_1 \\ t_2 & 0 & T_1 \\ \blacksquare & 1 & T_1 \end{pmatrix}_{\{h_2, t_1, t_2, h_1\}} \right\},\right.$$

$$\left.\left\{ \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & \frac{-1+T_1}{T_1} \\ t_2 & 0 & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}_{\{t_2, t_1, h_2, h_1\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & \frac{-1+T_1}{T_1} \\ t_2 & 0 & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}_{\{t_1, h_2, h_1, t_2\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & \frac{-1+T_1}{T_1} \\ t_2 & 0 & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}_{\{h_2, h_1, t_2, t_1\}}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & \frac{-1+T_1}{T_1} \\ t_2 & 0 & \frac{1}{T_1} \\ \blacksquare & 1 & \frac{1}{T_1} \end{pmatrix}_{\{h_1, t_2, t_1, h_2\}} \right\}\right\}$$

In[∘]:=
```
Γ /: Insert[γ1_Γ, γ2_Γ, k_] := Γ@<|
    ς → Flatten[Insert[γ1@ς, γ2@ς, k]],
    ω → γ1@ω γ2@ω,
    σ → γ1@σ + γ2@σ,
    λ → γ1@λ + γ2@λ
    |>
Γ /: Insert[γ2_Γ, k_][γ1_Γ] := Insert[γ1, γ2, k]
```

In[∘]:=
```
R1,2 // Insert[R3,4, 2]
```

Out[∘]=

$\{t_1, t_3, t_4, h_3, h_4, t_2, h_1, h_2\}$

1	h_1	h_2	h_3	h_4
t_1	1	$1-T_1$	0	0
t_2	0	T_1	0	0
t_3	0	0	1	$1-T_3$
t_4	0	0	0	T_3
	1	T_1	1	T_3

In[∘]:=
```
ComposeList[{Insert[R3,4, 5], dm2,3→5},
 R1,2]
```

Out[∘]=

{

$\{t_1, t_2, h_1, h_2\}$

1	h_1	h_2
t_1	1	$1-T_1$
t_2	0	T_1
	1	T_1

,

$\{t_1, t_2, h_1, h_2, t_3, t_4, h_3, h_4\}$

1	h_1	h_2	h_3	h_4
t_1	1	$1-T_1$	0	0
t_2	0	T_1	0	0
t_3	0	0	1	$1-T_3$
t_4	0	0	0	T_3
	1	T_1	1	T_3

,

$\{t_1, t_5, h_1, t_4, h_5, h_4\}$

1	h_1	h_4	h_5
t_1	1	0	$1-T_1$
t_4	0	T_5	0
t_5	0	$1-T_5$	T_1
	0	0	0

}