Pensieve header: Testing the common program for all w-meta-calculi. Continues pensieve://2014-07/MetaCalculi/.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/Projects/MetaCalculi/Archive/"];
<< KnotTheory`
<< "MetaCalculi-Program-150518.m"
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

General

```
SXForm[L = Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

```
SXForm[{Loop[1, 2, 3, 4], Loop[5, 6, 7, 8], Loop[9, 10, 11, 12]},
 Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]
```

```
Z[L]
```

```
dm[9, 12, 9][
 dm[9, 11, 9][dm[9, 10, 9][dm[5, 8, 5][dm[5, 7, 5][dm[5, 6, 5][dm[1, 4, 1][dm[1, 3, 1][
        dm[1, 2, 1][Xm[1, 6] Xm[5, 10] Xm[9, 2] Xp[3, 8] Xp[7, 12] Xp[11, 4]]]]]]]]]]
```

α-Calculus

$\{Xp_{ab}, Xm_{ab}\}$ // A

$$\left\{\begin{pmatrix} 1 & h[b] \\ t[a] & \frac{-1+e^{c_a}}{c_a} \end{pmatrix}, \begin{pmatrix} 1 & h[b] \\ t[a] & \frac{e^{-c_a}(1-e^{c_a})}{c_a} \end{pmatrix}\right\}$$

$\{Xm_{51}\, Xm_{62}\, Xp_{34}$ // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
 $Xp_{61}\, Xm_{24}\, Xm_{35}$ // A // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]$\}$

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[2] & \frac{e^{-c_2}(1-e^{c_2})}{c_2} & 0 \\ t[3] & \frac{e^{-c_2}(-1+e^{c_3})}{c_3} & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}\right\}$$

$\alpha = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // A

1	h[1]	h[3]	h[5]	h[7]	h[9]	h[11]	h[13]	h[15]
t[2]	0	0	0	$\frac{e^{-c_2}(1-e^{c_2})}{c_2}$	0	0	0	0
t[4]	0	0	0	0	0	$\frac{e^{-c_4}(1-e^{c_4})}{c_4}$	0	0
t[6]	0	0	0	0	0	0	$\frac{-1+e^{c_6}}{c_6}$	0
t[8]	0	$\frac{e^{-c_8}(1-e^{c_8})}{c_8}$	0	0	0	0	0	0
t[10]	0	0	0	0	0	0	0	$\frac{-1+e^{c_{10}}}{c_{10}}$
t[12]	$\frac{e^{-c_{12}}(1-e^{c_{12}})}{c_{12}}$	0	0	0	0	0	0	0
t[14]	0	0	0	0	$\frac{-1+e^{c_{14}}}{c_{14}}$	0	0	0
t[16]	0	0	$\frac{-1+e^{c_{16}}}{c_{16}}$	0	0	0	0	0

$\alpha = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}$ // A;
Do[α = α // dm[1, k, 1], {k, 2, 16}]; α

$$\begin{pmatrix} e^{-3c_1}\left(-1+4e^{c_1}-8e^{2c_1}+11e^{3c_1}-8e^{4c_1}+4e^{5c_1}-e^{6c_1}\right) \\ t[1] \end{pmatrix}$$

Testing R3

{(Xp$_{12}$ // A) ** (Xp$_{13}$ // A) ** (Xp$_{23}$ // A), (Xp$_{23}$ // A) ** (Xp$_{13}$ // A) ** (Xp$_{12}$ // A)}

$$\left\{\begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2}\end{pmatrix}, \begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2}\end{pmatrix}\right\}$$

Testing the KV Solution

The Hard R4 Equation

Print /@ {(Xp$_{23}$ // A) ** (Xp$_{13}$ // A) ** (V // A), (V // A) ** ((Xp$_{13}$ // A) // dΔ[1, 1, 2])};

$$\left(\begin{array}{ccc} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\ t[1] & \frac{-\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ t[2] & \frac{\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\ \cdots}{\cdots} \end{array}\right.$$

$$\left(\begin{array}{ccc} \frac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\ t[1] & \frac{-\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ t[2] & \frac{\sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}\,c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\ \cdots}{\cdots} \end{array}\right.$$

```
(Xp23 // A) ** (Xp13 // A) ** (V // A) == (V // A) ** ((Xp13 // A) // dΔ[1, 1, 2]) // Simplify
```

True

Unitarity

```
(V // A) ** (Vi // A)  // αCollect[Simplify]
```

(1)

qΔ (“renormalized cabling”)

```
qΔ0[z_, x_, y_][α_A] := Module[{b, a},
  Times[
      V // A // dσ[1 → b[x], 2 → b[y]],
      α // dΔ[z, x, y],
      V // A // dA[1] // dA[2] // dσ[1 → a[x], 2 → a[y]]
     ] // dm[b[x], x, x] // dm[b[y], y, y] // dm[x, a[x], x] // dm[y, a[y], y]
 ]
```

```
Xp13 // A // qΔ0[1, 1, 2] // αCollect[Simplify]
```

$$\left(\begin{array}{cc} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{array}\right)$$

```
(Xp₂₃ // A) ** (Xp₁₃ // A)
```

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{pmatrix}$$

```
Xp₁₃ // A // qΔ[1, 1, 2] // αCollect[Simplify]
```

$$\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-e^{c_2}+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{-1+e^{c_2}}{c_2} \end{pmatrix}$$

```
Clear[ω, α, θ, ϕ, Ξ];
α0 = A[ω, {t[a], t[S]}.(α θ; ϕ Ξ).{h[a], h[S]}];
α1 = α0 // qΔ0[a, x, y] // αCollect[Simplify]
```

$$\begin{pmatrix} \omega & h[S] & h[x] & h[\ldots \\ t[S] & \Xi & \phi & \ldots \\ t[x] & \frac{-e^{c_y}\theta c_x+e^{c_x+c_y}\theta c_x-e^{c_y}\theta c_y+e^{c_x+c_y}\theta c_y}{-c_x+e^{c_x+c_y}c_x} & \frac{-e^{c_y}\alpha c_x+e^{c_x+c_y}\alpha c_x-e^{c_y}\alpha c_y+e^{c_x+c_y}\alpha c_y}{-c_x+e^{c_x+c_y}c_x} & \frac{-e^{c_y}\alpha c_x+e^{c_x+c_y}\alpha c_x \ldots}{-c_x+e^{\ldots}} \\ t[y] & \frac{-\theta c_x+e^{c_y}\theta c_x-\theta c_y+e^{c_y}\theta c_y}{-c_y+e^{c_x+c_y}c_y} & \frac{-\alpha c_x+e^{c_y}\alpha c_x-\alpha c_y+e^{c_y}\alpha c_y}{-c_y+e^{c_x+c_y}c_y} & \frac{-\alpha c_x+e^{c_y}\alpha c_x \ldots}{-c_y+e^{\ldots}} \end{pmatrix}$$

[illegible]

```
αCollect[(Simplify[#] /. cx + cy → ca) &][α1]
```

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[a] & 0 & 0 & 0 \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y}\theta c_a+e^{c_x+c_y}\theta c_a}{-c_x+e^{c_a}c_x} & \frac{-e^{c_y}\alpha c_a+e^{c_x+c_y}\alpha c_a}{-c_x+e^{c_a}c_x} & \frac{-e^{c_y}\alpha c_a+e^{c_x+c_y}\alpha c_a}{-c_x+e^{c_a}c_x} \\ t[y] & \frac{-\theta c_a+e^{c_y}\theta c_a}{-c_y+e^{c_a}c_y} & \frac{-\alpha c_a+e^{c_y}\alpha c_a}{-c_y+e^{c_a}c_y} & \frac{-\alpha c_a+e^{c_y}\alpha c_a}{-c_y+e^{c_a}c_y} \end{pmatrix}$$

```
Clear[ω, α, θ, ϕ, Ξ];
α0 = A[ω, {t[a], t[S]}.(α θ; ϕ Ξ).{h[a], h[S]}];
α0 // qΔ[a, x, y] // αCollect[Simplify]
```

$$\begin{pmatrix} \omega & h[S] & h[x] & h[y] \\ t[S] & \Xi & \phi & \phi \\ t[x] & \frac{-e^{c_y}\theta c_x+e^{c_x+c_y}\theta c_x-e^{c_y}\theta c_y+e^{c_x+c_y}\theta c_y}{-c_x+e^{c_x+c_y}c_x} & \frac{-e^{c_y}\alpha c_x+e^{c_x+c_y}\alpha c_x-e^{c_y}\alpha c_y+e^{c_x+c_y}\alpha c_y}{-c_x+e^{c_x+c_y}c_x} & \frac{-e^{c_y}\alpha c_x+e^{c_x+c_y}\alpha c_x-e^{c_y}\alpha c_y+e^{c_x+c_y}\alpha c_y}{-c_x+e^{c_x+c_y}c_x} \\ t[y] & \frac{-\theta c_x+e^{c_y}\theta c_x-\theta c_y+e^{c_y}\theta c_y}{-c_y+e^{c_x+c_y}c_y} & \frac{-\alpha c_x+e^{c_y}\alpha c_x-\alpha c_y+e^{c_y}\alpha c_y}{-c_y+e^{c_x+c_y}c_y} & \frac{-\alpha c_x+e^{c_y}\alpha c_x-\alpha c_y+e^{c_y}\alpha c_y}{-c_y+e^{c_x+c_y}c_y} \end{pmatrix}$$

Γ-Calculus

```
{Xpab, Xmab} // Γ
```

$$\left\{\begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & 1-T_a \\ s_b & 0 & T_a \\ \Sigma & 1 & T_a \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_b \\ s_a & 1 & \frac{-1+T_a}{T_a} \\ s_b & 0 & \frac{1}{T_a} \\ \Sigma & 1 & \frac{1}{T_a} \end{pmatrix}\right\}$$

Meta-Associativity

$$n = 4;\ \gamma 0 = \Gamma\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

γ0 // dm[1, 2, 1] // dm[1, 3, 1]

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{3}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{4}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

γ0 // dm[2, 3, 2] // dm[1, 2, 1]

$$\begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}) & s_1 \\ s_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{3}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ s_4 & \frac{\alpha_{41}-\alpha_{12}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}-\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\alpha_{11}\,\alpha_{42}+\alpha_{13}\,\alpha_{21}\,\alpha_{42}-\alpha_{11}\,\alpha_{23}\,\alpha_{42}+\alpha_{21}\,\alpha_{43}-\alpha_{12}\,\alpha_{21}\,\alpha_{4}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ \Sigma & \sigma_1\,\sigma_2\,\sigma_3 \end{pmatrix}$$

(γ0 // dm[1, 2, 1] // dm[1, 3, 1]) == (γ0 // dm[2, 3, 2] // dm[1, 2, 1])

True

Cyclicity of tr

$$n = 3;\ \gamma 0 = \Gamma\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

{γ0 // dm[1, 2, 1], γ0 // dm[1, 2, 1] // tr[1]}

$$\left\{\begin{pmatrix} -\omega\,(-1+\alpha_{12}) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{12}} \\ s_3 & \frac{-\alpha_{31}+\alpha_{12}\,\alpha_{31}-\alpha_{11}\,\alpha_{32}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{32}-\alpha_{33}+\alpha_{12}\,\alpha_{33}}{-1+\alpha_{12}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\ \begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}) & \\ s_3 & \frac{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha}{} \\ \Gamma & \end{pmatrix}\right.$$

{γ0 // dm[2, 1, 1], γ0 // dm[2, 1, 1] // tr[1]}

$$\left\{\begin{pmatrix} -\omega\,(-1+\alpha_{21}) & s_1 & s_3 \\ s_1 & \frac{-\alpha_{12}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{21}} & \frac{-\alpha_{13}+\alpha_{13}\,\alpha_{21}-\alpha_{11}\,\alpha_{23}}{-1+\alpha_{21}} \\ s_3 & \frac{-\alpha_{22}\,\alpha_{31}-\alpha_{32}+\alpha_{21}\,\alpha_{32}}{-1+\alpha_{21}} & \frac{-\alpha_{23}\,\alpha_{31}-\alpha_{33}+\alpha_{21}\,\alpha_{33}}{-1+\alpha_{21}} \\ \Gamma & \sigma_1\,\sigma_2 & \sigma_3 \end{pmatrix},\ \begin{pmatrix} \omega\,(1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}) & \\ s_3 & \frac{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha}{} \\ \Gamma & \end{pmatrix}\right.$$

```
(γ0 // dm[1, 2, 1] // tr[1]) == (γ0 // dm[2, 1, 1] // tr[1])
```

True

Testing the MVA

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$$\left(\begin{array}{cc} \frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9} & s_1 \\ s_1 & \frac{T_9\ (1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9 \ldots}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+ \ldots} \\ s_5 & \frac{T_1\ (-1+T_5)\ (1-T_1+T_1 T_5)\ (-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+ \ldots} \\ s_9 & \frac{(-1+T_5)\ T_5\ (-1+T_9)\ (1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)\ (T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+ \ldots} \\ \Gamma & 1 \end{array}\right.$$

```
trZ[L_] := Module[{γ},
  γ = Z[Γ, L];
  Do[
   γ = γ // tr[k],
   {k, Most[γ // dL]}];
  γ[ω] / (T_Last[γ//dL] - 1)]
```

```
trZ[Link["L10a4"]]
```

$$\frac{(-1+T_1)\ (-1+T_5)\ \left(1-3 T_5+3 T_5^2-3 T_5^3+T_5^4\right)}{T_5^2}$$

```
MultivariableAlexander[Link["L10a4"]][T]
```

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$-\frac{1}{\sqrt{T[1]}\ T[2]^{5/2}} (-1+T[1])\ (-1+T[2])\ \left(1-3 T[2]+3 T[2]^2-3 T[2]^3+T[2]^4\right)$$

```
Factor[1/trZ[#] (MultivariableAlexander[#][T] /. T[i_] :> T_Skeleton[#][[i,1]])] & /@
 AllLinks[{2, 7}]
```

$$\left\{-T_1^2 T_3,\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\ T_5^{3/2},\ -T_1^{3/2} \sqrt{T_5},\ -T_1^2 T_7^2,\ -T_1^2 T_7^2,\ \frac{\sqrt{T_1}\ \sqrt{T_9}}{\sqrt{T_5}},\ T_1^{3/2} T_5^{3/2} T_9^{3/2},\ \frac{\sqrt{T_1}}{\sqrt{T_5}\ \sqrt{T_9}},\ -\frac{1}{\sqrt{T_1}\ \sqrt{T_5}},\ -T_1^{3/2} T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1 T_7^2,\ -\frac{1}{T_7},\ \frac{T_1^{3/2} \sqrt{T_5}}{\sqrt{T_9}},\ -\sqrt{T_1}\ T_5^{5/2},\ -\frac{T_5^{3/2}}{\sqrt{T_1}}\right\}$$

The Mirror Properties

n = 3; γ0 = Γ$\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

γ0 // Mirror // dm[1, 2, 1]

$$\begin{pmatrix} -\frac{\omega\,(\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{23}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{21}\alpha_{33}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33})}{\sigma_1\sigma_2\sigma_3} & s_1 \\ s_1 & \frac{\alpha_{13}\alpha_{32}+\alpha_{33}-\alpha_1}{-\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{23}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}} \\ s_3 & \frac{\alpha_{13}\alpha_{22}+\alpha_{23}-\alpha_1}{\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{23}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}} \\ \Gamma & \frac{1}{\sigma_1\sigma_2} \end{pmatrix}$$

γ0 // dm[1, 2, 1] // Mirror

$$\begin{pmatrix} -\frac{\omega\,(\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{23}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{21}\alpha_{33}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33})}{\sigma_1\sigma_2\sigma_3} & s_1 \\ s_1 & \frac{\alpha_{13}\alpha_{32}+\alpha_{33}-\alpha_1}{-\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{23}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}} \\ s_3 & \frac{\alpha_{13}\alpha_{22}+\alpha_{23}-\alpha_1}{\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{23}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}} \\ \Gamma & \frac{1}{\sigma_1\sigma_2} \end{pmatrix}$$

(γ0 // Mirror // dm[1, 2, 1]) == (γ0 // dm[1, 2, 1] // Mirror)

True

{n = 2; γ0 = Γ$\left[\omega,\ \sum_{a=0}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$**,**

t1 = γ0 // Mirror // dS[1], t2 = γ0 // dS[1] // Mirror, t1 == t2}

$$\left\{\begin{pmatrix} \omega & s_1 & s_2 \\ s_1 & \alpha_{11} & \alpha_{12} \\ s_2 & \alpha_{21} & \alpha_{22} \\ \Gamma & \sigma_1 & \sigma_2 \end{pmatrix},\ \begin{pmatrix} \frac{\omega\,\alpha_{22}}{\sigma_2} & s_1 & s_2 \\ s_1 & \frac{-\alpha_{12}\alpha_{21}+\alpha_{11}\alpha_{22}}{\alpha_{22}} & -\frac{\alpha_{21}}{\alpha_{22}} \\ s_2 & \frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{\omega\,\alpha_{22}}{\sigma_2} & s_1 & s_2 \\ s_1 & \frac{-\alpha_{12}\alpha_{21}+\alpha_{11}\alpha_{22}}{\alpha_{22}} & \frac{\alpha_{21}}{\alpha_{22}} \\ s_2 & -\frac{\alpha_{12}}{\alpha_{22}} & \frac{1}{\alpha_{22}} \\ \Gamma & \sigma_1 & \frac{1}{\sigma_2} \end{pmatrix},\ -\frac{\alpha_{21}}{\alpha_{22}} == \frac{\alpha_{21}}{\alpha_{22}}\ \&\&\ \frac{\alpha_{12}}{\alpha_{22}} == -\frac{\alpha_{12}}{\alpha_{22}}\right\}$$

Column Sums

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

```
γ0 // dm[a, b, c]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S\\ s_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\\ \Sigma & \sigma_a\,\sigma_b & \sigma_S \end{pmatrix}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\end{pmatrix}\ \text{// Simplify}$$

$$\left\{\frac{(-1+\beta)\,\gamma + (-1+\beta)\,\phi - \alpha\,(\delta+\psi)}{-1+\beta},\ \frac{\epsilon - \beta\,\epsilon + \delta\,\theta + \Xi - \beta\,\Xi + \theta\,\psi}{1-\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\end{pmatrix}\ /.\ \{\alpha \to s1-\gamma-\phi,\ \delta \to s2-\beta-\psi,\ \Xi \to s3-\theta-\epsilon\}\ \text{// Simplify}$$

$$\left\{\frac{s1\,(-s2+\beta) + (-1+s2)\,(\gamma+\phi)}{-1+\beta},\ \frac{s3\,(-1+\beta) + \theta - s2\,\theta}{-1+\beta}\right\}$$

$$\{1, 1\}.\begin{pmatrix} \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta}\\ \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta}\end{pmatrix}\ /.\ \{\alpha \to s1-\gamma-\phi,\ \delta \to s2-\beta-\psi,\ \Xi \to s3-\theta-\epsilon\}\ /.$$
$$s1\ |\ s2\ |\ s3 \to 1\ \text{// Simplify}$$

{1, 1}

Tangle Concatenation; Γ-inversion

$$\mathbf{n = 3;}\ \Big\{\ \gamma 1 = \Gamma\Big[\omega 1,\ \sum_{a=0}^{n} h_a\, \sigma 1_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \alpha_{10\,a+b}\Big],$$

$$\gamma 2 = \Gamma\Big[\omega 2,\ \sum_{a=0}^{n} h_a\, \sigma 2_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \beta_{10\,a+b}\Big],$$

$$\text{FullStitch}[\gamma 1, \gamma 2],\ \gamma 1 ** \gamma 2,\ \text{FullStitch}[\gamma 1, \gamma 2] == \gamma 1 ** \gamma 2\Big\}$$

$$\left\{\begin{pmatrix} \omega 1 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \Sigma & \sigma 1_1 & \sigma 1_2 & \sigma 1_3 \end{pmatrix},\ \begin{pmatrix} \omega 2 & s_1 & s_2 & s_3 \\ s_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ s_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ s_3 & \beta_{31} & \beta_{32} & \beta_{33} \\ \Sigma & \sigma 2_1 & \sigma 2_2 & \sigma 2_3 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega 1\ \omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11}+\alpha_{21}\beta_{12}+\alpha_{31}\beta_{13} & \alpha_{12}\beta_{11}+\alpha_{22}\beta_{12}+\alpha_{32}\beta_{13} & \alpha_{13}\beta_{11}+\alpha_{23}\beta_{12}+\alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21}+\alpha_{21}\beta_{22}+\alpha_{31}\beta_{23} & \alpha_{12}\beta_{21}+\alpha_{22}\beta_{22}+\alpha_{32}\beta_{23} & \alpha_{13}\beta_{21}+\alpha_{23}\beta_{22}+\alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31}+\alpha_{21}\beta_{32}+\alpha_{31}\beta_{33} & \alpha_{12}\beta_{31}+\alpha_{22}\beta_{32}+\alpha_{32}\beta_{33} & \alpha_{13}\beta_{31}+\alpha_{23}\beta_{32}+\alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega 1\ \omega 2 & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}\beta_{11}+\alpha_{21}\beta_{12}+\alpha_{31}\beta_{13} & \alpha_{12}\beta_{11}+\alpha_{22}\beta_{12}+\alpha_{32}\beta_{13} & \alpha_{13}\beta_{11}+\alpha_{23}\beta_{12}+\alpha_{33}\beta_{13} \\ s_2 & \alpha_{11}\beta_{21}+\alpha_{21}\beta_{22}+\alpha_{31}\beta_{23} & \alpha_{12}\beta_{21}+\alpha_{22}\beta_{22}+\alpha_{32}\beta_{23} & \alpha_{13}\beta_{21}+\alpha_{23}\beta_{22}+\alpha_{33}\beta_{23} \\ s_3 & \alpha_{11}\beta_{31}+\alpha_{21}\beta_{32}+\alpha_{31}\beta_{33} & \alpha_{12}\beta_{31}+\alpha_{22}\beta_{32}+\alpha_{32}\beta_{33} & \alpha_{13}\beta_{31}+\alpha_{23}\beta_{32}+\alpha_{33}\beta_{33} \\ \Sigma & \sigma 1_1\,\sigma 2_1 & \sigma 1_2\,\sigma 2_2 & \sigma 1_3\,\sigma 2_3 \end{pmatrix},\ \text{True}\right\}$$

$$\gamma 1^{-1}$$

$$\left(\begin{matrix} \frac{1}{\omega 1} & s_1 & s_2 \\ s_1 & \frac{\alpha_{23}\alpha_{32}-\alpha_{22}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{32}-\alpha_{12}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}} \\ s_2 & \frac{\alpha_{23}\alpha_{31}-\alpha_{21}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{13}\alpha_{31}-\alpha_{11}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}} \\ s_3 & \frac{\alpha_{22}\alpha_{31}-\alpha_{21}\alpha_{32}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33}} & \frac{\alpha_{12}\alpha_{31}-\alpha_{11}\alpha_{32}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}} \\ \Sigma & \frac{1}{\sigma 1_1} & \frac{1}{\sigma 1_2} \end{matrix}\right.$$

$$\gamma 1 ** \gamma 1^{-1}$$

$$\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & 0 \\ s_2 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 1 \\ \Sigma & 1 & 1 & 1 \end{pmatrix}$$

```
γ1 // dS[1] // dS[2] // dS[3]
```

$$\left(\begin{matrix} -\frac{\omega 1\,(\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}\alpha_{22}\alpha_{33})}{\sigma 1_1\,\sigma 1_2\,\sigma 1_3} & s_1 \\ s_1 & \frac{\alpha_{23}\alpha_{32}-\alpha_{22}\alpha_{33}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}} \\ s_2 & \frac{\alpha_{23}\alpha_{31}-\alpha_{21}\alpha_{33}}{-\alpha_{13}\alpha_{22}\alpha_{31}+\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{11}\alpha_{23}\alpha_{32}-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{11}} \\ s_3 & \frac{\alpha_{22}\alpha_{31}-\alpha_{21}\alpha_{32}}{\alpha_{13}\alpha_{22}\alpha_{31}-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}+\alpha_{11}\alpha_{23}\alpha_{32}+\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{11}} \\ \Sigma & \frac{1}{\sigma 1_1} \end{matrix}\right.$$

$(\gamma 1\ \texttt{//}\ \mathbf{dS}[1]\ \texttt{//}\ \mathbf{dS}[2]\ \texttt{//}\ \mathbf{dS}[3]) == \gamma 1^{-1}\ \texttt{//}\ \mathbf{Simplify}$

$$-\frac{1}{\sigma 1_1\, \sigma 1_2\, \sigma 1_3}\omega 1\ (\alpha_{13}\ (\alpha_{22}\,\alpha_{31} - \alpha_{21}\,\alpha_{32}) + \alpha_{12}\ (-\alpha_{23}\,\alpha_{31} + \alpha_{21}\,\alpha_{33}) + \alpha_{11}\ (\alpha_{23}\,\alpha_{32} - \alpha_{22}\,\alpha_{33})) == \frac{1}{\omega 1}$$

Other

R3

$\{Xm_{51}\, Xm_{62}\, Xp_{34}\ \texttt{//}\ \Gamma\ \texttt{//}\ dm[1, 4, 1]\ \texttt{//}\ dm[2, 5, 2]\ \texttt{//}\ dm[3, 6, 3],$
$Xp_{61}\, Xm_{24}\, Xm_{35}\ \texttt{//}\ \Gamma\ \texttt{//}\ dm[1, 4, 1]\ \texttt{//}\ dm[2, 5, 2]\ \texttt{//}\ dm[3, 6, 3]\}$

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

$\gamma = Xm_{12,1}\, Xm_{27}\, Xm_{83}\, Xm_{4,11}\, Xp_{16,5}\, Xp_{6,13}\, Xp_{14,9}\, Xp_{10,15}\ \texttt{//}\ \Gamma$

1	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
s_1	$\frac{1}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s_2	0	1	0	0	0	0	$\frac{-1+T_2}{T_2}$	0	0	0	0	0	0	0	0	0
s_3	0	0	$\frac{1}{T_8}$	0	0	0	0	0	0	0	0	0	0	0	0	0
s_4	0	0	0	1	0	0	0	0	0	0	$\frac{-1+T_4}{T_4}$	0	0	0	0	0
s_5	0	0	0	0	T_{16}	0	0	0	0	0	0	0	0	0	0	0
s_6	0	0	0	0	0	1	0	0	0	0	0	0	$1-T_6$	0	0	0
s_7	0	0	0	0	0	0	$\frac{1}{T_2}$	0	0	0	0	0	0	0	0	0
s_8	0	0	$\frac{-1+T_8}{T_8}$	0	0	0	0	1	0	0	0	0	0	0	0	0
s_9	0	0	0	0	0	0	0	0	T_{14}	0	0	0	0	0	0	0
s_{10}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$1-T_{10}$	0
s_{11}	0	0	0	0	0	0	0	0	0	0	$\frac{1}{T_4}$	0	0	0	0	0
s_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
s_{13}	0	0	0	0	0	0	0	0	0	0	0	0	T_6	0	0	0
s_{14}	0	0	0	0	0	0	0	0	$1-T_{14}$	0	0	0	0	1	0	0
s_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	T_{10}	0
s_{16}	0	0	0	0	$1-T_{16}$	0	0	0	0	0	0	0	0	0	0	1
Σ	$\frac{1}{T_{12}}$	1	$\frac{1}{T_8}$	1	T_{16}	1	$\frac{1}{T_2}$	1	T_{14}	1	$\frac{1}{T_4}$	1	T_6	1	T_{10}	1

```
Do[γ = γ // dm1k→1, {k, 2, 10}]; γ
```

$\frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2}$	s_1	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_1
s_1	$\frac{T_{14}(-T_1+T_1^2+T_{16})}{T_{12}(T_1^2+T_{16}-T_1 T_{16})}$	$\frac{(-1+T_1)(1-T_1+T_1^2) T_{14} T_{16}}{T_1(T_1^2+T_{16}-T_1 T_{16})}$	0	$-\frac{(-1+T_1)(1-T_1+T_1^2) T_{14}}{T_1^2+T_{16}-T_1 T_{16}}$	0	$1-T_1$	0
s_{11}	0	$\frac{1}{T_1}$	0	0	0	0	0
s_{12}	$\frac{-1+T_{12}}{T_{12}}$	0	1	0	0	0	0
s_{13}	0	0	0	T_1	0	0	0
s_{14}	$-\frac{(-1+T_{14})(-T_1+T_1^2+T_{16})}{T_{12}(T_1^2+T_{16}-T_1 T_{16})}$	$-\frac{(-1+T_1)(1-T_1+T_1^2)(-1+T_{14}) T_{16}}{T_1(T_1^2+T_{16}-T_1 T_{16})}$	0	$\frac{(-1+T_1)(1-T_1+T_1^2)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}}$	1	0	0
s_{15}	0	0	0	0	0	T_1	0
s_{16}	$-\frac{T_1(-1+T_{16})}{T_{12}(T_1^2+T_{16}-T_1 T_{16})}$	$-\frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	$\frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}}$	0	0	1
Σ	$\frac{T_{14} T_{16}}{T_1^2 T_{12}}$	$\frac{1}{T_1}$	1	T_1	1	T_1	1

8_17

```
γ = Xm12,1 Xm27 Xm83 Xm4,11 Xp16,5 Xp6,13 Xp14,9 Xp10,15 // Γ;
Do[γ = γ // dm[1, k, 1], {k, 2, 16}]; γ
```

8_17

$-\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3}$	s_1
s_1	1
Σ	1

```
Z[Γ, Link["L6a4"]]
```

KnotTheory::loading : Loading precomputed data in PD4Links`.

$\frac{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+T_1 T_5 T_9)}{T_1 T_5 T_9}$	s_1
s_1	$\frac{T_9(1-2 T_1+T_1^2-T_5+3 T_1 T_5-T_1^2 T_5-T_9+2 T_1 T_9-T_1^2 T_9+T_5 T_9-2 T_1 T_5 T_9}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+}$
s_5	$\frac{T_1(-1+T_5)(1-T_1+T_1 T_5)(-1+T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+}$
s_9	$\frac{(-1+T_5) T_5 (-1+T_9)(1-2 T_1-T_9+T_1 T_9)}{(1-T_1-T_5+T_1 T_5-T_9+T_1 T_9+T_5 T_9)(T_1+T_5-T_1 T_5+T_9-T_1 T_9-T_5 T_9+}$
Σ	1

```
MVA[Γ, Link["L6a4"]]
```

$$-\frac{(-1+T_1)(-1+T_5)(-1+T_9)}{T_1 T_5}$$

$$\texttt{Factor}\left[\frac{1}{\texttt{MVA}[\Gamma,\ \#]}(\texttt{MultivariableAlexander}[\#][\texttt{T}]\ \texttt{/.}\ \texttt{T}[i\_] \mapsto \texttt{T}_{\texttt{Skeleton}[\#][[i,1]]})\right]\ \texttt{\&}\ \texttt{/@}\ \texttt{AllLinks}[\{2,\ 8\}]$$

KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.

$$\Big\{-T_1^2 T_3,\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -T_1^{3/2}\sqrt{T_5},\ -T_1^2 T_7^2,\ -T_1^2 T_7^2,\ -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2},$$
$$-\frac{\sqrt{T_1}\sqrt{T_5}}{T_9^{3/2}},\ -\sqrt{T_1}\sqrt{T_5},\ -T_1^{3/2} T_5^{7/2},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -\frac{\sqrt{T_1}}{T_5^{3/2}},\ -T_1 T_7^2,\ -\frac{1}{T_7},\ -\frac{T_1^{3/2}\sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2} T_5^{7/2},$$
$$-\sqrt{T_1}\, T_5^{5/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -\frac{\sqrt{T_1}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\, T_5^{3/2},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -\frac{T_1^{3/2}}{\sqrt{T_5}},\ -T_1^{3/2} T_5^{7/2},\ -\frac{T_1}{T_7},$$
$$-T_1 T_7,\ -T_1^2 T_7^3,\ -T_1^2 T_7^3,\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{5/2} T_9^{5/2},\ -T_1^{3/2} T_5^{3/2}\sqrt{T_9},\ -\sqrt{T_1},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$-\frac{\sqrt{T_1}}{T_{11}^2},\ -\frac{\sqrt{T_1}\, T_5}{T_{11}},\ -\frac{T_1^{3/2}\sqrt{T_5}}{T_{13}^{3/2}},\ -T_1^{3/2} T_5^{3/2} T_9^{3/2} T_{13}^{3/2},\ -T_1^{3/2} T_5^{3/2},\ -\sqrt{T_1}\sqrt{T_5},\ -T_1^{3/2} T_5^2 T_{11}^2,$$
$$-\sqrt{T_1}\, T_5^2 T_{11},\ -\sqrt{T_1}\, T_5^{3/2} T_{11}^{3/2},\ -T_1^{3/2} T_5^{5/2}\sqrt{T_{13}},\ -\frac{\sqrt{T_1}\sqrt{T_5}}{\sqrt{T_9}\sqrt{T_{13}}},\ -\sqrt{T_1}\sqrt{T_5}\sqrt{T_9}\sqrt{T_{13}}\Big\}$$

α↔Γ Convertions

```
{Xp[1, 2] // Γ, Xp[1, 2] // A // Γ}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix},\ \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // A, Xm[1, 2] // Γ // A}
```

$$\left\{\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1}(1-e^{c_1})}{c_1} \end{pmatrix},\ \begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{-c_1}(1-e^{c_1})}{c_1} \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\gamma 0 = \Gamma\left[\omega,\ h_a\sigma_a + h_b\sigma_b + h_S\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right];$$

$$\left\{\gamma 0,\ \gamma 0\ \texttt{//}\ \texttt{A},\ \gamma 0\ \texttt{//}\ \texttt{A}\ \texttt{//}\ \Gamma,\ (\gamma 0\ \texttt{//}\ \texttt{A}\ \texttt{//}\ \Gamma)\ \texttt{/.}\ \{\alpha \to 1-\gamma-\phi,\ \delta \to 1-\beta-\psi,\ \Xi \to 1-\theta-\epsilon\}\right\}$$

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix},\ \begin{pmatrix} \omega & h[a] & h[b] & h[S] \\ t[a] & \frac{-\alpha+\sigma_a}{c_a} & -\frac{\beta}{c_a} & -\frac{\theta}{c_a} \\ t[b] & -\frac{\gamma}{c_b} & \frac{-\delta+\sigma_b}{c_b} & -\frac{\epsilon}{c_b} \\ t[S] & -\frac{\phi}{c_S} & -\frac{\psi}{c_S} & \frac{-\Xi+\sigma_S}{c_S} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & 1-\alpha-\gamma-\phi+\sigma_a & 1-\beta-\delta-\psi+\sigma_b & 1-\epsilon-\theta-\Xi+\sigma_S \end{pmatrix},\ \begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & 1-\gamma-\phi & \beta & \theta \\ s_b & \gamma & 1-\beta-\psi & \epsilon \\ s_S & \phi & \psi & 1-\epsilon-\theta \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
γ0 = Γ[ω, ha σa + hb σb + hS σS, {ta, tb, tS}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{ha, hb, hS}];
{γ0 // dm[a, b, c] // A, γ0 // A // dm[a, b, c]} /.
  {α → 1 - γ - ϕ, δ → 1 - β - ψ, Ξ → 1 - θ - ϵ}
```

$$\left\{\begin{pmatrix} \omega-\beta\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\phi-\psi+\gamma\psi+\phi\psi-\sigma_a\sigma_b+\beta\sigma_a\sigma_b}{-c_c+\beta c_c} & \frac{\epsilon-\beta\epsilon+\theta-\beta\theta-\theta\psi}{-c_c+\beta c_c} \\ t[S] & \frac{\phi-\beta\phi+\psi-\gamma\psi-\phi\psi}{-c_S+\beta c_S} & \frac{1-\beta-\epsilon+\beta\epsilon-\theta+\beta\theta+\theta\psi-\sigma_S+\beta\sigma_S}{-c_S+\beta c_S}\end{pmatrix}, \begin{pmatrix} \omega-\beta\omega & h[c] & h[S] \\ t[c] & \frac{1-\beta-\phi+\beta\phi-\psi+\gamma\psi+\phi\psi-\sigma_a\sigma_b+\beta\sigma_a\sigma_b}{-c_c+\beta c_c} & \frac{\epsilon-\beta\epsilon+\theta-\beta\theta-\theta\psi}{-c_c+\beta c_c} \\ t[S] & \frac{\phi-\beta\phi+\psi-\gamma\psi-\phi\psi}{-c_S+\beta c_S} & \frac{1-\beta-\epsilon+\beta\epsilon-\theta+\beta\theta+\theta\psi-\sigma_S+\beta\sigma_S}{-c_S+\beta c_S}\end{pmatrix}\right\}$$

The KV solution in Γ, starting from α

```
V // A // αCollect[FullSimplify]
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] & \\ t[1] & \frac{-\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \\ t[2] & \frac{\sqrt{2}\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}-2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}} & \frac{-e^{\frac{c_1}{2}}c_1\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}+e^{\frac{3c_1}{2}+c_2}c_1\sqrt{\cdots}}{\cdots} \end{pmatrix}$$

```
V // A // Γ //
 ΓCollect[Assuming[T1 > 0 && T2 > 0, (# /. {Sinh[x_] :> (e^x - e^-x)/2}) // FullSimplify] &]
```

$$\begin{pmatrix} \frac{\left(\frac{-1+T_1}{\text{Log}[T_1]}\right)^{1/4}\left(\frac{-1+T_2}{\text{Log}[T_2]}\right)^{1/4}}{\left(\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}\right)^{1/4}} & s_1 & s_2 \\ s_1 & \frac{\text{Log}[T_1]\left(\text{Log}[T_2]\sqrt{\frac{(-1+T_1)T_1}{\text{Log}[T_1]}}\sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}-\sqrt{\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}}+T_1\sqrt{\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}}\right)}{\text{Log}[T_1T_2](-1+T_1)\sqrt{\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}}} & -\frac{\text{Log}[T_1]\left(\sqrt{\frac{(-1+T_1)T_1}{\text{Log}[T_1]}}\,T_2-\sqrt{\cdots}\right.}{\text{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}\sqrt{\cdots}} \\ s_2 & \frac{\text{Log}[T_2]\left(-T_1\sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}+\sqrt{\frac{(-1+T_1)T_1}{\text{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}}\right)}{\text{Log}[T_1T_2]\sqrt{\frac{(-1+T_1)T_1}{\text{Log}[T_1]}}\sqrt{\frac{-1+T_1T_2}{\text{Log}[T_1T_2]}}} & \frac{\text{Log}[T_1]\sqrt{\frac{(-1+T_1)T_1}{\text{Log}[T_1]}}\,T_2+\text{Log}[T_2]\sqrt{\cdots}}{\text{Log}[T_1T_2]\sqrt{\frac{-1+T_2}{\text{Log}[T_2]}}\sqrt{\cdots}} \\ \Sigma & 1 & \sqrt{T_1} \end{pmatrix}$$

$$\text{ΓSimp} = \text{Assuming}\left[T_1 > 1 \;\&\&\; T_2 > 1, \left(\# \;/.\; \left\{\text{Sinh}[x\_] \mapsto \frac{e^x - e^{-x}}{2}\right\}\right) \;//\; \text{FullSimplify}\right] \;\&;$$

V // A // Γ

$$\begin{pmatrix} \left(\frac{\text{Log}[T_1 T_2]\,(-1+T_1)\,(-1+T_2)}{\text{Log}[T_1]\,\text{Log}[T_2]\,(-1+T_1 T_2)}\right)^{1/4} & s_1 & s_2 \\ s_1 & \frac{\text{Log}[T_1]+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,T_1\,(-1+T_2)}{(-1+T_1)\,(-1+T_1 T_2)}}}{\text{Log}[T_1 T_2]} & \frac{\text{Log}[T_1]\left(-1+T_2\left(T_1-\sqrt{\frac{\text{Log}[T_2]\,\text{Log}[T \ldots}{\text{Lo}\ldots}}\right.\right.}{\text{Log}[T_1 T_2]\,(-1+\ldots} \\ s_2 & -\frac{\text{Log}[T_2]\left(-1+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_1 T_2]\,T_1\,(-1+T_2)}{\text{Log}[T_2]\,(-1+T_1)\,(-1+T_1 T_2)}}\right)}{\text{Log}[T_1 T_2]} & \frac{-\text{Log}[T_2]+T_2\left(\text{Log}[T_2]\,T_1+\sqrt{\frac{\text{Log}[T_1]^3\,\text{Log}[T_2]\,(-1+T_1)\,T_1\,(-1\ldots}{\text{Log}[T_1 T_2]\,(-1+T_2)}}\right.}{\text{Log}[T_1 T_2]\,(-1+\ldots} \\ \Sigma & 1 & \sqrt{T_1} \end{pmatrix}$$

Γ[V] ** Γ[Vi]

$$\begin{pmatrix} 1 & s_1 \\ s_1 & \frac{-\text{Log}[T_1 T_2]+T_2\left(\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_1)\,T_1\,(-1+T_2)}{-1+T_1 T_2}}+T_1\left(\text{Log}[T_1 T_2]+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}-T_1\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(\ldots}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}\right.\right.}{\text{Log}[T_1 T_2]\,(-1+T_1 T_2)} \\ s_2 & \frac{(-1+T_2)\left(-(-1+T_1)\,T_1\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_1)\,T_1\,(-1+T_2)}{-1+T_1 T_2}}\right)}{\text{Log}[T_1 T_2]\,(-1+T_1)\,(-1+T_1 T_2)} \\ \Sigma & 1 \end{pmatrix}$$

$$\frac{1}{\text{Log}[T_1 T_2]\,(-1+T_1 T_2)}\left(-\text{Log}[T_1 T_2]+T_2\left(\sqrt{\left(\frac{1}{-1+T_1 T_2}\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_1)\,T_1\,(-1+T_2)\right)}+T_1\left(\text{Log}[T_1 T_2]+\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}-T_1\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}\right)\right)\right) \;//\; \text{PowerExpand} \;//\; \text{Simplify}$$

1

$$\left((-1+T_2)\left(-(-1+T_1)\,T_1\sqrt{\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_2)}{(-1+T_1)\,T_1\,(-1+T_1 T_2)}}+\sqrt{\left(\frac{\text{Log}[T_1]\,\text{Log}[T_2]\,\text{Log}[T_1 T_2]\,(-1+T_1)\,T_1\,(-1+T_2)}{-1+T_1 T_2}\right)}\right)\right)\Big/\left(\text{Log}[T_1 T_2]\,(-1+T_1)\,(-1+T_1 T_2)\right) \;//\; \text{PowerExpand} \;//\; \text{Simplify}$$

0

dS and dA for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
((γ0 // A // dS[a] // Γ) == (γ0 // A // dA[a] // Γ)) /. {α → 1 - ϕ, Ξ → 1 - θ}
```

True

```
(γ0 // A // dS[a] // Γ)
```

$$\begin{pmatrix} \frac{(-1+\phi)\ \omega}{-1+\alpha+\phi-\sigma_a} & s_a & s_S \\ s_a & -\frac{1}{-1+\phi} & -\frac{\theta}{-1+\phi} \\ s_S & \frac{\phi}{-1+\phi} & \frac{-1+\theta+\phi}{-1+\phi} \\ \Sigma & -\frac{1}{-1+\alpha+\phi-\sigma_a} & -\frac{1-\alpha-\theta+\alpha\,\theta-\Xi+\alpha\,\Xi-\phi+\theta\,\phi+\Xi\,\phi+\sigma_a-\theta\,\sigma_a-\Xi\,\sigma_a+\sigma_S-\alpha\,\sigma_S-\phi\,\sigma_S+\sigma_a\,\sigma_S}{-1+\alpha+\phi-\sigma_a} \end{pmatrix}$$

```
(γ0 // A // dA[a] // Γ) /. {ϕ → 1 - α, Ξ → 1 - θ} // ΓCollect
```

$$\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_a & s_S \\ s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

```
(γ0 // dA[a]) /. {ϕ → 1 - α, Ξ → 1 - θ} // ΓCollect
```

$$\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_a & s_S \\ s_a & \frac{1}{\alpha} & \frac{\theta}{\alpha} \\ s_S & \frac{-1+\alpha}{\alpha} & \frac{\alpha-\theta}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}$$

```
(γ0 // A // dA[a] // Γ) == (γ0 // dA[a]) /. {ϕ → 1 - α, Ξ → 1 - θ} // Simplify
```

True

dΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, ha σa + hS σS, {ta, tS}.(α θ; ϕ Ξ).{ha, hS}];
((γ0 // A // dΔ[a, b, c] // Γ) /. {α → 1 - ϕ, Ξ → 1 - θ}) // ΓCollect
```

$$\begin{pmatrix} \omega & s_b & s_c & s_S \\ s_b & -\frac{-\text{Log}[T_b]+\phi\,\text{Log}[T_b]-\text{Log}[T_c]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & -\frac{\text{Log}[T_b]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_b]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ s_c & -\frac{\text{Log}[T_c]\,(-1+\phi+\sigma_a)}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\text{Log}[T_c]-\phi\,\text{Log}[T_c]+\text{Log}[T_b]\,\sigma_a}{\text{Log}[T_b]+\text{Log}[T_c]} & \frac{\theta\,\text{Log}[T_c]}{\text{Log}[T_b]+\text{Log}[T_c]} \\ s_S & \phi & \phi & 1-\theta \\ \Sigma & \sigma_a & \sigma_a & \sigma_S \end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, ha σa + hS σS, {ta, tS}.(α θ; ϕ Ξ).{ha, hS}];
γ0 // A // dΔ[a, b, c] // dS[c] // dm[b, c, a]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity + $\frac{(\Xi - \theta\phi - \Xi\phi - \sigma_S + \phi\sigma_S)\,t[S]}{-c_S + \phi c_S}$ encountered. ≫

$$\left(\frac{-\omega+\phi\,\omega}{-1+\alpha+\phi-\sigma_a} \right)$$

qΔ for Γ, starting from α

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, ha σa + hS σS, {ta, tS}.(α θ; ϕ Ξ).{ha, hS}];
γ0 // A // qΔ[a, b, c] // Γ
```

$$\begin{pmatrix} \omega & s_b & s_c & \\ s_b & -\frac{1-\alpha-\phi+\alpha\,T_c-T_b\,T_c+\phi\,T_b\,T_c+\sigma_a-T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & \frac{\theta\,(}{-} \\ s_c & \frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b\,T_c} & -\frac{1-\phi-\alpha\,T_c-T_b\,T_c+\alpha\,T_b\,T_c+\phi\,T_b\,T_c+T_c\,\sigma_a-T_b\,T_c\,\sigma_a}{-1+T_b\,T_c} & \frac{\theta}{-} \\ s_S & \phi & \phi & \\ \Sigma & 1-\alpha-\phi+\sigma_a & 1-\alpha-\phi+\sigma_a & 1-(\end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // qΔ[a, b, c]
```

$$\begin{pmatrix} \omega & s_b & s_c & s_S \\ s_b & \frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c \sigma_a}{-1+T_b T_c} & \frac{(-1+T_b)\, T_c\, (\alpha-\sigma_a)}{-1+T_b T_c} & \frac{\theta\, (-1+T_b)\, T_c}{-1+T_b T_c} \\ s_c & \frac{(-1+T_c)\, (\alpha-\sigma_a)}{-1+T_b T_c} & \frac{-\alpha+\alpha T_c-T_c \sigma_a+T_b T_c \sigma_a}{-1+T_b T_c} & \frac{\theta\, (-1+T_c)}{-1+T_b T_c} \\ s_S & \phi & \phi & \Xi \\ \Sigma & \sigma_a & \sigma_a & \sigma_S \end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
Simplify[
  (γ0 // qΔ[a, b, c])[[3]] == (γ0 // A // qΔ[a, b, c] // Γ)[[3]] /. {α → 1 - ϕ, θ → 1 - Ξ}]
```

True

---

qΔ tests for Γ

```
{t1 = Xp13 // Γ // qΔ[1, 1, 2], t2 = (ϵ[1] Xp23 // Γ) ** (ϵ[2] Xp13 // Γ), t1 == t2}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1+T_1)\, T_2 \\ s_2 & 0 & 1 & 1-T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & -(-1+T_1)\, T_2 \\ s_2 & 0 & 1 & 1-T_2 \\ s_3 & 0 & 0 & T_1 T_2 \\ \Sigma & 1 & 1 & T_1 T_2 \end{pmatrix}, \text{True}\right\}$$

```
{t1 = Xm13 // Γ // qΔ[1, 1, 2], t2 = (ϵ[2] Xm13 // Γ) ** (ϵ[1] Xm23 // Γ), t1 == t2}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & 1 & 0 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & 1 & \frac{-1+T_2}{T_1 T_2} \\ s_3 & 0 & 0 & \frac{1}{T_1 T_2} \\ \Sigma & 1 & 1 & \frac{1}{T_1 T_2} \end{pmatrix}, \text{True}\right\}$$

```
{t1 = Xp3,1 // Γ // qΔ[1, 1, 2], t2 = (ϵ[1] Xp3,2 // Γ) ** (ϵ[2] Xp3,1 // Γ), t1 == t2}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1-T_3 & 1-T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & T_3 & 0 & 0 \\ s_2 & 0 & T_3 & 0 \\ s_3 & 1-T_3 & 1-T_3 & 1 \\ \Sigma & T_3 & T_3 & 1 \end{pmatrix}, \text{True}\right\}$$

```
{t1 = Xm3,1 // Γ // qΔ[1, 1, 2], t2 = (ϵ[2] Xm3,1 // Γ) ** (ϵ[1] Xm3,2 // Γ), t1 == t2}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{1}{T_3} & 0 & 0 \\ s_2 & 0 & \frac{1}{T_3} & 0 \\ s_3 & \frac{-1+T_3}{T_3} & \frac{-1+T_3}{T_3} & 1 \\ \Sigma & \frac{1}{T_3} & \frac{1}{T_3} & 1 \end{pmatrix}, \text{True}\right\}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
{γ0 = Γ[ω, ha σa + hb σb + hS σS, {ta, tb, tS}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{ha, hb, hS}], γ0 // dm[a, b, c]}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta} \\ \Sigma & \sigma_a\sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
γ0 // dm[a, b, c] // qΔ[c, c1, c2]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_{c1} & \cdots \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)(-1+T_{c1}T_{c2})} & (-1+T_{c1})\,\cdots \\ s_{c2} & \frac{(-1+T_{c2})(-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)(-1+T_{c1}T_{c2})} & \gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_c\cdots \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Sigma & \sigma_a\sigma_b & \end{pmatrix}$$

```
γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] // dm[a2, b2, c2]
```

$$\begin{pmatrix} -(-1+\beta)\,\omega & s_{c1} & \cdots \\ s_{c1} & \frac{\gamma T_{c2}-\beta\gamma T_{c2}+\alpha\delta T_{c2}-\gamma T_{c1} T_{c2}+\beta\gamma T_{c1} T_{c2}-\alpha\delta T_{c1} T_{c2}+\sigma_a\sigma_b-\beta\sigma_a\sigma_b-T_{c2}\sigma_a\sigma_b+\beta T_{c2}\sigma_a\sigma_b}{(-1+\beta)(-1+T_{c1}T_{c2})} & (-1+T_{c1})\,\cdots \\ s_{c2} & \frac{(-1+T_{c2})(-\gamma+\beta\gamma-\alpha\delta+\sigma_a\sigma_b-\beta\sigma_a\sigma_b)}{(-1+\beta)(-1+T_{c1}T_{c2})} & \gamma-\beta\gamma+\alpha\delta-\gamma T_{c2}+\beta\gamma T_{c2}-\alpha\delta T_c\cdots \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \\ \Sigma & \sigma_a\sigma_b & \end{pmatrix}$$

```
(γ0 // dm[a, b, c] // qΔ[c, c1, c2]) ==
  (γ0 // qΔ[a, a1, a2] // qΔ[b, b1, b2] // dm[a1, b1, c1] //
    dm[a2, b2, c2]) // Simplify
```

True

dS tests for Γ

```
{Xp[1, 2] // Γ, Xm[1, 2] // Γ // dS[1], Xm[1, 2] // Γ // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // Γ, Xp[1, 2] // Γ // dS[1], Xp[1, 2] // Γ // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

```
Xp[1, 2] // Γ // dS[1] // dS[2]
```

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}$$

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma\left[\omega, h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S, \{t_a, t_b, t_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{h_a, h_b, h_S\}\right],\right.$$
$$\left. t1 = \gamma 0 \;//\; dm[a, b, c] \;//\; dS[c],\; t2 = \gamma 0 \;//\; dS[a] \;//\; dS[b] \;//\; dm[b, a, c],\; t1 == t2\right\}$$

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Sigma & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b} & s_C & s_S \\ s_C & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b} & s_C & s_S \\ s_C & \frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta} \\ s_S & -\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta} & \frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta} \\ \Sigma & \frac{1}{\sigma_a\,\sigma_b} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
```

$$\gamma 0 = \Gamma\left[\omega, h_a\,\sigma_a + h_S\,\sigma_S, \{t_a, t_S\}.\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix}.\{h_a, h_S\}\right];$$

```
FullSimplify[{1, 1}.dS[a][γ0][A], And @@ Thread[{1, 1}.γ0[A] == {1, 1}]]
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{1-\phi}{\alpha}\right\}$$

```
{1, 1}.dS[a][γ0][A] // Simplify
```

$$\left\{\frac{1-\phi}{\alpha}, \frac{\theta+\alpha\,\Xi-\theta\,\phi}{\alpha}\right\}$$

```
And @@ Thread[{1, 1}.γ0[A] == {1, 1}]
```

$\alpha + \phi == 1 \;\&\&\; \theta + \Xi == 1$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
γ0 // dS[a] // dS[a]
```

$$\begin{pmatrix} \omega & s_a & s_S \\ s_a & \alpha & \theta \\ s_S & \phi & \Xi \\ \Sigma & \sigma_a & \sigma_S \end{pmatrix}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α[T_a] θ[T_a]; ϕ[T_a] Ξ[T_a]).{h_a, h_S}];
{(γ0 // dη[a]) (ϵ[a] // Γ), γ0 // qΔ[a, b, c] // dS[c] // dm[b, c, a],
 γ0 // qΔ[a, b, c] // dS[c] // dm[c, b, a],
 γ0 // qΔ[a, b, c] // dS[b] // dm[b, c, a], γ0 // qΔ[a, b, c] // dS[b] // dm[c, b, a]}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & 0 \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & \theta[1] \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \omega & s_a & s_S \\ s_a & 1 & \theta[1] \\ s_S & 0 & \Xi[1] \\ \Sigma & 1 & \sigma_S \end{pmatrix}, \begin{pmatrix} \frac{\omega\,\alpha[1]}{\sigma_a} & s_a & s_S \\ s_a & 1 & \frac{\theta[1]}{\alpha[1]} \\ s_S & 0 & \frac{\alpha[1]\,\Xi[1]-\theta[1]\,\phi[1]}{\alpha[1]} \\ \Sigma & 1 & \sigma_S \end{pmatrix}\right\}$$

```
{Xp[S, a] // Γ, Xp[S, a] // Γ // qΔ[a, b, c] // dS[c] // dm[c, b, a]}
```

$$\left\{\begin{pmatrix} 1 & s_a & s_S \\ s_a & T_S & 0 \\ s_S & 1-T_S & 1 \\ \Sigma & T_S & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_a & s_S \\ s_a & 1 & 0 \\ s_S & 0 & 1 \\ \Sigma & 1 & 1 \end{pmatrix}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
γ0 = Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}];
{t1 = γ0 // qΔ[a, b, c] // dS[b] // dS[c],
 t2 = γ0 // dS[a] // qΔ[a, c, b], Simplify[t1 == t2]}
```

$$\left\{\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{\alpha\,\omega}{\sigma_a} & s_b & s_c & s_S \\ s_b & \frac{-\alpha\,T_b+\alpha\,T_b\,T_c-\sigma_a+T_b\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & -\frac{(-1+T_b)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,(-1+T_b)}{\alpha\,(-1+T_b\,T_c)} \\ s_c & -\frac{T_b\,(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{-\alpha+\alpha\,T_b-T_b\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a} & \frac{\theta\,T_b\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)} \\ s_S & -\frac{\phi}{\alpha} & -\frac{\phi}{\alpha} & \frac{\alpha\,\Xi-\theta\,\phi}{\alpha} \\ \Sigma & \frac{1}{\sigma_a} & \frac{1}{\sigma_a} & \sigma_S \end{pmatrix}, \text{True}\right\}$$

dA tests for Γ

```
{Xp[1, 2] // Γ, (Xm[1, 2] // Γ) /. T_1 → 1/T_1, Xm[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Sigma & 1 & T_1 \end{pmatrix}\right\}$$

```
{Xm[1, 2] // Γ, (Xp[1, 2] // Γ) /. T_1 → 1/T_1, Xp[1, 2] // Γ // dA[1] // dA[2]}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & \frac{-1+T_1}{T_1} \\ s_2 & 0 & \frac{1}{T_1} \\ \Sigma & 1 & \frac{1}{T_1} \end{pmatrix}\right\}$$

Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];

$$\left\{\gamma 0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_b\,\sigma_b+h_S\,\sigma_S,\ \{t_a,t_b,t_S\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{h_a,h_b,h_S\}\right],\right.$$
$$\left.t1=\gamma 0\ /\!/\ \mathrm{dm}[a,b,c]\ /\!/\ \mathrm{dA}[c],\ t2=\gamma 0\ /\!/\ \mathrm{dA}[a]\ /\!/\ \mathrm{dA}[b]\ /\!/\ \mathrm{dm}[b,a,c],\ t1=\!=t2\right\}$$

$$\left\{\begin{pmatrix}\omega&s_a&s_b&s_S\\s_a&\alpha&\beta&\theta\\s_b&\gamma&\delta&\epsilon\\s_S&\phi&\psi&\Xi\\\Sigma&\sigma_a&\sigma_b&\sigma_S\end{pmatrix},\ \begin{pmatrix}-\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b}&s_c&s_S\\s_c&\frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta}\\s_S&-\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta}\\\Sigma&\frac{1}{\sigma_a\,\sigma_b}&\sigma_S\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}-\frac{(-\gamma+\beta\gamma-\alpha\delta)\,\omega}{\sigma_a\,\sigma_b}&s_c&s_S\\s_c&\frac{-1+\beta}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\epsilon+\beta\epsilon-\delta\theta}{-\gamma+\beta\gamma-\alpha\delta}\\s_S&-\frac{-\phi+\beta\phi-\alpha\psi}{-\gamma+\beta\gamma-\alpha\delta}&\frac{-\gamma\Xi+\beta\gamma\Xi-\alpha\delta\Xi+\epsilon\phi-\beta\epsilon\phi+\delta\theta\phi+\alpha\epsilon\psi-\gamma\theta\psi}{-\gamma+\beta\gamma-\alpha\delta}\\\Sigma&\frac{1}{\sigma_a\,\sigma_b}&\sigma_S\end{pmatrix},\ \mathrm{True}\right\}$$

Clear[α, θ, ϕ, Ξ, ω];

$$\gamma 0=\Gamma\left[\omega,\ h_a\,\sigma_a+h_S\,\sigma_S,\ \{t_a,t_S\}.\begin{pmatrix}\alpha&\theta\\\phi&\Xi\end{pmatrix}.\{h_a,h_S\}\right];$$

$$\{t1=\gamma 0\ /\!/\ \mathrm{q}\Delta[a,b,c]\ /\!/\ \mathrm{dA}[b]\ /\!/\ \mathrm{dA}[c],$$
$$t2=\gamma 0\ /\!/\ \mathrm{dA}[a]\ /\!/\ \mathrm{q}\Delta[a,b,c],\ \mathrm{Simplify}[t1=\!=t2]\}$$

$$\left\{\begin{pmatrix}\frac{\alpha\,\omega}{\sigma_a}&s_b&s_c&s_S\\s_b&\frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&-\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{\theta\,(-1+T_b)\,T_c}{\alpha\,(-1+T_b\,T_c)}\\s_c&-\frac{(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{\theta\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)}\\s_S&-\frac{\phi}{\alpha}&-\frac{\phi}{\alpha}&\frac{\alpha\,\Xi-\theta\,\phi}{\alpha}\\\Sigma&\frac{1}{\sigma_a}&\frac{1}{\sigma_a}&\sigma_S\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\frac{\alpha\,\omega}{\sigma_a}&s_b&s_c&s_S\\s_b&\frac{-\alpha+\alpha\,T_c-T_c\,\sigma_a+T_b\,T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&-\frac{(-1+T_b)\,T_c\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{\theta\,(-1+T_b)\,T_c}{\alpha\,(-1+T_b\,T_c)}\\s_c&-\frac{(-1+T_c)\,(\alpha-\sigma_a)}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{-\alpha\,T_c+\alpha\,T_b\,T_c-\sigma_a+T_c\,\sigma_a}{\alpha\,(-1+T_b\,T_c)\,\sigma_a}&\frac{\theta\,(-1+T_c)}{\alpha\,(-1+T_b\,T_c)}\\s_S&-\frac{\phi}{\alpha}&-\frac{\phi}{\alpha}&\frac{\alpha\,\Xi-\theta\,\phi}{\alpha}\\\Sigma&\frac{1}{\sigma_a}&\frac{1}{\sigma_a}&\sigma_S\end{pmatrix},\ \mathrm{True}\right\}$$

$$n=4;\ \gamma 0=\Gamma\left[\omega,\ \sum_{a=0}^{n}h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n}t_a\,h_b\,\alpha_{10\,a+b}\right]$$

$$\begin{pmatrix}\omega&s_1&s_2&s_3&s_4\\s_1&\alpha_{11}&\alpha_{12}&\alpha_{13}&\alpha_{14}\\s_2&\alpha_{21}&\alpha_{22}&\alpha_{23}&\alpha_{24}\\s_3&\alpha_{31}&\alpha_{32}&\alpha_{33}&\alpha_{34}\\s_4&\alpha_{41}&\alpha_{42}&\alpha_{43}&\alpha_{44}\\\Sigma&\sigma_1&\sigma_2&\sigma_3&\sigma_4\end{pmatrix}$$

```
γ0 // dA[1] // dA[2] // dA[3] // dA[4]
```

$$\left(\omega\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}-\alpha \ldots\right.$$

```
(γ0 // dA[1] // dA[2] // dA[3] // dA[4]) ** γ0
```

$$\left(\omega^2\ (\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}-\alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}-\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}+\alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41}+\alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}-\alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41}-\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}+\alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}+\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}- \ldots\right.$$

$$\begin{aligned}(&\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41} - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41} - \alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41} + \alpha_{12}\,\alpha_{24}\,\alpha_{33}\,\alpha_{41} + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41} - \alpha_{12}\,\alpha_{23}\,\alpha_{34}\,\alpha_{41} -\\ &\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42} + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42} + \alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42} - \alpha_{11}\,\alpha_{24}\,\alpha_{33}\,\alpha_{42} - \alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42} +\\ &\alpha_{11}\,\alpha_{23}\,\alpha_{34}\,\alpha_{42} + \alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43} - \alpha_{12}\,\alpha_{24}\,\alpha_{31}\,\alpha_{43} - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43} + \alpha_{11}\,\alpha_{24}\,\alpha_{32}\,\alpha_{43} +\\ &\alpha_{12}\,\alpha_{21}\,\alpha_{34}\,\alpha_{43} - \alpha_{11}\,\alpha_{22}\,\alpha_{34}\,\alpha_{43} - \alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44} + \alpha_{12}\,\alpha_{23}\,\alpha_{31}\,\alpha_{44} + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44} -\\ &\alpha_{11}\,\alpha_{23}\,\alpha_{32}\,\alpha_{44} - \alpha_{12}\,\alpha_{21}\,\alpha_{33}\,\alpha_{44} + \alpha_{11}\,\alpha_{22}\,\alpha_{33}\,\alpha_{44}) == \mathrm{Det}\left[\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14}\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24}\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34}\\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}\end{pmatrix}\right]\end{aligned}$$

```
True
```

Θ tests

```
Θ[1, 2] // A
```

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1 + e^{\frac{c_2}{2}}\, c_1 - e^{\frac{c_1}{2}+\frac{c_2}{2}}\, c_2 + e^{\frac{c_2}{2}}\, c_2}{c_1^2 + c_1\, c_2} & \frac{-1 + e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1 + c_2} \\ t[2] & \frac{-1 + e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1 + c_2} & \frac{e^{\frac{c_1}{2}}\, c_1 - e^{\frac{c_1}{2}+\frac{c_2}{2}}\, c_1 - c_2 + e^{\frac{c_1}{2}}\, c_2}{c_1\, c_2 + c_2^2} \end{pmatrix}$$

(V // A) ** (Θ[1, 2] // A)

$\dfrac{2^{1/4}\left(\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}\right)^{1/4}}$	h[1]
t[1]	$\dfrac{-\sqrt{2}\, e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}}\, c_2 - 2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2\, e^{\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2\, e^{\frac{c_2}{2}} \ldots}{2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1^2\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 c_2\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}$
t[2]	$\dfrac{\sqrt{2}\, e^{\frac{c_1}{2}+\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}} - 2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}} + 2\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_2\sqrt{\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}$

(Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1])

$\dfrac{2^{1/4}\left(\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}\right)^{1/4}}$	h[1
t[1]	$\dfrac{\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 - e^{\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 - e^{c_1+c_2}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 + e^{c_1+\frac{3c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 - e^{\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_2 + e^{c_1+\frac{3c_2}{2}} \ldots}{-\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1^2 + e^{c_1+c_2}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1^2 - \ldots}$
t[2]	$\dfrac{\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} - e^{c_1+c_2}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}} + \sqrt{2}\, e^{c_1+c_2} c_1\sqrt{\frac{\text{Sinh}[\frac{c_2}{2}]}{c_2}} \ldots}{-\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 + e^{c_1+c_2}\sqrt{\frac{\text{Sinh}[\frac{c_1}{2}]}{c_1}}\, c_1 - \ldots}$

(V // A) ** (Θ[1, 2] // A) == (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify

True

{t1 = Θ[1, 2] // A // Γ, t2 = Θi[1, 2] // A // Γ, t1 ** t2}

{

1	s_1	s_2
s_1	$\frac{\text{Log}[T_1]+\text{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]}$	$-\frac{\text{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]}$
s_2	$-\frac{\text{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{\text{Log}[T_1]+\text{Log}[T_2]}$	$\frac{\text{Log}[T_2]+\text{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{\text{Log}[T_1]+\text{Log}[T_2]}$
Σ	$\sqrt{T_2}$	$\sqrt{T_1}$

,

1	s_1	s_2
s_1	$\frac{\text{Log}[T_2]+\text{Log}[T_1]\sqrt{T_1}\sqrt{T_2}}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}}$	$\frac{\text{Log}[T_1]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}}$
s_2	$\frac{\text{Log}[T_2]\left(-1+\sqrt{T_1}\sqrt{T_2}\right)}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}}$	$\frac{\text{Log}[T_1]+\text{Log}[T_2]\sqrt{T_1}\sqrt{T_2}}{(\text{Log}[T_1]+\text{Log}[T_2])\sqrt{T_1}\sqrt{T_2}}$
Σ	$\frac{1}{\sqrt{T_2}}$	$\frac{1}{\sqrt{T_1}}$

,

1	s_1	s_2
s_1	1	0
s_2	0	1
Σ	1	1

}

```
(V // A) ** (Θi[1, 2] // A) ==
  (Xm[2, 1] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // FullSimplify
```

```
True
```

```
(Θ[1, 2] // A) == (Vi // A) ** (Xp[1, 2] // A) ** (V // A // dσ[1 → 2, 2 → 1]) // Simplify
```

Simplify::time :

Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

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Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

Simplify::time :

Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

General::stop : Further output of Simplify::time will be suppressed during this calculation. ≫

```
$Aborted
```

```
(Xp[2, 1] // A) == (V // A // dσ[1 → 2, 2 → 1]) ** (Θ[1, 2] // A) ** (Vi // A) // Simplify
```

```
True
```

Γb-Calculus

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
```

$$\left\{\gamma 0 = \Gamma b\left[\omega,\ h_a\,\sigma_a + h_b\,\sigma_b + h_S\,\sigma_S,\ \{t_a, t_b, t_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{h_a, h_b, h_S\}\right],\ \gamma 0\ //\ \Gamma\ //\ dm[a, b, c]\ //\ \Gamma b\right\}$$

$$\left\{\begin{pmatrix}\omega & s_a & s_b & s_S\\ s_a & \alpha & \beta & \theta\\ s_b & \gamma & \delta & \epsilon\\ s_S & \phi & \psi & \Xi\\ \Sigma & \sigma_a & \sigma_b & \sigma_S\end{pmatrix},\ \begin{pmatrix}-\beta+\omega & s_c & s_S\\ s_c & \frac{-\beta\,\gamma+\alpha\,\delta+\gamma\,\omega}{\omega} & \frac{-\beta\,\epsilon+\delta\,\theta+\epsilon\,\omega}{\omega}\\ s_S & \frac{-\beta\,\phi+\alpha\,\psi+\phi\,\omega}{\omega} & \frac{-\beta\,\Xi+\theta\,\psi+\Xi\,\omega}{\omega}\\ \Sigma & \sigma_a\,\sigma_b & \sigma_S\end{pmatrix}\right\}$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &]
```

$$\begin{pmatrix}\frac{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])^{1/4}\,(-1+T_1)^{1/4}\,(-1+T_2)^{1/4}}{\mathrm{Log}[T_1]^{1/4}\,\mathrm{Log}[T_2]^{1/4}\,(-1+T_1\,T_2)^{1/4}} & \frac{\mathrm{Log}[T_1]^{1/4}\left(\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\,\sqrt{-1+T_2}-\sqrt{\mathrm{Log}[T_1]}\,\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\ T\right.}{}\\ s_1 & \\ s_2 & \\ \Sigma & \end{pmatrix}$$

```
V // A // Γb // ΓbCollect[FullSimplify[PowerExpand[#]] &] // ΓbCollect[
  Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a :> e^(c_a)]]] &]
```

$$\begin{pmatrix} \frac{((c_1+c_2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(c_1 c_2(-1+T_1T_2))^{1/4}} & & \\ s_1 & \sqrt{c_1(c_1+c_2)(-1+T_2)} - T_1\sqrt{c_1(c_1+c_2)(-1+T_2)} - T_1\sqrt{c_1(c_1+c_2)(-1+T_2)}\; T_2 + T_1^2\sqrt{c_1(c_1+c_2)(-1+T_2 \ldots} & \ldots(-1 \\ s_2 & & \frac{c_2^{1/4}(-1+T_2)^{1/\ldots}}{\ldots} \\ \Sigma & & \end{pmatrix}$$

```
V // A // Γ // ΓCollect[FullSimplify[PowerExpand[#]] &] // ΓCollect[
  Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a :> e^(c_a)]]] &]
```

$$\begin{pmatrix} \frac{((c_1+c_2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(c_1 c_2(-1+T_1T_2))^{1/4}} & s_1 & \\ s_1 & \frac{c_1\sqrt{c_1c_2(-1+T_2)}+c_2\sqrt{c_1c_2(-1+T_2)}+c_1\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1T_2}}{(c_1+c_2)\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1T_2}} & \frac{-\sqrt{c_1c_2(c_1+c_2)(-1+T_1)}\ldots}{(c_1+c_2)\sqrt{\ldots}} \\ s_2 & \frac{-c_1\sqrt{c_1c_2(-1+T_2)}-c_2\sqrt{c_1c_2(-1+T_2)}+c_2\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1T_2}}{(c_1+c_2)\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1T_2}} & \frac{\sqrt{c_1c_2(c_1+c_2)(-1+T_1)}\ldots}{(c_1+c_2)\sqrt{\ldots}} \\ \Sigma & 1 & \end{pmatrix}$$

```
V // A // Γ // ΓCollect[FullSimplify[PowerExpand[#]] &] // ΓCollect[
  Assuming[d1 > 0 && d2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a :> e^(d_a^2)]]] &]
```

$$\begin{pmatrix} \frac{((d_1^2+d_2^2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(d_1^2 d_2^2(-1+T_1T_2))^{1/4}} & s_1 & \ldots \\ s_1 & \frac{d_1\left(d_1^2 d_2\sqrt{-1+T_2}+d_2^3\sqrt{-1+T_2}+d_1\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1T_2)}\right)}{(d_1^2+d_2^2)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1T_2)}} & \frac{d_1\left(-d_2\sqrt{(d_1^2+d_2^2)(-1+T_1)}\sqrt{\ldots}\right.}{(d_1^2+d_2^2)\sqrt{-1+\ldots}} \\ s_2 & -\frac{d_2\left(d_1^3\sqrt{-1+T_2}+d_1d_2^2\sqrt{-1+T_2}-d_2\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1T_2)}\right)}{(d_1^2+d_2^2)\sqrt{\frac{-1+T_1}{T_1}}\sqrt{(d_1^2+d_2^2)(-1+T_1T_2)}} & \frac{d_2\left(d_1\sqrt{(d_1^2+d_2^2)(-1+T_1)}\sqrt{\ldots}\right.}{(d_1^2+d_2^2)\sqrt{-1+\ldots}} \\ \Sigma & 1 & \sqrt{\ldots} \end{pmatrix}$$

Γ1-Calculus

```
Clear[α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ, ω];
{γ0 = Γ1[ω, h_a σ_a + h_b σ_b + h_S σ_S, {t_a, t_b, t_S}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{h_a, h_b, h_S}], γ0 // dm[a, b, c]}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_b & s_S \\ s_a & \alpha & \beta & \theta \\ s_b & \gamma & \delta & \epsilon \\ s_S & \phi & \psi & \Xi \\ \Gamma1 & \sigma_a & \sigma_b & \sigma_S \end{pmatrix}, \begin{pmatrix} -(-1+\beta)\,\omega & s_c & s_S \\ s_c & \frac{-\gamma+\beta\gamma-\alpha\delta}{-1+\beta} & \frac{-\epsilon+\beta\epsilon-\delta\theta}{-1+\beta} \\ s_S & \frac{-\phi+\beta\phi-\alpha\psi}{-1+\beta} & \frac{-\Xi+\beta\Xi-\theta\psi}{-1+\beta} \\ \Gamma1 & \sigma_a\sigma_b & \sigma_S \end{pmatrix}\right\}$$

```
Clear[α, θ, ϕ, Ξ, ω];
{γ0 = Γ1[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}], γ0 // qΔ[a, b, c],
 Γ[ω, h_a σ_a + h_S σ_S, {t_a, t_S}.(α θ; ϕ Ξ).{h_a, h_S}] // qΔ[a, b, c]}
```

$$\left\{\begin{pmatrix} \omega & s_a & s_S \\ s_a & \alpha & \theta \\ s_S & \phi & \Xi \\ \Gamma 1 & \sigma_a & \sigma_S \end{pmatrix}, \begin{pmatrix} \omega & s_b & s_c & s_S \\ s_b & \frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c \sigma_a}{-1+T_b T_c} & \frac{(-1+T_c)\, T_c\, (\alpha-\sigma_a)}{-1+T_b T_c} & \theta T_c \\ s_c & \frac{(-1+T_b)\,(\alpha-\sigma_a)}{-1+T_b T_c} & \frac{-\alpha+\alpha T_c-T_c \sigma_a+T_b T_c \sigma_a}{-1+T_b T_c} & \theta \\ s_S & \frac{\phi\,(-1+T_b)}{-1+T_b T_c} & \frac{\phi\,(-1+T_c)}{-1+T_b T_c} & \Xi \\ \Gamma 1 & \sigma_a & \sigma_a & \sigma_S \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & s_b & s_c & s_S \\ s_b & \frac{-\alpha T_c+\alpha T_b T_c-\sigma_a+T_c \sigma_a}{-1+T_b T_c} & \frac{(-1+T_b)\, T_c\, (\alpha-\sigma_a)}{-1+T_b T_c} & \frac{\theta\,(-1+T_b)\, T_c}{-1+T_b T_c} \\ s_c & \frac{(-1+T_c)\,(\alpha-\sigma_a)}{-1+T_b T_c} & \frac{-\alpha+\alpha T_c-T_c \sigma_a+T_b T_c \sigma_a}{-1+T_b T_c} & \frac{\theta\,(-1+T_c)}{-1+T_b T_c} \\ s_S & \phi & \phi & \Xi \\ \Gamma & \sigma_a & \sigma_a & \sigma_S \end{pmatrix}\right\}$$

```
{Xp[1, 2] // Γ, Xp[1, 2] // Γ1}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_1 \\ s_2 & 0 & T_1 \\ \Gamma & 1 & T_1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-T_2 \\ s_2 & 0 & T_1 \\ \Gamma 1 & 1 & T_1 \end{pmatrix}\right\}$$

R3

```
{Xm51 Xm62 Xp34 // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3],
 Xp61 Xm24 Xm35 // Γ1 // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]}
```

R3

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_1}{T_2} & \frac{-1+T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ s_1 & \frac{T_3}{T_2} & 0 & 0 \\ s_2 & \frac{-1+T_1}{T_2} & \frac{1}{T_3} & 0 \\ s_3 & -\frac{-1+T_1}{T_2} & \frac{-1+T_2}{T_3} & 1 \\ \Gamma 1 & \frac{T_3}{T_2} & \frac{1}{T_3} & 1 \end{pmatrix}\right\}$$

```
V // A // Γ1 // Γ1Collect[FullSimplify[PowerExpand[#]] &]
```

$$\begin{pmatrix} \frac{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])^{1/4}\,(-1+T_1)^{1/4}\,(-1+T_2)^{1/4}}{\mathrm{Log}[T_1]^{1/4}\,\mathrm{Log}[T_2]^{1/4}\,(-1+T_1 T_2)^{1/4}} & s_1 \\ s_1 & \frac{\sqrt{\mathrm{Log}[T_1]}\left(\mathrm{Log}[T_1]\sqrt{\mathrm{Log}[T_2]}\sqrt{-1+T_2}+\mathrm{Log}[T_2]^{3/2}\sqrt{-1+T_2}+\sqrt{\mathrm{Log}[T_1]}\sqrt{\mathrm{Log}[T_1]\cdots}\right.}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])^{3/2}\sqrt{\frac{-1+T_1}{T_1}}\sqrt{-1+T_1 T_2}} \\ s_2 & -\frac{\sqrt{\mathrm{Log}[T_2]}\left(\sqrt{\mathrm{Log}[T_1]}\sqrt{\mathrm{Log}[T_1]+\mathrm{Log}[T_2]}\sqrt{-1+T_1}\sqrt{T_1}\sqrt{-1+T_2}+\sqrt{\mathrm{Log}[T_2]}\sqrt{-1+\cdots}\right.}{(\mathrm{Log}[T_1]+\mathrm{Log}[T_2])\,(-1+T_2)\sqrt{-1+T_1 T_2}} \\ \Sigma & 1 \end{pmatrix}$$

```
V // A // Γ1 // Γ1Collect[FullSimplify[PowerExpand[#]] &] // Γ1Collect[
  Assuming[c1 > 0 && c2 > 0, FullSimplify[# /. Log[x_] :> Log[x /. T_a_ :> e^(c_a)]]] &]
```

$$\begin{pmatrix} \frac{((c_1+c_2)(-1+T_1))^{1/4}(-1+T_2)^{1/4}}{(c_1 c_2(-1+T_1 T_2))^{1/4}} & s_1 & \\ s_1 & \frac{c_1\sqrt{c_1 c_2(-1+T_2)}+c_2\sqrt{c_1 c_2(-1+T_2)}+c_1\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}}{(c_1+c_2)\sqrt{\frac{(c_1+c_2)(-1+T_1)}{T_1}}\sqrt{-1+T_1 T_2}} & -\frac{\sqrt{c_1(-1+T_2)}\left(c_2^{3/2}\sqrt{\ }\right.}{} \\ s_2 & \frac{-\sqrt{c_1 c_2(c_1+c_2)(-1+T_1)}\sqrt{T_1}\sqrt{-1+T_2}-c_2\sqrt{-1+T_1 T_2}+c_2 T_1\sqrt{-1+T_1 T_2}}{(c_1+c_2)(-1+T_2)\sqrt{-1+T_1 T_2}} & \sqrt{\ } \\ \Sigma & 1 & \end{pmatrix}$$