

Pensieve header: The free-Lie meta-group-action structure for <http://www.math.toronto.edu/~drorbn/papers/KBH/>.

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SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\KBH"];
<< CellExport.m
<< FreeLie.m

Randomμ[d_, m_, n_] := Module[{bas},
  bas = AllLyndonWords[{d}, Characters[StringTake["1234567890", m]]];
  μ[
    MakeCWSeries[RandomInteger[{-2, 2}, Length[bas]].bas /. LW → CW],
    Sum[h[j] MakeLieSeries[RandomInteger[{-2, 2}, Length[bas]].bas], {j, n}]
  ]
];
hL[μ_] := Union[Cases[μ, h[s_] → s, Infinity]];
μ[ω_, λ_][d_] := μ[ω[d], λ /. s_LieSeries → s[d]];
μ[ω1_, λ1_] ≡ μ[ω2_, λ2_] :=
  ω1 ≡ ω2 && (And @@ ((D[λ1, h[#]] ≡ D[λ2, h[#]] & /@ hL[{λ1, λ2}]));
LieDerivation[der_][μ[ω_, λ_]] := μ[ω // der, Collect[λ, _h, der]];
LieMorphism[mor_][μ[ω_, λ_]] := μ[ω // mor, Collect[λ, _h, mor]];

{Randomμ[1, 3, 3], Randomμ[3, 3, 3]}
{μ[CWS[-2 CW[1] + CW[3], 0, 0], h[1] LS[2 ⟨2⟩ + 2 ⟨3⟩, 0, 0] +
  h[2] LS[-⟨1⟩ + 2 ⟨2⟩ + 2 ⟨3⟩, 0, 0] + h[3] LS[⟨1⟩ + ⟨2⟩ + 2 ⟨3⟩, 0, 0]],
  μ[CWS[-2 CW[2] + CW[3], 2 CW[12] + 2 CW[13] + 2 CW[23],
    -CW[112] - 2 CW[113] - CW[122] + 2 CW[132] - 2 CW[133] + CW[233]],
  h[1] LS[-2 ⟨1⟩ + 2 ⟨2⟩ + 2 ⟨3⟩, -⟨12⟩, 2 ⟨113⟩ + ⟨122⟩ + ⟨123⟩ - 2 ⟨132⟩ +
    ⟨133⟩ + ⟨223⟩ + ⟨233⟩] + h[2] LS[2 ⟨1⟩ + ⟨2⟩ + 2 ⟨3⟩, -⟨12⟩ + 2 ⟨13⟩,
    2 ⟨113⟩ - ⟨122⟩ - ⟨123⟩ - 2 ⟨132⟩ - 2 ⟨133⟩ - ⟨223⟩ + 2 ⟨233⟩] +
  h[3] LS[⟨1⟩ - 2 ⟨2⟩ - 2 ⟨3⟩, -⟨12⟩ + ⟨13⟩, 2 ⟨112⟩ + 2 ⟨132⟩ - 2 ⟨133⟩ - ⟨233⟩]]}

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μ /: μ[ω1_, λ1_] μ[ω2_, λ2_] := μ[
  AddCWSeries[ω1, ω2],
  Collect[λ1+λ2, _h,
    (# /. ls1_LieSeries + ls2_LieSeries => AddLieSeries[ls1, ls2]) &
  ]
];

tm[x_, y_, z_][μ[ω_, λ_]] := μ[ω, λ] // LieMorphism[{<x> → <z>, <y> → <z>}];
hm[x_, y_, z_][μ[ω_, λ_]] := μ[ω, Plus[
  λ /. {h[x] → 0, h[y] → 0},
  h[z] BCH[D[λ, h[x]], D[λ, h[y]]]
]];

hta[x_, y_, z_][μ[ω_, λ_]] := Module[{λx, Adλx},
  λx = MakeLieSeries[D[λ, h[x]]];
  Adλx = LieMorphism[{LW[y] → Ad[ScaleLieSeries[-1, λx]][LW[z]]}];
  μ[
    AddCWSeries[StableApply[Adλx, ω], J[LW[y], λx]],
    Collect[λ, _h, StableApply[Adλx, #] &
  ]
];

hta[x_, y_][μ[ω_, λ_]] :=
  μ[ω, λ] // hta[x, y, <"z">] // LieMorphism[{LW["z"] → LW[y]}];
dm[x_, y_, z_][μ_] := μ // hta[y, x] // tm[x, y, z] // hm[x, y, z];
Rp[x_, y_] := μ[MakeCWSeries[0], h[y] MakeLieSeries[<x>]];
Rm[x_, y_] := μ[MakeCWSeries[0], h[y] MakeLieSeries[-<x>]];

n = $SeriesShowDegree = $SeriesCompareDegree = 4;
Print /@ {
  μ0 = Randomμ[3, 2, 2],
  μ0 // hta[1, 1],
  μ1 = ReplacePart[μ0, 1 → MakeCWSeries[0]],
  μ1 // hta[1, 1]
};

μ[CWS[-2 CW[1], -CW[12], CW[112] - 2 CW[122], 0],
  h[1] LS[2 <1> - 2 <2>, -<12>, <112> - <122>, 0] + h[2] LS[<1> - 2 <2>, 0, <122>, 0]]

μ[CWS[0, -2 CW[12],  $\frac{CW[112]}{3} - \frac{8 CW[122]}{3}$ ,  $CW[1112] + \frac{3 CW[1122]}{2} + CW[1222]$ ],
  h[1] LS[2 <1> - 2 <2>, 3 <12>, -<112> + <122>,  $-\frac{4 \langle 1112 \rangle}{3} + \frac{\langle 1122 \rangle}{3} - \frac{4 \langle 1222 \rangle}{3}$ ] +
  h[2] LS[<1> - 2 <2>, 2 <12>, -<112> + 3 <122>,  $-\frac{2 \langle 1112 \rangle}{3} - \frac{4 \langle 1122 \rangle}{3} + \frac{10 \langle 1222 \rangle}{3}$ ]]

μ[CWS[0, 0, 0, 0],
  h[1] LS[2 <1> - 2 <2>, -<12>, <112> - <122>, 0] + h[2] LS[<1> - 2 <2>, 0, <122>, 0]]

μ[CWS[2 CW[1], -CW[12],  $-\frac{2 CW[112]}{3} - \frac{2 CW[122]}{3}$ ,  $CW[1112] - \frac{CW[1122]}{2} + 2 CW[1212] + CW[1222]$ ],
  h[1] LS[2 <1> - 2 <2>, 3 <12>, -<112> + <122>,  $-\frac{4 \langle 1112 \rangle}{3} + \frac{\langle 1122 \rangle}{3} - \frac{4 \langle 1222 \rangle}{3}$ ] +
  h[2] LS[<1> - 2 <2>, 2 <12>, -<112> + 3 <122>,  $-\frac{2 \langle 1112 \rangle}{3} - \frac{4 \langle 1122 \rangle}{3} + \frac{10 \langle 1222 \rangle}{3}$ ]]

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Testing tm

```

n = $SeriesShowDegree = $SeriesCompareDegree = 3;
Print /@ {μ0 = Randomμ[n, 4, 1],
  μ0 // tm[1, 2, 2],
  μ0 // tm[2, 3, 3],
  t1 = μ0 // tm[1, 2, 2] // tm[2, 3, 3],
  t2 = μ0 // tm[2, 3, 3] // tm[1, 3, 3],
  t1 ≡ t2
};

μ[CWS[2 CW[1] - CW[2] + CW[4], 2 CW[12] - CW[13] + 2 CW[14] + 2 CW[23] - 2 CW[24] + CW[34],
  -CW[112] - 2 CW[113] - 2 CW[114] + CW[124] - 2 CW[132] + CW[133] + 2 CW[134] -
  2 CW[144] + 2 CW[223] - CW[224] - CW[233] - CW[243] + 2 CW[244] + 2 CW[334]],
h[1] LS[-2 ⟨1⟩ - 2 ⟨2⟩ + 2 ⟨3⟩, -⟨12⟩ + ⟨14⟩ - ⟨23⟩ + 2 ⟨24⟩, 2 ⟨112⟩ + ⟨114⟩ + ⟨122⟩ +
  ⟨123⟩ - ⟨124⟩ - ⟨142⟩ - ⟨143⟩ + ⟨144⟩ - ⟨223⟩ + 2 ⟨224⟩ + 2 ⟨233⟩ + ⟨234⟩ - ⟨243⟩]]

μ[CWS[CW[2] + CW[4], 2 CW[22] + CW[23] + CW[34],
  -CW[222] - 2 CW[223] - 2 CW[224] + 2 CW[234] - CW[243] + 2 CW[334]],
h[1] LS[-4 ⟨2⟩ + 2 ⟨3⟩, -⟨23⟩ + 3 ⟨24⟩, 3 ⟨224⟩ + 2 ⟨233⟩ + ⟨234⟩ - 2 ⟨243⟩ + ⟨244⟩]]

μ[CWS[2 CW[1] - CW[3] + CW[4], CW[13] + 2 CW[14] + 2 CW[33] - CW[34],
  -3 CW[113] - 2 CW[114] - CW[133] + 3 CW[134] - 2 CW[144] + CW[333] + 2 CW[344]],
h[1] LS[-2 ⟨1⟩, -⟨13⟩ + ⟨14⟩ + 2 ⟨34⟩, 2 ⟨113⟩ + ⟨114⟩ + ⟨133⟩ - ⟨134⟩ - 2 ⟨143⟩ + ⟨144⟩ + 4 ⟨334⟩]]

μ[CWS[CW[3] + CW[4], 3 CW[33] + CW[34], -3 CW[333] + CW[334]],
h[1] LS[-2 ⟨3⟩, 3 ⟨34⟩, 6 ⟨334⟩ + ⟨344⟩]]

μ[CWS[CW[3] + CW[4], 3 CW[33] + CW[34], -3 CW[333] + CW[334]],
h[1] LS[-2 ⟨3⟩, 3 ⟨34⟩, 6 ⟨334⟩ + ⟨344⟩]]

True

t1 = μ0 // tm[1, 2, 2] // tm[2, 3, 3] // InputForm
μ[CWSeries[LieMorphismOnCWSeries$1866], h[1]*LieSeries[LieMorphismOnLieSeries$1868]]

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Testing hm

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Print /@ {μ0 = Randomμ[4, 2, 3],
  μ0 // hm[1, 2, 2],
  t1 = μ0 // hm[1, 2, 2] // hm[2, 3, 3],
  t2 = μ0 // hm[2, 3, 3] // hm[1, 3, 3],
  t1 ≡ t2
};

μ[CWS[2 CW[1] - CW[2], 0, -2 CW[122]], h[1] LS[2 ⟨1⟩ - ⟨2⟩, 2 ⟨12⟩, 2 ⟨112⟩ - 2 ⟨122⟩] +
  h[2] LS[⟨2⟩, 2 ⟨12⟩, -2 ⟨112⟩ + ⟨122⟩] + h[3] LS[⟨1⟩ - 2 ⟨2⟩, -⟨12⟩, -⟨112⟩ - ⟨122⟩]]

μ[CWS[2 CW[1] - CW[2], 0, -2 CW[122]],
  h[2] LS[2 ⟨1⟩, 5 ⟨12⟩,  $\frac{7 \langle 112 \rangle}{3} + \frac{4 \langle 122 \rangle}{3}$ ] + h[3] LS[⟨1⟩ - 2 ⟨2⟩, -⟨12⟩, -⟨112⟩ - ⟨122⟩]]

μ[CWS[2 CW[1] - CW[2], 0, -2 CW[122]], h[3] LS[3 ⟨1⟩ - 2 ⟨2⟩, 2 ⟨12⟩,  $-\frac{5 \langle 112 \rangle}{2} - 4 \langle 122 \rangle$ ]]

μ[CWS[2 CW[1] - CW[2], 0, -2 CW[122]], h[3] LS[3 ⟨1⟩ - 2 ⟨2⟩, 2 ⟨12⟩,  $-\frac{5 \langle 112 \rangle}{2} - 4 \langle 122 \rangle$ ]]

True

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Testing hta

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n = $SeriesShowDegree = $SeriesCompareDegree = 4;
Print /@ {μ0 = Randomμ[3, 2, 2], μ0 // hta[1, 1]};

μ[CWS[-CW[1] - 2 CW[2], CW[12], -CW[112], 0],
  h[1] LS[⟨1⟩ + ⟨2⟩, 0, -2 ⟨112⟩ - 2 ⟨122⟩, 0] + h[2] LS[2 ⟨1⟩ - 2 ⟨2⟩, -2 ⟨12⟩, 0, 0]]

μ[CWS[-2 CW[2],  $\frac{3 CW[12]}{2}$ ,  $\frac{5 CW[112]}{6} - \frac{11 CW[122]}{6}$ ,
  -  $\frac{23 CW[1112]}{24} - \frac{7 CW[1122]}{12} + \frac{29 CW[1212]}{12} - \frac{23 CW[1222]}{24}$ ],
  h[1] LS[⟨1⟩ + ⟨2⟩, -⟨12⟩, - $\frac{3 \langle 112 \rangle}{2} - \frac{3 \langle 122 \rangle}{2}$ ,  $\frac{11 \langle 1112 \rangle}{6} + \frac{10 \langle 1122 \rangle}{3} + \frac{11 \langle 1222 \rangle}{6}$ ] +
  h[2] LS[2 ⟨1⟩ - 2 ⟨2⟩, -4 ⟨12⟩, ⟨112⟩ + 3 ⟨122⟩,  $\frac{11 \langle 1112 \rangle}{3} + \frac{5 \langle 1122 \rangle}{3} - \frac{4 \langle 1222 \rangle}{3}$ ]]

Print /@ {μ0 = Randomμ[n, 3, 2],
  t1 = μ0 // hta[1, 1] // hta[1, 2] // tm[1, 2, 1],
  t2 = μ0 // tm[1, 2, 1] // hta[1, 1],
  t1 ≡ t2
};

μ[CWS[-CW[1] + 2 CW[2], -2 CW[13] - 2 CW[23],
  2 CW[112] + 2 CW[122] + CW[123] - 2 CW[132] + 2 CW[133] - 2 CW[233],
  CW[1112] + CW[1113] + 2 CW[1122] - CW[1133] + CW[1213] - CW[1222] - 2 CW[1223] -
  CW[1233] + CW[1322] + CW[1323] + 2 CW[1333] + 2 CW[2223] + 2 CW[2233]],
  h[1] LS[⟨1⟩ - ⟨3⟩, ⟨12⟩ - ⟨13⟩ + ⟨23⟩, 2 ⟨112⟩ + ⟨113⟩ - 2 ⟨122⟩ - ⟨133⟩ - ⟨233⟩,
  2 ⟨1112⟩ + ⟨1113⟩ + ⟨1122⟩ + 2 ⟨1123⟩ - 2 ⟨1132⟩ + ⟨1133⟩ + 2 ⟨1222⟩ - ⟨1223⟩ -
  2 ⟨1232⟩ + 2 ⟨1233⟩ + ⟨1322⟩ + 2 ⟨1323⟩ + 2 ⟨1332⟩ - ⟨1333⟩ - ⟨2233⟩ + 2 ⟨2333⟩] +
  h[2] LS[-2 ⟨1⟩ + ⟨2⟩, ⟨12⟩ + 2 ⟨23⟩, -2 ⟨113⟩ + ⟨122⟩ + 2 ⟨123⟩ - 2 ⟨132⟩ - ⟨133⟩ + ⟨223⟩,
  -⟨1113⟩ + ⟨1122⟩ + 2 ⟨1123⟩ - ⟨1132⟩ + ⟨1133⟩ - ⟨1222⟩ + 2 ⟨1223⟩ -
  2 ⟨1232⟩ + 2 ⟨1233⟩ + ⟨1323⟩ + 2 ⟨1332⟩ + 2 ⟨1333⟩ + 2 ⟨2223⟩ - 2 ⟨2333>]]

μ[CWS[2 CW[1], - $\frac{9 CW[13]}{2}$ , 4 CW[111] -  $\frac{11 CW[113]}{6} - \frac{11 CW[133]}{6}$ ,
  2 CW[1111] -  $\frac{85 CW[1113]}{24} + \frac{35 CW[1133]}{12} - \frac{7 CW[1313]}{12} + \frac{47 CW[1333]}{24}$ ],
  h[1] LS[⟨1⟩ - ⟨3⟩, ⟨13⟩,  $\frac{\langle 113 \rangle}{2} - \frac{3 \langle 133 \rangle}{2}$ ,  $\frac{37 \langle 1113 \rangle}{6} + \frac{7 \langle 1133 \rangle}{3} - \frac{5 \langle 1333 \rangle}{6}$ ] +
  h[2] LS[-⟨1⟩, ⟨13⟩,  $\frac{7 \langle 113 \rangle}{2} + \frac{\langle 133 \rangle}{2}$ ,  $\frac{53 \langle 1113 \rangle}{6} + \frac{5 \langle 1133 \rangle}{3} - \frac{\langle 1333 \rangle}{6}$ ]]

μ[CWS[2 CW[1], - $\frac{9 CW[13]}{2}$ , 4 CW[111] -  $\frac{11 CW[113]}{6} - \frac{11 CW[133]}{6}$ ,
  2 CW[1111] -  $\frac{85 CW[1113]}{24} + \frac{35 CW[1133]}{12} - \frac{7 CW[1313]}{12} + \frac{47 CW[1333]}{24}$ ],
  h[1] LS[⟨1⟩ - ⟨3⟩, ⟨13⟩,  $\frac{\langle 113 \rangle}{2} - \frac{3 \langle 133 \rangle}{2}$ ,  $\frac{37 \langle 1113 \rangle}{6} + \frac{7 \langle 1133 \rangle}{3} - \frac{5 \langle 1333 \rangle}{6}$ ] +
  h[2] LS[-⟨1⟩, ⟨13⟩,  $\frac{7 \langle 113 \rangle}{2} + \frac{\langle 133 \rangle}{2}$ ,  $\frac{53 \langle 1113 \rangle}{6} + \frac{5 \langle 1133 \rangle}{3} - \frac{\langle 1333 \rangle}{6}$ ]]

True

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n = $SeriesShowDegree = $SeriesCompareDegree = 5;
Print /@ {μ0 = ReplacePart[Randomμ[n, 2, 3], 1 → MakeCWSeries[0]],
  t1 = μ0 // hta[1, 1] // hta[2, 1] // hm[1, 2, 1],
  t2 = μ0 // hm[1, 2, 1] // hta[1, 1],
  t1 ≡ t2
};

μ[CWS[0, 0, 0, 0, 0], h[1] LS[-2 ⟨1⟩ + 2 ⟨2⟩, 0, 2 ⟨122⟩,
  2 ⟨1112⟩ + 2 ⟨1122⟩ - 2 ⟨1222⟩, -⟨11112⟩ - 2 ⟨11122⟩ - ⟨11222⟩ - 2 ⟨12222⟩] +
h[2] LS[-⟨1⟩ + 2 ⟨2⟩, -2 ⟨12⟩, ⟨112⟩ - ⟨122⟩, -2 ⟨1112⟩ + 2 ⟨1122⟩ + 2 ⟨1222⟩,
  -⟨11112⟩ - ⟨11122⟩ - 2 ⟨11212⟩ + 2 ⟨12222⟩] +
h[3] LS[0, ⟨12⟩, 2 ⟨122⟩, -⟨1122⟩ - ⟨1222⟩, ⟨11112⟩ + 2 ⟨11122⟩ - ⟨11212⟩ + ⟨11222⟩]]

μ[CWS[-3 CW[1], -3 CW[12], - $\frac{14 CW[112]}{3}$  + CW[122],
  - $\frac{43 CW[1112]}{12}$  +  $\frac{61 CW[1122]}{6}$  -  $\frac{34 CW[1212]}{3}$  +  $\frac{8 CW[1222]}{3}$ ,
  - $\frac{187 CW[11112]}{60}$  +  $\frac{454 CW[11122]}{15}$  -  $\frac{551 CW[11212]}{15}$  -  $\frac{14 CW[11222]}{3}$  + 18 CW[12122] -  $\frac{26 CW[12222]}{15}$ ],
h[3] LS[0, ⟨12⟩, -2 ⟨122⟩, -4 ⟨1122⟩ - ⟨1222⟩,
  ⟨11112⟩ -  $\frac{8 \langle 11122 \rangle}{3}$  +  $\frac{11 \langle 11212 \rangle}{3}$  + 10 ⟨11222⟩ + 9 ⟨12122⟩ +  $\frac{28 \langle 12222 \rangle}{3}$ ] +
h[1] LS[-3 ⟨1⟩ + 4 ⟨2⟩, 9 ⟨12⟩,  $\frac{73 \langle 112 \rangle}{6}$  - 9 ⟨122⟩,  $\frac{77 \langle 1112 \rangle}{6}$  -  $\frac{63 \langle 1122 \rangle}{2}$  -  $\frac{2 \langle 1222 \rangle}{3}$ ,
   $\frac{277 \langle 11112 \rangle}{24}$  -  $\frac{1309 \langle 11122 \rangle}{20}$  +  $\frac{163 \langle 11212 \rangle}{20}$  +  $\frac{379 \langle 11222 \rangle}{45}$  +  $\frac{2237 \langle 12122 \rangle}{90}$  + 6 ⟨12222⟩]]

μ[CWS[-3 CW[1], -3 CW[12], - $\frac{14 CW[112]}{3}$  + CW[122],
  - $\frac{43 CW[1112]}{12}$  +  $\frac{61 CW[1122]}{6}$  -  $\frac{34 CW[1212]}{3}$  +  $\frac{8 CW[1222]}{3}$ ,
  - $\frac{187 CW[11112]}{60}$  +  $\frac{454 CW[11122]}{15}$  -  $\frac{551 CW[11212]}{15}$  -  $\frac{14 CW[11222]}{3}$  + 18 CW[12122] -  $\frac{26 CW[12222]}{15}$ ],
h[1] LS[-3 ⟨1⟩ + 4 ⟨2⟩, 9 ⟨12⟩,  $\frac{73 \langle 112 \rangle}{6}$  - 9 ⟨122⟩,  $\frac{77 \langle 1112 \rangle}{6}$  -  $\frac{63 \langle 1122 \rangle}{2}$  -  $\frac{2 \langle 1222 \rangle}{3}$ ,
   $\frac{277 \langle 11112 \rangle}{24}$  -  $\frac{1309 \langle 11122 \rangle}{20}$  +  $\frac{163 \langle 11212 \rangle}{20}$  +  $\frac{379 \langle 11222 \rangle}{45}$  +  $\frac{2237 \langle 12122 \rangle}{90}$  + 6 ⟨12222⟩] +
h[3] LS[0, ⟨12⟩, -2 ⟨122⟩, -4 ⟨1122⟩ - ⟨1222⟩,
  ⟨11112⟩ -  $\frac{8 \langle 11122 \rangle}{3}$  +  $\frac{11 \langle 11212 \rangle}{3}$  + 10 ⟨11222⟩ + 9 ⟨12122⟩ +  $\frac{28 \langle 12222 \rangle}{3}$ ]]
True

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Testing dm

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$SeriesShowDegree = 3;
$SeriesCompareDegree = n = 5;
Timing[Print /@ {μ0 = Randomμ[n, 4, 4],
  t1 = μ0 // dm[1, 2, 1] // dm[1, 3, 1],
  t2 = μ0 // dm[2, 3, 2] // dm[1, 2, 1],
  t1 ≡ t2
};]

μ[CWS[-2 CW[1] + 2 CW[2] - CW[3], -CW[12] - 2 CW[23] + CW[24],
  -CW[112] + CW[114] + CW[122] + 2 CW[123] + CW[133] - CW[134] + CW[142] - CW[143] -
  CW[144] + 2 CW[224] - CW[233] - 2 CW[234] + CW[244] - 2 CW[334] - CW[344]],
h[1] LS[-2 ⟨2⟩ - 2 ⟨4⟩, 2 ⟨13⟩ + 2 ⟨14⟩ + ⟨23⟩ - 2 ⟨24⟩ - ⟨34⟩, -2 ⟨114⟩ + 2 ⟨122⟩ - ⟨123⟩ - ⟨124⟩ +
  2 ⟨132⟩ + 2 ⟨133⟩ + 2 ⟨134⟩ + ⟨142⟩ + 2 ⟨224⟩ + 2 ⟨233⟩ - 2 ⟨234⟩ - 2 ⟨244⟩ + 2 ⟨334⟩ - 2 ⟨344⟩] +
h[2] LS[2 ⟨1⟩ + 2 ⟨2⟩ + ⟨3⟩ + 2 ⟨4⟩, ⟨12⟩ - 2 ⟨13⟩ - 2 ⟨23⟩ + ⟨24⟩ - ⟨34⟩, 2 ⟨113⟩ - 2 ⟨114⟩ - ⟨122⟩ +
  ⟨133⟩ + ⟨134⟩ - ⟨142⟩ + 2 ⟨143⟩ - ⟨223⟩ + 2 ⟨224⟩ + ⟨234⟩ + ⟨243⟩ - 2 ⟨244⟩ + ⟨334⟩ - 2 ⟨344⟩] +
h[3] LS[⟨1⟩ - 2 ⟨2⟩ - 2 ⟨3⟩ - ⟨4⟩, ⟨12⟩ + ⟨13⟩ + 2 ⟨14⟩ - 2 ⟨23⟩ - 2 ⟨24⟩ - 2 ⟨34⟩,
  -⟨112⟩ - ⟨113⟩ - ⟨122⟩ + ⟨132⟩ + ⟨133⟩ + ⟨134⟩ + 2 ⟨142⟩ + 2 ⟨144⟩ +
  2 ⟨223⟩ - 2 ⟨224⟩ + 2 ⟨233⟩ + 2 ⟨234⟩ - 2 ⟨243⟩ + ⟨244⟩ + ⟨344⟩] +
h[4] LS[-2 ⟨2⟩ + ⟨3⟩ + 2 ⟨4⟩, ⟨13⟩ - 2 ⟨14⟩ - ⟨24⟩ - ⟨34⟩,
  2 ⟨112⟩ - ⟨113⟩ + 2 ⟨114⟩ + ⟨122⟩ - 2 ⟨123⟩ - ⟨124⟩ - ⟨132⟩ + ⟨133⟩ - ⟨134⟩ +
  ⟨142⟩ - ⟨223⟩ + 2 ⟨224⟩ - ⟨233⟩ - ⟨234⟩ - 2 ⟨243⟩ + ⟨244⟩ + 2 ⟨334⟩ + 2 ⟨344⟩]]

μ[CWS[0, 4 CW[11] +  $\frac{11 CW[14]}{2}$ , - $\frac{7 CW[111]}{3}$  +  $\frac{7 CW[114]}{6}$  +  $\frac{CW[144]}{6}$ ],
h[4] LS[-⟨1⟩ + 2 ⟨4⟩, -6 ⟨14⟩, -2 ⟨114⟩ + 3 ⟨144⟩] +
h[1] LS[-⟨4⟩, - $\frac{13 \langle 14 \rangle}{2}$ ,  $\frac{17 \langle 114 \rangle}{2}$  -  $\frac{45 \langle 144 \rangle}{4}$ ]]

μ[CWS[0, 4 CW[11] +  $\frac{11 CW[14]}{2}$ , - $\frac{7 CW[111]}{3}$  +  $\frac{7 CW[114]}{6}$  +  $\frac{CW[144]}{6}$ ],
h[4] LS[-⟨1⟩ + 2 ⟨4⟩, -6 ⟨14⟩, -2 ⟨114⟩ + 3 ⟨144⟩] +
h[1] LS[-⟨4⟩, - $\frac{13 \langle 14 \rangle}{2}$ ,  $\frac{17 \langle 114 \rangle}{2}$  -  $\frac{45 \langle 144 \rangle}{4}$ ]]

True
{43.93, Null}

```

Testing R-Moves

```

(Rp[1, 2] Rm[3, 4]) // dm[1, 3, 1] // dm[2, 4, 2]
μ[CWS[0, 0, 0], h[1] LS[0, 0, 0] + h[2] LS[0, 0, 0]]

(Rp[1, 2] Rm[3, 4]) // dm[1, 3, 1] // dm[4, 2, 2]
μ[CWS[0, 0, 0], h[1] LS[0, 0, 0] + h[2] LS[0, 0, 0]]

t1 = Rp[1, 2] Rp[4, 3] Rp[5, 6] // dm[1, 4, 1] // dm[2, 5, 2] // dm[3, 6, 3]
μ[CWS[0, 0, 0],
  h[1] LS[0, 0, 0] + h[2] LS[⟨1⟩, 0, 0] + h[3] LS[⟨1⟩ + ⟨2⟩,  $\frac{\langle 12 \rangle}{2}$ ,  $\frac{\langle 112 \rangle}{12}$  +  $\frac{\langle 122 \rangle}{12}$ ]]

```

```

t2 = Rp[2, 3] Rp[1, 4] Rp[5, 6] // dm[1, 5, 1] // dm[2, 6, 2] // dm[3, 4, 3]
μ[CWS[0, 0, 0],
h[1] LS[0, 0, 0] + h[2] LS[⟨1⟩, 0, 0] + h[3] LS[⟨1⟩ + ⟨2⟩,  $\frac{\langle 12 \rangle}{2}$ ,  $\frac{\langle 112 \rangle}{12}$  +  $\frac{\langle 122 \rangle}{12}$ ]]]
{t1[7], t2[7], t1[7] == t2[7]}
{μ[0, h[3] (  $\frac{\langle 1111112 \rangle}{30240}$  -  $\frac{\langle 1111122 \rangle}{5040}$  +  $\frac{\langle 1111212 \rangle}{10080}$  +  $\frac{\langle 1111222 \rangle}{3780}$  +  $\frac{\langle 1112112 \rangle}{10080}$  +  $\frac{\langle 1112122 \rangle}{1680}$  +
 $\frac{\langle 1112212 \rangle}{1260}$  +  $\frac{\langle 1112222 \rangle}{3780}$  +  $\frac{\langle 1121122 \rangle}{2016}$  -  $\frac{\langle 1121212 \rangle}{5040}$  +  $\frac{13 \langle 1121222 \rangle}{15120}$  +  $\frac{\langle 1122122 \rangle}{10080}$  -
 $\frac{\langle 1122212 \rangle}{1512}$  -  $\frac{\langle 1122222 \rangle}{5040}$  +  $\frac{\langle 1212122 \rangle}{1260}$  -  $\frac{\langle 1212222 \rangle}{2016}$  -  $\frac{\langle 1221222 \rangle}{5040}$  +  $\frac{\langle 1222222 \rangle}{30240}$  )],
μ[0, h[3] (  $\frac{\langle 1111112 \rangle}{30240}$  -  $\frac{\langle 1111122 \rangle}{5040}$  +  $\frac{\langle 1111212 \rangle}{10080}$  +  $\frac{\langle 1111222 \rangle}{3780}$  +  $\frac{\langle 1112112 \rangle}{10080}$  +  $\frac{\langle 1112122 \rangle}{1680}$  +
 $\frac{\langle 1112212 \rangle}{1260}$  +  $\frac{\langle 1112222 \rangle}{3780}$  +  $\frac{\langle 1121122 \rangle}{2016}$  -  $\frac{\langle 1121212 \rangle}{5040}$  +  $\frac{13 \langle 1121222 \rangle}{15120}$  +  $\frac{\langle 1122122 \rangle}{10080}$  -
 $\frac{\langle 1122212 \rangle}{1512}$  -  $\frac{\langle 1122222 \rangle}{5040}$  +  $\frac{\langle 1212122 \rangle}{1260}$  -  $\frac{\langle 1212222 \rangle}{2016}$  -  $\frac{\langle 1221222 \rangle}{5040}$  +  $\frac{\langle 1222222 \rangle}{30240}$  )], True}

```

817

```

Print /@ {
μ0 = Rm[12, 1] Rm[2, 7] Rm[8, 3] Rm[4, 11] Rp[16, 5] Rp[6, 13] Rp[14, 9] Rp[10, 15],
Do[μ0 = μ0 // dm[1, k, 1], {k, 2, 16}]; μ0
}
μ[CWS[0, 0, 0],
h[1] LS[-⟨b⟩, 0, 0] + h[7] LS[-⟨2⟩, 0, 0] + h[3] LS[-⟨8⟩, 0, 0] + h[11] LS[-⟨4⟩, 0, 0] +
h[5] LS[⟨f⟩, 0, 0] + h[13] LS[⟨6⟩, 0, 0] + h[9] LS[⟨d⟩, 0, 0] + h[15] LS[⟨0⟩, 0, 0]]]
μ[CWS[0, -CW[11], 0], h[1] LS[0, 0, 0]]
{Null, Null}
$RecursionLimit = 2^16;
μ0[{4}]
μ[CWS[0, -CW[11], 0, - $\frac{31 \text{ CW}[1111]}{12}$ ], h[1] LS[0, 0, 0, 0]]]
μ0[{6}]
μ[CWS[0, -CW[11], 0, - $\frac{31 \text{ CW}[1111]}{12}$ , 0, - $\frac{1351 \text{ CW}[111111]}{360}$ ], h[1] LS[0, 0, 0, 0, 0, 0]]]
μ0[{8}]
μ[CWS[0, -CW[11], 0, - $\frac{31 \text{ CW}[1111]}{12}$ , 0, - $\frac{1351 \text{ CW}[111111]}{360}$ , 0, - $\frac{123271 \text{ CW}[11111111]}{20160}$ ],
h[1] LS[0, 0, 0, 0, 0, 0, 0, 0]]]

```

```
Print /@ {β = -  $\frac{1 - 4 X + 8 X^2 - 11 X^3 + 8 X^4 - 4 X^5 + X^6}{X^3}$  /. X → ex,
  Series[β, {x, 0, 8}],
  Series[Log[β], {x, 0, 9}]
};
-e-3x (1 - 4 ex + 8 e2x - 11 e3x + 8 e4x - 4 e5x + e6x)
1 - x2 -  $\frac{25 x^4}{12}$  -  $\frac{481 x^6}{360}$  -  $\frac{1109 x^8}{4032}$  + O[x]9
-x2 -  $\frac{31 x^4}{12}$  -  $\frac{1351 x^6}{360}$  -  $\frac{123271 x^8}{20160}$  + O[x]10
```

Runs for the KBH paper

```
Rp[u, x]
```

```
μ[CWS[0, 0, 0], h[x] LS[⟨u⟩, 0, 0]]
```

```
Format[CW[w_], StandardForm] := "[" <> w <> "]"
```

ExampleZeta

```
T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[2, 1, 1] // dm[4, b, b] // dm[1, a, a] // dm[3, a, a];
ζ[{4}]
```

ExampleZeta

```
μ[CWS[-[a], -[ab], - $\frac{[aab]}{2}$  -  $\frac{[abb]}{2}$ , - $\frac{[aaab]}{6}$  +  $\frac{7[aabb]}{4}$  -  $\frac{5[abab]}{2}$  -  $\frac{[abbb]}{6}$ ],
  h[b] LS[⟨a⟩, 0, -⟨aab⟩, - $\frac{\langle aaab \rangle}{2}$  +  $\frac{\langle aabb \rangle}{2}$ ] +
  h[a] LS[-⟨a⟩ + ⟨b⟩,  $\frac{3\langle ab \rangle}{2}$ ,  $\frac{13\langle aab \rangle}{12}$  -  $\frac{13\langle abb \rangle}{12}$ ,  $\frac{\langle aaab \rangle}{2}$  -  $\frac{17\langle aabb \rangle}{8}$  +  $\frac{\langle abbb \rangle}{2}$ ]]]
```


TheTrivialBalloon

```
Graphics3D[{
  Red, Thickness[0.02], Line[{{0, -2, 0}, {0, -1, 0}}],
  Thick, Line[Table[
    {Cos[ $\theta$ ], Sin[ $\theta$ ], 0},
    { $\theta$ , 0, 2  $\pi$ , 2  $\pi$  / 72}
  ]],
  RGBColor[1, 0.65, 0.65], Sphere[{0, 0, 0}, 1]
},
  Axes  $\rightarrow$  True, AxesLabel  $\rightarrow$  {x, y, t}, LabelStyle  $\rightarrow$  Directive[Large],
  Ticks  $\rightarrow$  {{}, {}, {-1, 1}},
  Lighting  $\rightarrow$  "Neutral"
]
```

TheTrivialBalloon

