

Balloons and Hoops and their Universal Finite-Type Invariant, BF Theory, and an Ultimate Alexander Invariant

1
2
3

Dror Bar-Natan 4

Received: 11 May 2014 / Accepted: 5 August 2014 5
© Institute of Mathematics, Vietnam Academy of Science and Technology (VAST) and Springer 6
Science+Business Media Singapore 2015 7

Abstract Balloons are 2D spheres. Hoops are 1D loops. Knotted balloons and hoops (KBH) in 4-space behave much like the first and second homotopy groups of a topological space—hoops can be composed as in π_1 , balloons as in π_2 , and hoops “act” on balloons as π_1 acts on π_2 . We observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops. We give an ansatz for a tree and wheel (that is, free Lie and cyclic word)-valued invariant ζ of (ribbon) KBHs in terms of the said compositions and action and we explain its relationship with finite-type invariants. We speculate that ζ is a complete evaluation of the BF topological quantum field theory in 4D. We show that a certain “reduction and repackaging” of ζ is an “ultimate Alexander invariant” that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least wasteful in a computational sense. If you believe in categorification, that should be a wonderful playground. 8
9
10
11
12
13
14
15
16
17
18
19
20

Keywords 2-knots · Tangles · Virtual knots · w-tangles · Ribbon knots · Finite type invariants · BF theory · Alexander polynomial · Meta-groups · Meta-monoids 21
22

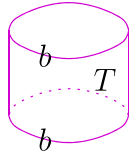
Mathematics Subject Classification (2010) 57M25 23

Web resources for this paper are available at [Web/]:=<http://www.math.toronto.edu/~drorbn/papers/KBH/>, including an electronic version, source files, computer programs, lecture handouts and lecture videos. *Throughout this paper, we follow the notational conventions and notations outlined in Section 10.5.*

D. Bar-Natan (✉)
Department of Mathematics, University of Toronto,
Toronto, Ontario M5S 2E4, Canada
e-mail: drorbn@math.toronto.edu; <http://www.math.toronto.edu/~drorbn>

24 1 Introduction

25 **Riddle 1.1** The set of homotopy classes of maps of a tube $T = S^1 \times [0, 1]$ into a based
 26 topological space (X, b) which map the rim $\partial T = S^1 \times \{0, 1\}$ of T to the basepoint b
 27 is a group with an obvious “stacking” composition; we call that group $\pi_T(X)$. Homotopy
 28 theorists often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$ but seldom, if ever, do they
 29 study $\pi_T(X) = [T, X]$. Why?



30 The solution of this riddle is on page 13. Whatever it may be, the moral is that it is better
 31 to study the homotopy classes of circles and spheres in X rather than the homotopy classes of
 32 tubes in X , and by morphological transfer, it is better to study isotopy classes of embeddings
 33 of circles and spheres into some ambient space than isotopy classes of embeddings of tubes
 34 into the same space.

35 In [4, 5], Zsuzsanna Dancso and I studied the finite-type knot theory of ribbon tubes
 36 in \mathbb{R}^4 and found it to be closely related to deep results by Alekseev and Torossian [1] on
 37 the Kashiwara-Vergne conjecture and Drinfel’d’s associators. At some point, we needed a
 38 computational tool with which to make and to verify conjectures.

39 This paper started in being that computational tool. After a lengthy search, I found a
 40 language in which all the operations and equations needed for [4, 5] could be expressed
 41 and computed. Upon reflection, it turned out that the key to that language was to work with
 42 knotted balloons and hoops, meaning spheres and circles, rather than with knotted tubes.

43 Then, I realized that there may be independent interest in that computational tool. For
 44 (ribbon) knotted balloons and hoops in \mathbb{R}^4 (\mathcal{K}^{rbh} , Section 2) in themselves form a lovely
 45 algebraic structure (a meta-monoid-action (MMA), Section 3), and the “tool” is really a
 46 well-behaved invariant ζ . More precisely, ζ is a “homomorphism ζ of the MMA \mathcal{K}_0^{rbh} to
 47 the MMA M of trees and wheels” (trees in Section 4 and wheels in Section 5). Here, \mathcal{K}_0^{rbh}
 48 is a variant of \mathcal{K}^{rbh} defined using generators and relations (Definition 3.5). Assuming a
 49 sorely missing Reidemeister theory for ribbon-knotted tubes in \mathbb{R}^4 (Conjecture 3.7), \mathcal{K}_0^{rbh}
 50 is actually equal to \mathcal{K}^{rbh} .

51 The invariant ζ has a rather concise definition that uses only basic operations written in
 52 the language of free Lie algebras. In fact, a nearly complete definition appears within Fig. 4,
 53 with lesser extras in Figs. 5 and 1. These definitions are relatively easy to implement on a
 54 computer, and as that was my original goal, the implementation along with some computa-
 55 tional examples is described in Section 6. Further computations, more closely related to [1]
 56 and to [4, 5], will be described in [3].

57 In Section 7, we sketch a conceptual interpretation of ζ . Namely, we sketch the statement
 58 and the proof of the following theorem:

59 **Theorem 1.2** *The invariant ζ is (the logarithm of) a universal finite type invariant of the*
 60 *objects in \mathcal{K}_0^{rbh} (assuming Conjecture 3.7, of ribbon-knotted balloons and hoops in \mathbb{R}^4).*

While the formulae defining ζ are reasonably simple, the proof that they work using only notions from the language of free Lie algebras involves some painful computations—the more reasonable parts of the proof are embedded within Sections 4 and 5, and the less reasonable parts are postponed to Section 10.4. An added benefit of the results of Section 7 is that they constitute an alternative construction of ζ and an alternative proof of its invariance—the construction requires more words than the free Lie construction, yet the proof of invariance becomes simpler and more conceptual.

In Section 8, we discuss the relationship of ζ with the BF topological quantum field theory, and in Section 9, we explain how a certain reduction of ζ becomes a system of formulae for the (multivariable) Alexander polynomial which, in some senses, is better than any previously available formula.

Section 10 is for “odds and ends”—things worth saying, yet those that are better postponed to the end. This includes the details of some definitions and proofs, some words about our conventions, and an attempt at explaining how I think about “meta” structures.

Remark 1.3 Nothing of substance places this paper in \mathbb{R}^4 . Everything works just as well in \mathbb{R}^d for any $d \geq 4$, with “balloons” meaning $(d-2)$ -dimensional spheres and “hoops” always meaning 1-dimensional circles. We have only specialized to $d = 4$ only for reasons of concreteness.

2 The Objects

2.1 Ribbon-Knotted Balloons and Hoops

This paper is about ribbon-knotted balloons (S^2 's) and hoops (or loops, or S^1 's) in \mathbb{R}^4 or, equivalently, in S^4 . Throughout this paper, T and H will denote finite¹ (not necessarily disjoint) sets of “labels”, where the labels in T label the balloons (though for reasons that will become clear later, they are also called “tail labels” and the things they label are sometimes called “tails”), and the labels in H label the hoops (though they are sometimes called “head labels” and they sometimes label “heads”).

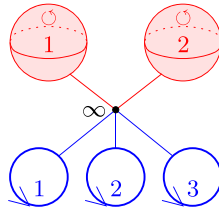
Definition 2.1 A $(T; H)$ -labelled ribbon-knotted balloons and hoops (rKBH) is a ribbon² up-to-isotopy embedding into \mathbb{R}^4 or into S^4 of $|T|$ -oriented 2-spheres labelled by the elements of T (the balloons), of $|H|$ -oriented circles labelled by the elements of H (the hoops), and of $|T| + |H|$ strings (namely, intervals) connecting the $|T|$ balloons and the $|H|$ hoops to some fixed base point, often denoted ∞ . Thus a $(\underline{2}; \underline{3})$ -labelled³ rKBH, for

¹The bulk of the paper easily generalizes to the case where H (not T !) is infinite, though nothing is gained by allowing H to be infinite.

²The adjective “ribbon” will be explained in Definition 2.4.

³See “notational conventions”, Section 10.5.

94 example, is a ribbon up-to-isotopy embedding into \mathbb{R}^4 or into S^4 of the space drawn below.
 95 Let $\mathcal{K}^{rbh}(T; H)$ denote the set of all $(T; H)$ -labelled rKBHs.



96 Recall that 1D objects cannot be knotted in 4D space. Hence, the hoops in an rKBH
 97 are not in themselves knotted, and hence an rKBH may be viewed as a knotting of the
 98 $|T|$ balloons in it, along with a choice of $|H|$ elements of the fundamental group of the
 99 complement of the balloons. Likewise, the $|T| + |H|$ strings in an rKBH only matter as
 100 homotopy classes of paths in the complement of the balloons. In particular, they can be
 101 modified arbitrarily in the vicinity of ∞ , and hence they don't even need to be specified
 102 near ∞ —it is enough that we know that they emerge from a small neighbourhood of ∞
 103 (small enough so as to not intersect the balloons) and that each reaches its balloon or its
 104 hoop.

105 Conveniently, we often pick our base point to be the point ∞ of the formula
 106 $S^4 = \mathbb{R}^4 \cup \{\infty\}$ and hence, we can draw rKBHs in \mathbb{R}^4 (meaning, of course, that we draw
 107 in \mathbb{R}^2 and adopt conventions on how to lift these drawings to \mathbb{R}^4).

108 We will usually reserve the labels x, y and z for hoops; the labels u, v and w for balloons
 109 and the labels a, b and c for things that could be either balloons or hoops. With almost no
 110 risk of ambiguity, we also use x, y and z , along also with t , to denote the coordinates of \mathbb{R}^4 .
 111 Thus, \mathbb{R}^2_{xy} is the xy plane within \mathbb{R}^4 , $\mathbb{R}^3_{\rho_{xy}}$ is the hyperplane perpendicular to the z -axis and
 112 \mathbb{R}^4_{txyz} is just another name for \mathbb{R}^4 .

113 Examples 2.2 and 2.3 are more than just examples, for they introduce much notation that
 114 we use later on.

115 *Example 2.2* The first four examples of rKBHs are the “four generators” shown in Fig. 1:

- 116 • $h\epsilon_x$ is an element of $\mathcal{K}^{rbh}(\emptyset; \{x\})$ (more precisely, $\mathcal{K}^{rbh}(\emptyset; \{x\})$). It has a single hoop
 117 extending from near ∞ and back to near ∞ , and as indicated above, we didn't bother

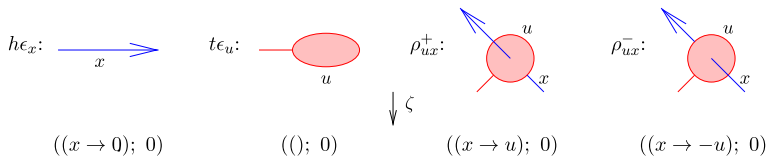
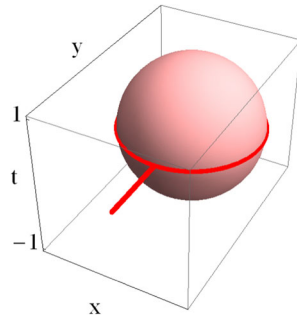


Fig. 1 The four generators $h\epsilon_x$, $t\epsilon_u$, ρ_{ux}^+ and ρ_{ux}^- , drawn in \mathbb{R}^3_{xyz} (ρ_{ux}^\pm differ in the direction in which x pierces u —from below at ρ_{ux}^+ and from above at ρ_{ux}^-). The lower part of the figure previews the values of the main invariant ζ discussed in this paper on these generators. More later, in Section 5

to indicate how it closes near ∞ and how it is connected to ∞ with an extra piece of string. Clearly, $h \in \epsilon_x$ is the “unknotted hoop”.

- $t \in u$ is an element of $\mathcal{K}^{bh}(u; \cdot)$. As a picture in \mathbb{R}^3_{xyz} , it looks like a simplified tennis racket, consisting of a handle, a rim, and a net. To interpret a tennis racket in \mathbb{R}^4 , we embed \mathbb{R}^3_{xyz} into \mathbb{R}^4_{txyz} as the hyperplane $[t = 0]$, and inside it, we place the handle and the rim as they were placed in \mathbb{R}^3_{xyz} . We also make two copies of the net, the “upper” copy and the “lower” copy. We place the upper copy so that its boundary is the rim and so that its interior is pushed into the $[t > 0]$ half-space (relative to the pictured $[t = 0]$ placement) by an amount proportional to the distance from the boundary. Similarly, we place the lower copy, except we push it into the $[t < 0]$ half space. Thus, the two nets along with the rim make a 2-sphere in \mathbb{R}^4 , which is connected to ∞ using the handle. Clearly, $t \in u$ is the “unknotted balloon” (see below). We orient $t \in u$ by adopting the conventions that surfaces drawn in the plane are oriented counterclockwise (unless otherwise noted) and that when pushed to 4D, the upper copy retains the original orientation while the lower copy reverses it.

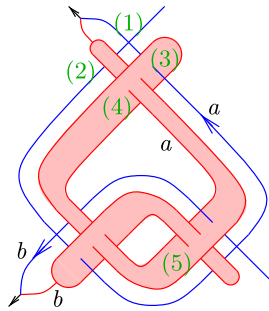


Warning: the vertical direction here is the “time” coordinate t , so this is an \mathbb{R}^3_{txy} picture.

- ρ_{ux}^+ is an element of $\mathcal{K}^{bh}(u; x)$. It is the 4D analogue of the “positive Hopf link”. Its picture in Fig. 1 should be interpreted in much the same way as the previous two—what is displayed should be interpreted as a 3D picture using standard conventions (what’s hidden is “below”), and then it should be placed within the $[t = 0]$ copy of \mathbb{R}^3_{xyz} in \mathbb{R}^4 . This done, the racket’s net should be split into two copies, one to be pushed to $[t > 0]$ and the other to $[t < 0]$. In \mathbb{R}^3_{xyz} , it appears as if the hoop x intersects the balloon u right in the middle. Yet in \mathbb{R}^4 , our picture represents a legitimate knot as the hoop is embedded in $[t = 0]$, the nets are pushed to $[t \neq 0]$, and the apparent intersection is eliminated.
- ρ_{ux}^- is the “negative Hopf link”. It is constructed out of its picture in exactly the same way as ρ_{ux}^+ . We postpone to Section 10.1 the explanation of why ρ_{ux}^+ is “positive” and ρ_{ux}^- is “negative”.

145 *Example 2.3* Below is a somewhat more sophisticated example of an rKBH with
 146 two balloons labelled a and b and two hoops labelled with the same labels
 147 (hence it is an element of $\mathcal{K}^{rbh}(a, b; a, b)$). It should be interpreted using the
 148 same conventions as in the previous example, though some further comments are in
 149 order:

- 150 • The “crossing” marked (1) below is between two hoops and in 4D it mat-
 151 ters not if it is an overcrossing or an undercrossing. Hence, we did not bother
 152 to indicate which of the two it is. A similar comment applies in two other
 places.



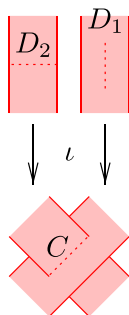
153

- 154 • Likewise, crossing (2) is between a 1D strand and a thin tube, and its sense is immate-
 155 rial. For no real reason, we’ve drawn the strand “under” the tube, but had we drawn it
 156 “over”, it would be the same rKBH. A similar comment applies in two other places.
- 157 • Crossing (3) is “real” and is similar to ρ^- in the previous example. Two other crossings
 158 in the picture are similar to ρ^+ .
- 159 • Crossing (4) was not seen before, though its 4D meaning should be clear from our
 160 interpretation rules: nets are pushed up (or down) along the t coordinate by an amount
 161 proportional to the distance from the boundary. Hence, the wider net in crossing (4)
 162 gets pushed more than the narrower one, and hence, in 4D, they do not intersect even
 163 though their projections to 3D do intersect, as the figure indicates. A similar comment
 164 applies in two other places.
- 165 • Our example can be simplified a bit using isotopies. Most notably, crossing (5) can be
 166 eliminated by pulling the narrow “\” finger up and out of the wider “/” membrane. Yet
 167 note that a similar feat cannot be achieved near (3) and (4). Over there, the wider “/”
 168 finger cannot be pulled down and away from the narrower “\” membrane and strand
 169 without a singularity along the way.

170 We can now complete Definition 2.1 by providing the definition of “ribbon
 171 embedding”.

172 **Definition 2.4** We say that an embedding of a collection of 2-spheres S_i into \mathbb{R}^4 (or into
 173 S^4) is a “ribbon” if it can be extended to an immersion ι of a collection of 3-balls B_i
 174 whose boundaries are the S_i s, so that the singular set $\Sigma \subset \mathbb{R}^4$ of ι consists of transverse

self-intersections, and so that each connected component C of Σ is a “ribbon singularity”: $\iota^{-1}(C)$ consists of two closed disks D_1 and D_2 , with D_1 embedded in the interior of one of the B_i and with D_2 embedded with its interior in the interior of some B_j and with its boundary in $\partial B_j = S_j$. A dimensionally reduced illustration is below. The ribbon condition does not place any restriction on the hoops of an rKBH.

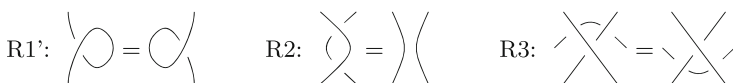


It is easy to verify that all the examples above are ribbon, and that all the operations we define below preserve the ribbon condition.

There is much literature about ribbon knots in \mathbb{R}^4 . See, e.g. [4, 5, 11, 12, 15, 26, 27].

2.2 Usual Tangles and the Map δ

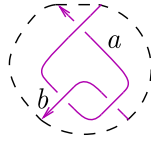
For the purposes of this paper, a “usual tangle”,⁴ or a “u-tangle”, is a “framed pure labelled tangle in a disk”. In detail, it is a piece of an oriented knot diagram drawn in a disk, having no closed components and with its components labelled by the elements of some set S , with all regarded modulo the Reidemeister moves R1’, R2 and R3:



The set of all tangles with components labelled by S is denoted as $u\mathcal{T}(S)$. An example of a member of $u\mathcal{T}(a, b)$ is below. Note that our u-tangles do not have a specific “up” direction so they do not form a category, and that the condition “no closed components” prevents them from being a planar algebra. In fact, $u\mathcal{T}$ carries almost no interesting algebraic structure. Yet it contains knots (as 1-component tangles) and more generally, by restricting to a subset, it

⁴Better English would be “ordinary tangle”, but I want the short form to be “u-tangle”, which fits better with the “v-tangles” and “w-tangles” that arise later in this paper.

193 contains “pure tangles” or “string links” [9]. And in the next section, $u\mathcal{T}$ will be generalized
 194 to $v\mathcal{T}$ and to $w\mathcal{T}$, which do carry much interesting structure.



194 There is a map $\delta: u\mathcal{T}(S) \rightarrow \mathcal{K}^{rbh}(S; S)$. The picture should precede the words, and it
 195 appears as the left half of Fig. 2.

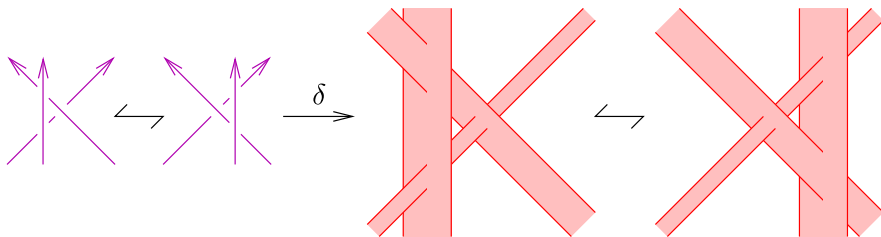
197 In words, if $T \in u\mathcal{T}(S)$, to make $\delta(T)$ we convert each strand $s \in S$ of T into
 198 a pair of parallel entities: a copy of s on the right and a band on the left (T is a planar
 199 diagram and s is oriented, so “left” and “right” make sense). We cap the resulting band
 200 near its beginning and near its end, connecting the cap at its end to ∞ (namely, to outside
 201 the picture) with an extra piece of string—so that when the bands are pushed to 4D in the
 202 usual way, they become balloons with strings. Finally, near the crossings of T we apply the
 following (sign-preserving) local rules:



203

204 **Proposition 2.5** *The map δ is well defined.*

205 *Proof* We need to check that the Reidemeister moves in $u\mathcal{T}$ are carried to isotopies in
 206 \mathcal{K}^{rbh} . We’ll only display the “band part” of the third Reidemeister move, as everything else
 207 is similar or easier:

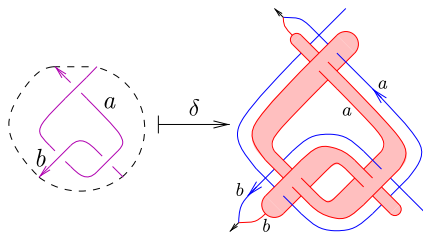


208 The fact that the two “band diagrams” above are isotopic before “inflation” to \mathbb{R}^4 , and
 209 hence also after, is visually obvious. □

210 2.3 The Fundamental Invariant and the Near-Injectivity of δ

211 The “Fundamental invariant” $\pi(K)$ of $K \in \mathcal{K}^{rbh}(u_i; x_j)$ is the triple $(\pi_1(K^c); m; l)$,
 212 where within this triple:

Fig. 2 A $T_0 \mapsto \delta(T_0)$ example, and its invariant ζ of Section 5 (computed to degree 3)



$$\begin{aligned}
 T_0 &= \mathbf{R}^- [3, a] \mathbf{R}^+ [b, 2] \mathbf{R}^+ [1, 4]; \\
 T_0 // \mathbf{dm}[2, 1, 1] // \mathbf{dm}[4, b, b] // \mathbf{dm}[1, a, a] // \\
 &\quad \mathbf{dm}[3, a, a] \\
 M \left[\left\{ a \rightarrow \text{LS} \left[-\overline{a} + \overline{b}, \frac{3}{2} \overline{ab}, \frac{13}{12} \overline{a \overline{a} b} - \frac{13}{12} \overline{a \overline{b} b} \right], \right. \right. \\
 &\quad \left. \left. b \rightarrow \text{LS} \left[\overline{a}, 0, -\overline{a \overline{a} b} \right] \right\}, \text{CWS} \left[-\overline{a}, -\overline{ab}, -\frac{\overline{aab}}{2} - \frac{\overline{abb}}{2} \right] \right]
 \end{aligned}$$

- The first entry is the fundamental group of the complement of the balloons of K , with basepoint taken to be at ∞ . 213
214
- The second entry m is the function $m: T \rightarrow \pi_1(K^c)$ which assigns to a balloon $u \in T$ its “base meridian” m_u —the path obtained by travelling along the string of u from ∞ to near the balloon, then Hopf-linking with the balloon u once in the positive direction much like in the generator ρ^+ of Fig. 1, and then travelling back to the basepoint again along the string of u . 215
216
217
218
219
- The third entry l is the function $l: H \rightarrow \pi_1(K^c)$ which assigns to hoop $x \in H$ its longitude l_x —it is simply the hoop x itself regarded as an element of $\pi_1(K^c)$. 220
221

Thus, for example, with $\langle \alpha \rangle$ denoting the group generated by a single element α and following the “notational conventions” of Section 10.5 for “inline functions”, 222
223

$$\begin{aligned}
 \pi(h \in_x) &= (1; (); (x \rightarrow 1)), & \pi(t \in_u) &= (\langle \alpha \rangle; (u \rightarrow \alpha); ()) \\
 \text{and } \pi(\rho_{ux}^\pm) &= (\langle \alpha \rangle; (u \rightarrow \alpha); (x \rightarrow \alpha^{\pm 1})).
 \end{aligned}$$
224

We leave the following proposition as an exercise for the reader: 225

Proposition 2.6 *If T is an \underline{n} -labelled u -tangle, then $\pi(\delta(T))$ is the fundamental group of the complement of T (within a 3D space!), followed by the list of meridians of T (placed near the outgoing ends of the components of T), followed by the list of longitudes of T .* 226
227
228

It is well known (e.g. [17, Theorem 6.1.7]) that knots are determined by the fundamental group of their complements, along with their “peripheral systems”, namely their meridians and longitudes regarded as elements of the fundamental groups of their complements. Thus we have the following: 229
230
231
232

Theorem 2.7 *When restricted to long knots (which are the same as knots), δ is injective.* 233

Remark 2.8 A similar map studied by Winter [30] is (sometimes) 2 to 1, as it retains less orientation information. 234
235

I expect that δ is also injective on arbitrary tangles and that experts in geometric topology would consider this trivial, but this result would be outside of my tiny puddle. 236
237



Fig. 3 The “overcrossing commute” (OC) relation and the gist of the proof that it is respected by δ , and the “undercrossing commute” (UC) relation and the gist of the reason why it is not respected by δ

238 2.4 The Extension to v/w-Tangles and the Near-Surjectivity of δ

239 The map δ can be extended to “virtual crossings” [16] using the local assignment

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \xrightarrow{\delta} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \quad (1)$$

240 In a few more words, u-tangles can be extended to “v-tangles” by allowing virtual crossings
 241 as on the left hand side of (1), and then modding out by the “virtual Reidemeister moves”
 242 and the “mixed move”/“detour move” of [16].⁵ One may then observe, as in Fig. 3, that δ
 243 respects those moves as well as the overcrossings commute relation (yet not the undercross-
 244 ings commute relation). Hence, δ descends to the space $w\mathcal{T}$ of w-tangles, which are the
 245 quotient of v-tangles by the overcrossings commute relation.

246 A topological-flavoured construction of δ appears in Section 10.2.

247 The newly extended $\delta: w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$ cannot possibly be surjective, for the rKBHs in
 248 its image always have an equal number of balloons as hoops, with the same labels. Yet, if
 249 we allow the deletion of components, δ becomes surjective:

250 **Theorem 2.9** For any KTG K , there is some w-tangle T so that K is obtained from $\delta(T)$ by
 251 the deletion of some of its components.

252 *Proof* (Sketch) This is a variant of Theorem 3.1 of Satoh’s [26]. Clearly, every knotting
 253 of 2-spheres in \mathbb{R}^4 can be obtained from a knotting of tubes by capping those tubes. Satoh
 254 shows that any knotting of tubes is in the image of a map he calls “tube”, which is identical
 255 to our δ except that our δ also includes the capping (good) and an extra hoop component for
 256 each balloon (harmless as they can be deleted). Finally, to get the hoops of K , simply put
 257 them in as extra strands in T , and then delete the spurious balloons that δ would produce
 258 next to each hoop. □

259 **3 The Operations**

260 3.1 The Meta-Monoid-Action

261 Loosely speaking, an rKBH K is a map of several S^1 ’s and several S^2 ’s into some ambient
 262 space. The former (the hoops of K) resemble elements of π_1 , and the latter (the balloons
 263 of K) resemble elements of π_2 . In general, in homotopy theory, π_1 and π_2 are groups, and
 264 further, there is an action of π_1 on π_2 . Thus, we find that on \mathcal{K}^{rbh} , there are operations that
 265 resemble the group multiplication of π_1 , and the group multiplication of π_2 , and the action
 266 of π_1 on π_2 .

⁵In [16], the mixed/detour move was yet unnamed, and was simply “move (c) of Fig. 2”.

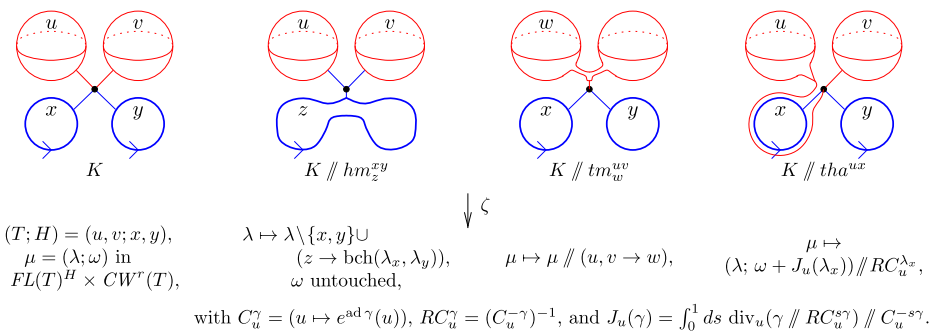


Fig. 4 An rKBH K and the three basic unary operators applied to it. We use schematic notation; K may have plenty more components, and it may actually be knotted. The lower part of the figure is a summary of the main invariant ζ defined in this paper. See Section 5

Let us describe these operations more carefully. Let $K \in \mathcal{K}^{rbh}(T; H)$.

267

- Analogously to the product in π_1 , there is the operation of “concatenating two hoops”. Specifically, if x and y are two distinct labels in H and z is a label not in H (except possibly equal to x or to y), we let⁶ $K // hm_z^{xy}$ be K with the x and y hoops removed and replaced with a single hoop labelled z that traces the path of them both. See Fig. 4. 268
269
270
271
272
- Analogously to the homotopy-theoretic product of π_2 , there is the operation of “merging two balloons”. Specifically, if u and v are two distinct labels in T and w is a label not in T (except possibly equal to u or to v), we let $K // tm_w^{uv}$ be K with the u and v balloons removed and replaced by a single two-lobed balloon (topologically, still a sphere!) labelled w which spans them both. See Fig. 4. 273
274
275
276
277
- Analogously to the homotopy-theoretic action of π_1 on π_2 , there is the operation tha^{ux} (tail by head action on u by x) of re-routing the string of the balloon u to go along the hoop x , as illustrated in Fig. 4. In balloon-theoretic language, after the isotopy which pulls the neck of u along its string, this is the operation of “tying the balloon”, commonly performed to prevent the leakage of air (though admittedly, this will fail in 4D). 278
279
280
281
282
283

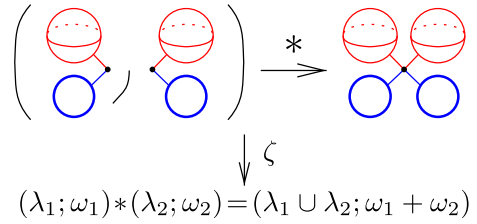
In addition, \mathcal{K}^{rbh} affords the further unary operations $t\eta^u$ (when $u \in T$) of “puncturing” the balloon u (implying, deleting it) and $h\eta^x$ (when $x \in H$) of “cutting” the hoop x (implying, deleting it). These two operations were already used in the statement and proof of Theorem 2.9. 284
285
286
287

In addition, \mathcal{K}^{rbh} affords the binary operation $*$ of “connected sum”, sketched in Fig. 5 (along with its ζ formulae of Section 5). Whenever we have disjoint label sets $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$, it is an operation $\mathcal{K}^{rbh}(T_1; H_1) \times \mathcal{K}^{rbh}(T_2; H_2) \rightarrow \mathcal{K}^{rbh}(T_1 \cup T_2; H_1 \cup H_2)$. We often suppress the $*$ symbol and write $K_1 K_2$ for $K_1 * K_2$. 288
289
290
291

Finally, there are re-labelling operations $h\sigma_b^a$ and $t\sigma_b^a$ on \mathcal{K}^{rbh} , which take a label a (either a head or a tail) and rename it b (provided b is “new”). 292
293

⁶See “notational conventions”, Section 10.5.

Fig. 5 Connected sums



294 **Proposition 3.1** *The operations $*$, $t\sigma_w^u$, $h\sigma_z^x$, $t\eta^u$, $h\eta^x$, hm_z^{xy} , tm_w^{uv} and $tha^{\mu x}$ and the*
 295 *special elements $t\epsilon_u$ and $h\epsilon_x$ have the following properties:*

- 296 • *If the labels involved are distinct, the unary operations all commute with each other.*
- 297 • *The re-labelling operations have some obvious properties and interactions:*
 298 $\sigma_b^a // \sigma_c^b = \sigma_c^a$, $hm_x^{xy} // h\sigma_z^x = hm_z^{xy}$, etc., *and similarly for the deletion operations*
 299 η^a .
- 300 • *$*$ is commutative and associative; where it makes sense, it bi-commutes with the unary*
 301 *operations ($(K_1 // hm_z^{xy}) * K_2 = (K_1 * K_2) // hm_z^{xy}$, etc.).*
- 302 • *$t\epsilon_u$ and $h\epsilon_x$ are “units”:*

$$303 \quad (K * t\epsilon_u) // tm_w^{uv} = K // t\sigma_w^v, \quad (K * t\epsilon_u) // tm_w^{vu} = K // t\sigma_w^v,$$

$$(K * h\epsilon_x) // hm_z^{xy} = K // h\sigma_z^y, \quad (K * h\epsilon_x) // hm_z^{yx} = K // h\sigma_z^y.$$

- 304 • *Meta-associativity of hm , similar to the associativity in π_1 :*

$$hm_x^{xy} // hm_x^{xz} = hm_y^{yz} // hm_x^{xy}. \tag{2}$$

- 305 • *Meta-associativity of tm , similar to the associativity in π_2 :*

$$tm_u^{uv} // tm_u^{uw} = tm_v^{vw} // tm_u^{uv}. \tag{3}$$

- 306 • *Meta-actions commute. The following is a special case of the first property above,*
 307 *yet it deserves special mention because later in this paper it will be the only such*
 308 *commutativity that is non-obvious to verify:*

$$tha^{\mu x} // tha^{\nu y} = tha^{\nu y} // tha^{\mu x}. \tag{4}$$

- 309 • *Meta-action axiom t , similar to $(uv)^x = u^x v^x$:*

$$tm_w^{uv} // tha^{wx} = tha^{\mu x} // tha^{\nu x} // tm_w^{uv}. \tag{5}$$

- 310 • *Meta-action axiom h , similar to $u^{xy} = (u^x)^y$:*

$$hm_z^{xy} // tha^{uz} = tha^{\mu x} // tha^{\nu y} // hm_z^{xy}. \tag{6}$$

311 *Proof* The first four properties say almost nothing and we did not even specify them in
 312 full.⁷ The remaining four deserve attention, especially in the light of the fact that the veri-
 313 fication of their analogues later in this paper will be non-trivial. Yet in the current context,
 314 their verification is straightforward. □

315 Later, we will seek to construct invariants of rKBHs by specifying their values on
 316 some generators and by specifying their behaviour under our list of operations. Thus, it is
 317 convenient to introduce a name for the algebraic structure of which \mathcal{K}^{rbh} is an instance:

⁷We feel that the clarity of this paper is enhanced by this omission.

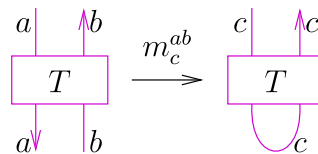
Definition 3.2 A meta-monoid-action (MMA) M is a collections of sets $M(T; H)$, one for each pair of finite sets of labels T and H , along with partially defined operations⁸ $*$, $t\sigma_v^u$, $h\sigma_y^x$, $t\eta^u$, $h\eta^x$, hm_z^{xy} , tm_w^{uv} and tha^{ux} , and with special elements $t\epsilon_u \in M(\{u\}; \emptyset)$ and $h\epsilon_x \in M(\emptyset; \{x\})$, which together satisfy the properties in Proposition 3.1.

For the rationale behind the name “meta-monoid-action” see Section 10.3. In Section 10.3.5, we note that \mathcal{K}^{rbh} in fact has the further structure making it a meta-group-action (or more precisely, a meta-Hopf-algebra-action).

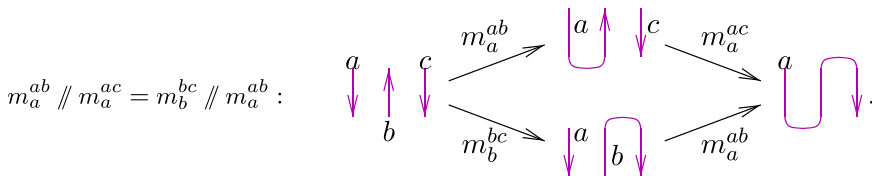
3.2 The Meta-Monoid of Tangles and the Homomorphism δ

Our aim in this section is to show that the map $\delta: w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$ of Sections 2.2 and 2.4, which maps w -tangles to knotted balloons and hoops, is a “homomorphism”. But first, we have to discuss the relevant algebraic structures on $w\mathcal{T}$ and on \mathcal{K}^{rbh} .

$w\mathcal{T}$ is a “meta-monoid” (see Section 10.3.2). Namely, for any finite set S of “strand labels” $w\mathcal{T}(S)$ is a set, and whenever we have a set S of labels and three labels $a \neq b$ and c not in it, we have the operation $m_c^{ab}: w\mathcal{T}(S \cup \{a, b\}) \rightarrow w\mathcal{T}(S \cup \{c\})$ of “concatenating strand a with strand b and calling the resulting strand c ”. See the picture below and note that while on $u\mathcal{T}$, the operation m_c^{ab} would be defined only if the head of a happens to be adjacent to the tail of b ; on $v\mathcal{T}$ and on $w\mathcal{T}$, this operation is always defined as the head of a can always be brought near the tail of b by adding some virtual crossings, if necessary. $w\mathcal{T}$ trivially also carries the rest of the necessary structure to form a meta-monoid—namely, strand relabelling operations σ_b^a , strand deletion operations η^a , and a disjoint union operation $*$, and units ϵ_a (tangles with a single unknotted strand labelled a).



It is easy to verify the associativity property (compare with (32) of Section 10.3.1):



It is also easy to verify that if a tangle $T \in w\mathcal{T}(a, b)$ is non-split, then $T \neq (T \parallel \eta^b) * (T \parallel \eta^a)$, so in the sense of Section 10.3.2, $w\mathcal{T}$ is non-classical.

⁸ tm_w^{uv} , for example, is defined on $M(T; H)$ exactly when $u, v \in T$ yet $w \notin T \setminus \{u, v\}$. All other operations behave similarly.

342 **Solution of Riddle 1.1** $\pi_T \cong \pi_1 \times \pi_2$ (a semi-direct product!), so if you know all about
 343 π_1 and π_2 (and the action of π_1 and π_2), you know all about π_T .

344 \mathcal{K}^{rbh} is an analogue of both π_1 and π_2 . In homotopy theory, the group π_1 acts on π_2 so
 345 one may form the semi-direct product $\pi_1 \times \pi_2$. In a similar manner, one may put a “com-
 346 bined” multiplication on that part of \mathcal{K}^{rbh} in which the balloons and the hoops are matched
 347 together. More precisely, given a finite set of labels S , let $\mathcal{K}^{b=h}(S) := \mathcal{K}^{rbh}(S; S)$ be the set
 348 of rKBHs whose balloons and whose hoops are both labelled with labels in S . Then
 349 define $dm_c^{ab} : \mathcal{K}^{b=h}(S \cup \{a, b\}) \rightarrow \mathcal{K}^{b=h}(S \cup \{c\})$ (the prefix d is for “diagonal” or
 350 “double”) by

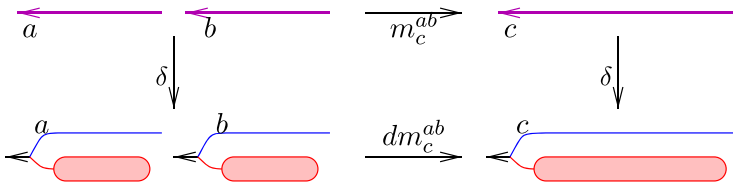
$$dm_c^{ab} = tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}. \tag{7}$$

351 It is a routine exercise to verify that the properties (2)–(6) of hm , tm and tha imply that dm
 352 is meta-associative:

$$dm_a^{ab} \parallel dm_a^{ac} = dm_b^{bc} \parallel dm_a^{ab}.$$

353 Thus, dm (along with diagonal η ’s and σ ’s and an unmodified $*$) puts a meta-monoid
 354 structure on $\mathcal{K}^{b=h}$.

355 **Proposition 3.3** $\delta : w\mathcal{T} \rightarrow \mathcal{K}^{b=h}$ is a meta-monoid homomorphism. (A rough picture is
 356 below: in the picture a and b are strands within the same tangle, and they may be knotted
 357 with each other and with possible further components of that tangle). \square



358 **3.3 Generators and Relations for \mathcal{K}^{rbh}**

359 It is always good to know that a certain algebraic structure is finitely presented. If we had a
 360 complete set of generators and relations for \mathcal{K}^{rbh} , for example, we could define a “homo-
 361 morphic invariant” of rKBHs by picking some target MMA \mathcal{M} (Definition 3.2), declaring
 362 the values of the invariant on the generators, and verifying that the relations are satisfied.
 363 Hence, it’s good to know the following:

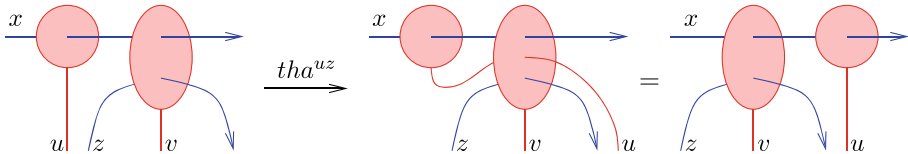
364 **Theorem 3.4** The MMA \mathcal{K}^{rbh} is generated (as an MMA) by the four rKBHs $h\epsilon_x$, $t\epsilon_u$, ρ_{ux}^+
 365 and ρ_{ux}^- of Fig. 1.

366 *Proof* By Theorem 2.9 and the fact that the MMA operations include component dele-
 367 tions $t\eta^\mu$ and $h\eta^x$, it follows that \mathcal{K}^{rbh} is generated by the image of δ . By the previous

proposition and the fact (7) that dm can be written in terms of the MMA operations of \mathcal{K}^{rbh} , it follows that \mathcal{K}^{rbh} is generated by the δ -images of the generators of $w\mathcal{T}$. But the generators of $w\mathcal{T}$ are the virtual crossing $\begin{smallmatrix} \times \\ a & b \end{smallmatrix}$ and the right-handed and left-handed crossings $\begin{smallmatrix} \nearrow \\ a & b \end{smallmatrix}$ and $\begin{smallmatrix} \searrow \\ a & b \end{smallmatrix}$; and so, the theorem follows from the following easily verified assertions:

$$\delta\left(\begin{smallmatrix} \times \\ a & b \end{smallmatrix}\right) = t\epsilon_a h\epsilon_a t\epsilon_b h\epsilon_b, \delta\left(\begin{smallmatrix} \nearrow \\ a & b \end{smallmatrix}\right) = \rho_{ab}^+ t\epsilon_b h\epsilon_a, \text{ and } \delta\left(\begin{smallmatrix} \searrow \\ a & b \end{smallmatrix}\right) = \rho_{ba}^- t\epsilon_a h\epsilon_b. \quad \square$$

We now turn to the study of relations. Our first is the hardest and most significant, the ‘‘Conjugation Relation’’, whose name is inspired by the group theoretic relation $vu^v = uv$ (here, u^v denotes group conjugation, $u^v = v^{-1}uv$). Consider the following equality:



Easily, the rKBH on the very left is $\rho_{ux}^+(\rho_{vy}^+\rho_{wz}^+ \parallel tm_v^{vw}) \parallel hm_x^{xy}$ and the one on the very right is $(\rho_{vx}^+\rho_{wz}^+ \parallel tm_v^{vw})\rho_{uy}^+ \parallel hm_x^{xy}$, and so

$$\rho_{ux}^+\rho_{vy}^+\rho_{wz}^+ \parallel tm_v^{vw} \parallel hm_x^{xy} \parallel tha^{uz} = \rho_{vx}^+\rho_{wz}^+\rho_{uy}^+ \parallel tm_v^{vw} \parallel hm_x^{xy}. \quad (8)$$

Definition 3.2 Let \mathcal{K}_0^{rbh} be the MMA freely generated by symbols $\rho_{ux}^\pm \in \mathcal{K}_0^{rbh}(u; x)$, modulo the following relations:

- Relabelling: $\rho_{ux}^\pm \parallel h\sigma_y^x \parallel t\sigma_v^u = \rho_{vy}^\pm$.
- Cutting and puncturing: $\rho_{ux}^\pm \parallel h\eta^x = t\epsilon_u$ and $\rho_{ux}^\pm \parallel t\eta^u = h\epsilon_x$.
- Inverses: $\rho_{ux}^+\rho_{vy}^- \parallel tm_w^{uv} \parallel hm_z^{xy} = t\epsilon_w h\epsilon_z$.
- Conjugation relations: for any $s_{1,2} \in \{\pm\}$,

$$\rho_{ux}^{s_1}\rho_{vy}^{s_2}\rho_{wz}^{s_2} \parallel tm_v^{vw} \parallel hm_x^{xy} \parallel tha^{uz} = \rho_{vx}^{s_2}\rho_{wz}^{s_2}\rho_{uy}^{s_1} \parallel tm_v^{vw} \parallel hm_x^{xy}.$$

- Tail commutativity: on any inputs, $tm_w^{uv} = tm_w^{vu}$.
- Framing independence:

$$\rho_{ux}^\pm \parallel tha^{ux} = \rho_{ux}^\pm. \quad (9)$$

The following proposition, whose proof we leave as an exercise, says that \mathcal{K}_0^{rbh} is a pretty good approximation to \mathcal{K}^{rbh} :

Proposition 3.3 The obvious maps $\pi = \mathcal{K}_0^{rbh} \rightarrow \mathcal{K}^{rbh}$ and $\delta = w\mathcal{T} \rightarrow \mathcal{K}_0^{rbh}$ are well defined. □

Conjecture 3.7 The projection $\pi : \mathcal{K}_0^{rbh} \rightarrow \mathcal{K}^{rbh}$ is an isomorphism.

We expect that there should be a Reidemeister-style combinatorial calculus of ribbon knots in \mathbb{R}^4 . The above conjecture is that the definition of \mathcal{K}_0^{rbh} is such a calculus. We expect that given any such calculus, the proof of the conjecture should be easy. In particular, the above conjecture is equivalent to the statement that the stated relations in the definition of

395 $w\mathcal{T}$ generate the relations in the kernel of Satoh’s Tube map δ_0 (see Section 10.2), and this
 396 is equivalent to the conjecture whose proof was attempted at [31]. Though I understood by
 397 private communication with B. Winter that [31] is presently flawed.

398 In the absence of a combinatorial description of \mathcal{K}^{rbh} , we replace it by \mathcal{K}_0^{rbh} throughout
 399 the rest of this paper. Hence, we construct invariants of elements of \mathcal{K}_0^{rbh} instead of invari-
 400 ants of genuine rKBHs. Yet note that the map $\delta = w\mathcal{T} \rightarrow \mathcal{K}_0^{rbh}$ is well-defined, so our
 401 invariants are always good enough to yield invariants of tangles and virtual tangles.

402 3.4 Example: The Fundamental Invariant

403 The fundamental invariant π of Section 2.3 is defined in a direct manner on \mathcal{K}^{rbh} and does
 404 not need to suffer from the difficulties of the previous section. Yet, it can also serve as an
 405 example for our approach for defining invariants on \mathcal{K}_0^{rbh} using generators and relations.

406 **Definition 3.8** Let $\Pi(T; H)$ denote the set of all triples $(G; m; l)$ of a group G along with
 407 functions $m \in G^T$ and $l \in G^H$, regarded modulo group isomorphisms with their obvious
 408 action on m and l .⁹ Define MMA operations $(*, t\sigma_v^u, h\sigma_y^x, t\eta^u, h\eta^x, tm_w^{uv}, hm_z^{xy}, tha^{ux})$ on
 409 $\Pi = \{\Pi(T; H)\}$ and units $t\epsilon_u$ and $h\epsilon_x$ as follows:

- 410 • $*$ is the operation of taking the free product $G_1 * G_2$ of groups and concatenating the
 411 lists of heads and tails:

$$(G_1; m_1; l_1) * (G_2; m_2; l_2) := (G_1 * G_2; m_1 \cup m_2; l_1 \cup l_2).$$

- 412 • $t\sigma_b^a / h\sigma_b^a$ relabels an element labelled a to be labelled b .
- 413 • $t\eta^u / h\eta^x$ removes the element labelled u / x .
- 414 • tm_w^{uv} “combines” u and v to make w . Precisely, it replaces the input group G with
 415 $G' = G / \langle m_u = m_v \rangle$, removes the tail labels u and v , and introduces a new tail, the
 416 element $m_u = m_v$ of G' and labels it w :

$$tm_w^{uv}(G; m; l) := (G / \langle m_u = m_v \rangle; (m \setminus \{u, v\}) \cup (w \rightarrow m_u); l).$$

- 417 • hm_z^{xy} replaces two elements in l by their product:

$$hm_z^{xy}(G; m; l) := (G, m, (l \setminus \{x, y\}) \cup (z \rightarrow l_x l_y)).$$

- 418 • The best way to understand the action of tha^{ux} is as “the thing that makes the funda-
 419 mental invariant π a homomorphism, given the geometric interpretation of tha^{ux} on
 420 \mathcal{K}^{rbh} in Section 3.1”. In formulae, this becomes

$$tha^{ux}(G; m; l) := (G * \langle \alpha \rangle / \langle m_u = l_x \alpha l_x^{-1} \rangle; (m \setminus u) \cup (u \rightarrow \alpha), l),$$

421 where α is some new element that is added to G .

- 422 • $t\epsilon_u = (\langle \alpha \rangle; (u \rightarrow \alpha); ())$ and $h\epsilon_x = (1; (); (x \rightarrow 1))$.

423 We state the following without its easy topological proof:

424 **Proposition 3.9** $\pi: \mathcal{K}^{rbh} \rightarrow \Pi$ is a homomorphism of MMAs. □

425 A consequence is that π can be computed on any rKBH starting from its values on the
 426 generators of \mathcal{K}^{rbh} as listed in Section 2.3 and then using the operations of Definition 3.8.

⁹I ignore set-theoretic difficulties. If you insist, you may restrict to countable groups or to finitely presented groups.

Comment 3.10 The fundamental groups of ribbon 2-knots are “labelled-oriented tree” (LOT) groups in the sense of Howie [13, 14]. Howie’s definition has an obvious extension to labelled-oriented forests (LOF), yielding a class of groups that may be called “LOF groups”. One may show that the fundamental groups of complements of rKBHs are always LOF groups. One may also show that the subset Π^{LOF} of Π in which the group component G is an LOF group is a sub-MMA of Π . Therefore $\pi = \mathcal{K}^{rbh} \rightarrow \Pi^{\text{LOF}}$ is also a homomorphism of MMAs; I expect it to be an isomorphism or very close to an isomorphism. Thus, much of the rest of this paper can be read as a “theory of homomorphic (in the MMA sense) invariants of LOF groups”. I don’t know how much it may extend to a similar theory of homomorphic invariants of bigger classes of groups.

4 The Free Lie Invariant

In this section, we construct ζ_0 , the “tree” part to our main tree-and-wheel-valued invariant ζ , by following the scheme of Section 3.3. Yet, before we succeed, it is useful to aim a bit higher and fail, and thus appreciate that even ζ_0 is not entirely trivial.

4.1 A Free Group Failure

If the balloon part of an rKBH K is unknotted, the fundamental group $\pi_1(K^c)$ of its complement is the free group generated by the meridians $(m_u)_{u \in T}$. The hoops of K are then elements in that group and hence, they can be written as words $(w_x)_{x \in H}$ in the m_u ’s and their inverses. Perhaps we can make an MMA \mathcal{W} out of lists (w_x) of free words in letters $m_u^{\pm 1}$ and use it to define a homomorphic invariant $W = \mathcal{K}^{rbh} \rightarrow \mathcal{W}$? All we need, it seems, is to trace how MMA operations on K affect the corresponding list (w_x) of words.

The beginning is promising. $*$ acts on pairs of lists of words by taking the union of those lists. hm_z^{xy} acts on a list of words by replacing w_x and w_y by their concatenation, now labelled z . tm_r^{pq} acts on $\bar{w} = (w_x)$ by replacing every occurrence of the letter m_p and every occurrence of the letter m_q in \bar{w} by a single new letter, m_r .

The problem is with tha^{ux} . Imitating the topology, tha^{ux} should act on $\bar{w} = (w_y)$ by replacing every occurrence of m_u in \bar{w} with $w_x \alpha w_x^{-1}$, where α is a new letter, destined to replace m_u . But w_x may also contain instances of m_u , so after the replacement, $m_u \mapsto \alpha^{w_x}$ is performed; it should be performed again to get rid of the m_u ’s that appear in the “conjugator” w_x . But new m_u ’s are then created, and the replacement should be carried out yet again. . . . The process clearly does not stop, and our attempt failed.

Yet, not all is lost. The latter and latter’s replacements occur within conjugators of conjugators, deeper and deeper into the lower central series of the free groups involved. Thus, if we replace free groups by some completion thereof in which deep members of the lower central series are “small”, the process becomes convergent. This is essentially what will be done in the next section.

4.2 A Free Lie Algebra Success

Given a set T , let $FL(T)$ denote the graded completion of the free Lie algebra on the generators in T (sometimes we will write “ FL ” for “ $FL(T)$ for some set T ”). We define a meta-monoid-action M_0 as follows. For any finite set T of “tail labels” and any finite set H or “head labels”, we let

$$M_0(T; H) := FL(T)^H$$

468 be the set of H -labelled arrays of elements of $FL(T)$. On $M_0 := \{M_0(T; H)\}$, we define
 469 operations as follows, starting from the trivial and culminating with the most interesting,
 470 tha^{ax} . All of our definitions are directly motivated by the “failure” of the previous section;
 471 in establishing the correspondence between the definitions below and the ones above, one
 472 should interpret $\lambda = (\lambda_x) \in M_0(T; H)$ as “a list of logarithms of a list of words (w_x) ”.

- 473 • $h\sigma_y^x$ is simply σ_y^x as explained in the conventions section, Section 10.5.
- 474 • $t\sigma_v^u$ is induced by the map $FL(T) \rightarrow FL((T \setminus u) \cup \{v\})$ in which the generator u is
 475 mapped to the generator v .
- 476 • $t\eta$ acts by setting one of the tail variables to 0, and $h\eta$ acts by dropping an array element.
 477 Thus, for $\lambda \in M_0(T; H)$,

$$\lambda \parallel t\eta^u = \lambda \parallel (u \mapsto 0) \quad \text{and} \quad \lambda \parallel h\eta^x = \eta \setminus x.$$

- 478 • If $\lambda_1 \in M_0(T_1; H_1)$ and $\lambda_2 \in M_0(T_2; H_2)$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$),
 479 then

$$\lambda_1 * \lambda_2 := (\lambda_1 \parallel \iota_1) \cup (\lambda_2 \parallel \iota_2)$$

480 where ι_i are the natural embeddings $\iota_i: FL(T_i) \hookrightarrow FL(T_1 \cup T_2)$, for $i = 1, 2$.

- 481 • If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel tm_w^{uv} := \lambda \parallel (u, v \mapsto w),$$

482 where $(u, v \mapsto w)$ denotes the morphism $FL(T) \rightarrow FL(T \setminus \{u, v\} \cup \{w\})$ defined
 483 by mapping the generators u and v to the generator w .

- 484 • If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel hm_z^{xy} := \lambda \setminus \{x, y\} \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y)),$$

485 where bch stands for the Baker-Campbell-Hausdorff formula:

$$\text{bch}(a, b) := \log(e^a e^b) = a + b + \frac{1}{2}[a, b] + \dots$$

- 486 • If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel tha^{ux} := \lambda \parallel (C_u^{-\lambda_x})^{-1} = \lambda \parallel RC_u^{\lambda_x} \tag{10}$$

487 In the above formula, $C_u^{-\lambda_x}$ denotes the automorphism of $FL(T)$ defined by mapping
 488 the generator u to its “conjugate” $e^{-\lambda_x} u e^{\lambda_x}$. More precisely, u is mapped to $e^{-\text{ad}^{\lambda_x}}(u)$,
 489 where ad denotes the adjoint action, and e^{ad} is taken in the formal sense. Thus

$$C_u^{-\lambda_x} : u \mapsto e^{-\text{ad}^{\lambda_x}}(u) = u - [\lambda_x, u] + \frac{1}{2}[\lambda_x, [\lambda_x, u]] - \dots \tag{11}$$

490 Also in (10), $RC_u^{\lambda_x} := (C_u^{-\lambda_x})^{-1}$ denotes the inverse of the automorphism $C_u^{-\lambda_x}$.

- 491 • $t\epsilon_u = ()$ and $h\epsilon_x = (x \rightarrow 0)$.

492 *Warning 4.1* When $\gamma \in FL$, the inverse of $C_u^{-\gamma}$ may not be C_u^γ . If γ does not contain
 493 the generator u , then indeed $C_u^{-\gamma} \parallel C_u^\gamma = I$. But in general, applying $C_u^{-\gamma}$ creates many
 494 new us , within the γ s that appear in the right hand side of (11), and the new us are then
 495 conjugated by C_u^γ instead of being left in place. Yet $C_u^{-\gamma}$ is invertible, so we simply name
 496 its inverse RC_u^γ .

The name “ RC ” stands either for “reverse conjugation” or for “repeated conjugation”. The rationale for the latter naming is that if $\alpha \in FL(T)$ and \bar{u} is a name for a new “temporary” free-Lie generator, then $RC_u^\gamma(\alpha)$ is the result of applying the transformation $u \mapsto e^{\text{ad}^\gamma}(\bar{u})$ repeatedly to α until it stabilizes (at any fixed degree, this will happen after a finite number of iterations), followed by the eventual renaming $\bar{u} \mapsto u$.

Comment 4.2 Some further insight into RC_u^γ can be obtained by studying the triangle below. The space at the bottom of the triangle is the quotient of the free Lie algebra on $T \cup \{\bar{u}\}$ (where \bar{u} is a new temporary generator) by either of the two relations shown there; these two relations are, of course, equivalent. The map ϕ is induced from the obvious inclusion of $FL(T)$ into $FL(T \cup \{\bar{u}\})$, and in the presence of the relation $\bar{u} = e^{-\text{ad}^\gamma} u$, it is clearly an isomorphism. The map $\bar{\phi}$ is likewise induced from the renaming of $u \mapsto \bar{u}$. It, too, is an isomorphism, but slightly less trivially—indeed, using the relation $u = e^{\text{ad}^\gamma} \bar{u}$ repeatedly, any element in $FL(T \cup \{\bar{u}\})$ can be written in form that does not include u , and hence is in the image of $\bar{\phi}$. It is clear that $C_u^{-\gamma} = \bar{\phi} \parallel \phi^{-1}$. Hence, $RC_u^\gamma = \phi \parallel \bar{\phi}^{-1}$, and as $\bar{\phi}^{-1}$ is described in terms of repeated applications of the relation $u = e^{\text{ad}^\gamma} \bar{u}$, it is clear that RC_u^γ indeed involves repeated conjugation as asserted in the previous paragraph.

$$\begin{array}{ccc}
 FL(T) & \begin{array}{c} \xleftarrow{C_u^{-\gamma}} \\ \xrightarrow{RC_u^\gamma} \end{array} & FL(T) \\
 \searrow \phi & & \swarrow \bar{\phi} \quad u \mapsto \bar{u} \\
 & & FL(T \cup \{\bar{u}\}) / \left(\begin{array}{c} \bar{u} = e^{-\text{ad}^\gamma} u \\ \text{and / or} \\ u = e^{\text{ad}^\gamma} \bar{u} \end{array} \right)
 \end{array}$$

Warning 4.3 Equation (10) does not say that $tha^{\mu x} = RC_u^{\lambda x}$ as abstract operations, only that they are equal when evaluated on λ . In general, it is not the case that $\mu \parallel tha^{\mu x} = \mu \parallel RC_u^{\lambda x}$ for arbitrary μ —the latter equality is only guaranteed if $\mu_x = \lambda_x$.

As another example of the difference, the operations hm_z^{xy} and $tha^{\mu x}$ do not commute—in fact, the composition $hm_z^{xy} \parallel tha^{\mu x}$ does not even make sense, for by the time $tha^{\mu x}$ is evaluated, its input does not have an entry labelled x . Yet, the commutativity

$$\lambda \parallel hm_z^{xy} \parallel RC_u^{\lambda x} = \lambda \parallel RC_u^{\lambda x} \parallel hm_z^{xy} \tag{12}$$

makes perfect sense and holds true, for the operation hm_z^{xy} only involves the heads/roots of trees, while $RC_u^{\lambda x}$ only involves their tails/leaves.

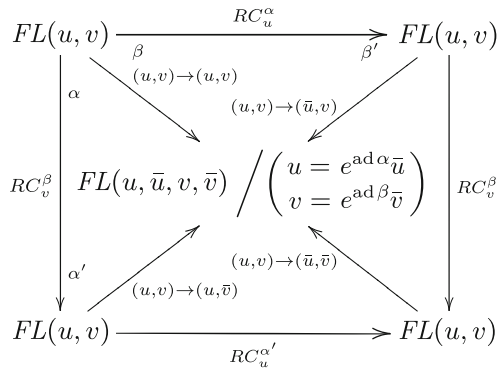
Theorem 4.4 M_0 , with the operations defined above, is a meta-monoid-action (MMA).

Proof Most MMA axioms are trivial to verify. The most important ones are the ones in (2) through (6). Of these, the meta-associativity of hm follows from the associativity of the bch formula, $bch(bch(\lambda_x, \lambda_y), \lambda_z) = bch(\lambda_x, bch(\lambda_y, \lambda_z))$, the meta-associativity of tm is trivial, and it remains to prove that meta-actions commute ((4); all other required commutativities are easy) and the meta-action axiom t (5) and h (6).

529 *Meta-actions commute* Expanding (4) using the above definitions and denoting $\alpha := \lambda_x$,
 530 $\beta = \lambda_y$, $\alpha' := \alpha \parallel RC_v^\beta$, and $\beta' := \beta \parallel RC_u^\alpha$, we see that we need to prove the
 531 identity

$$RC_u^\alpha \parallel RC_v^{\beta'} = RC_v^\beta \parallel RC_u^{\alpha'} \tag{13}$$

532 Consider the commutative diagram below. In it, $FL(u, v)$ means “the (completed) free
 533 Lie algebra with generators u and v , and some additional fixed collection of generators”,
 534 and likewise, for $FL(u, \bar{u}, v, \bar{v})$. The diagonal arrows are all substitution homomorphisms
 535 as indicated, and they are all isomorphisms. We put the elements α and β in the upper-left
 536 space, and by comparing with the diagram in Comment 4.2, we see that the upper horizontal
 537 map is RC_u^α and the left vertical map is RC_v^β . Therefore, β' is the image of β in the top left
 538 space, and α' is the image of α in the bottom left space. Therefore, again, using the diagram
 539 in Comment 4.2, the right vertical map is $RC_v^{\beta'}$ and the lower horizontal map is $RC_u^{\alpha'}$,
 and (13) follows from the commutativity of the external square in the diagram below.



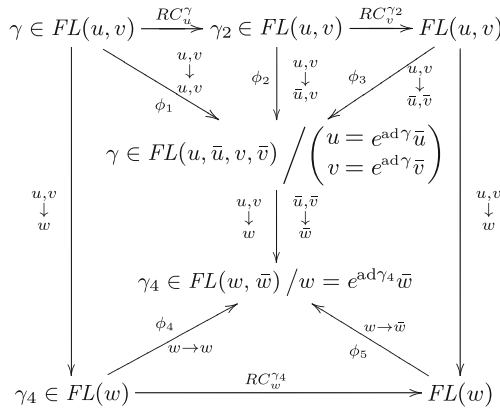
540 For later use, we record the fact that by reading all the horizontal and vertical arrows
 541 backwards, the above argument also proves the identity
 542

$$C_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} = C_v^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha} \tag{14}$$

543 *Meta-action axiom t.* Expanding (5) and denoting $\gamma := \lambda_x$, we need to prove the identity

$$tm_w^{uv} \parallel RC_w^\gamma \parallel t_w^{uv} = RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel tm_w^{uv} \tag{15}$$

544 Consider the diagram below. In it, the vertical and diagonal arrows are all substitution
 545 homomorphisms as indicated. The horizontal arrows are RC maps as indicated. The element
 546 γ lives in the upper left corner of the diagram, but equally makes sense in the upper of the
 547 central spaces. We denote its image via RC_u^γ by γ_2 , and think of it as an element of the
 548 middle space in the top row. Likewise, $\gamma_4 := \gamma \parallel tm_w^{uv}$ lives in both the bottom left space
 549 and the bottom of the two middle spaces.
 550



It requires a minimal effort to show that the map at the very centre of the diagram is well defined. The commutativity of the triangles in the diagram follows from Comment 4.2, and the commutativity of the trapezoids is obvious. Hence, the diagram is overall commutative. Reading it from the top left to the bottom right along the left and the bottom edges gives the left hand side of (15), and along the top and the right edges gives the right hand side.

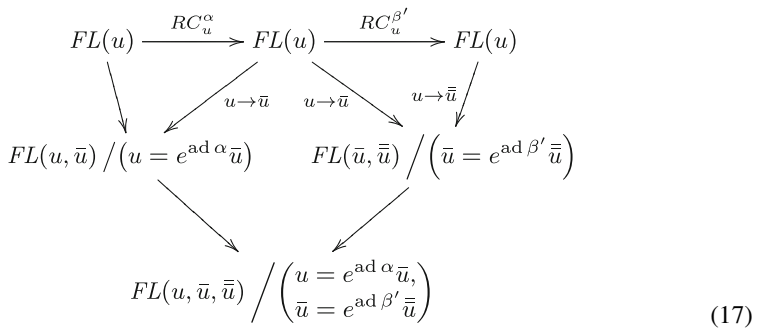
Meta-action axiom h Expanding (6), we need to prove

$$\lambda \parallel hm_z^{xy} \parallel RC_u^{\text{bch}(\lambda_x, \lambda_y)} = \lambda \parallel RC_u^{\lambda_x} \parallel RC_u^{\lambda_y} \parallel RC_u^{\lambda_x} \parallel hm_z^{xy}.$$

Using commutativities as in (12) and denoting $\alpha = \lambda_x$ and $\beta = \lambda_y$, we can cancel the hm_z^{xy} s, and we are left with

$$RC_u^{\text{bch}(\alpha, \beta)} \stackrel{?}{=} RC_u^\alpha \parallel RC_u^{\beta'}, \quad \text{where} \quad \beta' := \beta \parallel RC_u^\alpha. \tag{16}$$

This last equality follows from a careful inspection of the following commutative diagram:



Indeed, by the definition of RC_u^α , we have $\beta' = \beta$ modulo the relation $u = e^{\text{ad} \alpha} \bar{u}$. So, in the bottom space, $u = e^{\text{ad} \alpha} \bar{u} = e^{\text{ad} \alpha} e^{\text{ad} \beta'} \bar{\bar{u}} = e^{\text{ad} \alpha} e^{\text{ad} \beta} \bar{\bar{u}} = e^{\text{bch}(\text{ad} \alpha, \text{ad} \beta)} \bar{\bar{u}} = e^{\text{ad} \text{bch}(\alpha, \beta)} \bar{\bar{u}}$. Hence, if we concentrate on the three corners of (17), we see the diagram below, whose top row is both $RC_u^\alpha \parallel RC_u^{\beta'}$ and the definition of $RC_u^{\text{bch}(\alpha, \beta)}$.



$$\begin{array}{ccc}
 FL(u) & \dashrightarrow & FL(u) \\
 \searrow & & \swarrow \\
 & & u \rightarrow \bar{u} \\
 & & \swarrow \\
 & & FL(u, \bar{u}) / \left(u = e^{\text{ad bch}(\alpha, \beta) \bar{u}} \right)
 \end{array}
 \quad \square$$

564 It remains to construct $\zeta_0: \mathcal{K}_0^{rbh} \rightarrow M_0$ by proclaiming its values on the generators:
 565 $\zeta_0(t\epsilon_u) := ()$, $\zeta_0(h\epsilon_x) := (x \rightarrow 0)$, and $\zeta_0(\rho_{ux}^\pm) := (x \rightarrow \pm u)$.

565 **Proposition 4.5** ζ_0 is well defined; namely, the values above satisfy the relations in
 566 Definition 3.5.

567 *Proof* We only verify the conjugation relation (8), as all other relations are easy. On the
 568 left, we have

$$\begin{aligned}
 \rho_{ux}^+ \rho_{vy}^+ \rho_{wz}^+ \xrightarrow{\zeta_0} (x \rightarrow u, y \rightarrow v, z \rightarrow w) &\xrightarrow{tm_v^{vw}} (x \rightarrow u, y \rightarrow v, z \rightarrow v) \\
 &\xrightarrow{hm_x^{xy}} (x \rightarrow \text{bch}(u, v), z \rightarrow v) \xrightarrow{tha^{uz}} (x \rightarrow \text{bch}(e^{\text{ad } v}(u), v), z \rightarrow v),
 \end{aligned}$$

569 while on the right it is

$$\rho_{vx}^+ \rho_{wz}^+ \rho_{uy}^+ \xrightarrow{\zeta_0} (x \rightarrow v, y \rightarrow u, z \rightarrow w) \xrightarrow{tm_v^{vw} // hm_x^{xy}} (x \rightarrow \text{bch}(v, u), z \rightarrow v),$$

570 and the equality follows because $\text{bch}(e^{\text{ad } v}(u), v) = \log(e^v e^u e^{-v} \cdot e^v) = \text{bch}(v, u)$. \square

571 As we shall see in Section 7, ζ_0 is related to the tree part of the Kontsevitch integral.
 572 Thus, by finite-type folklore [2, 10], when evaluated on string links (i.e., pure tangles) ζ_0
 573 should be equivalent to the collection of all Milnor μ invariants [23]. No proof of this fact
 574 will be provided here.

575 **5 The Wheel-Valued Spice and the Invariant ζ**

576 This is perhaps the most important section of this paper. In it, we construct the wheel part of
 577 the full trees-and-wheels MMA M and the full tree-and-wheels invariant $\zeta: \mathcal{K}^{rbh} \rightarrow M$.

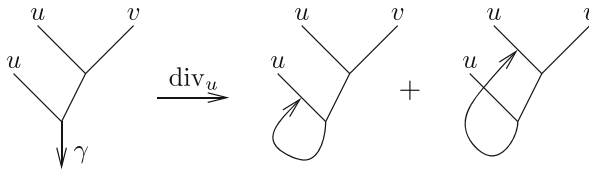
578 **5.1 Cyclic Words, div_u , and J_u**

579 The target MMA, M , of the extended invariant ζ is an extension of M_0 by “wheels”, or
 580 equally well, by “cyclic words”, and the main difference between M and M_0 is the addi-
 581 tion of a wheel-valued “spice” term $J_u(\lambda_x)$ to the meta-action $tha^{\mu x}$. We first need the
 582 “infinitesimal version” div_u of J_u .

583 Recall that if T is a set (normally, of tail labels), we denote by $FL(T)$ the graded
 584 completion of the free Lie algebra on the generators in T . Similarly, we denote by
 585 $FA(T)$ the graded completion of the free associative algebra on the generators in T , and
 586 by $CW(T)$ the graded completion of the vector space of cyclic words on T , namely,
 587 $CW(T) := FA(T)/\{uw = wu : u \in T, w \in FA(T)\}$. Note that the last is a vector space
 588 quotient—we mod out by the vector-space span of $\{uw = wu\}$, and not by the ideal gen-
 589 erated by that set. Hence, CW is not an algebra and not “commutative”; merely, the words

in it are invariant under cyclic permutations of their letters. We often call the elements of CW “wheels”. Denote by tr the projection $\text{tr} : FA \rightarrow CW$ and by ι the standard inclusion $\iota : FL(T) \rightarrow FA(T)$ (ι is defined to be the identity on letters in T , and is then extended to the rest of FL using $\iota([\lambda_1, \lambda_2]) := \iota(\lambda_1)\iota(\lambda_2) - \iota(\lambda_2)\iota(\lambda_1)$). Note that operations defined by “letter substitutions” make sense on FA and on CW . In particular, the operation RC_u^γ of Section 4.2 makes sense on FA and on CW .

The inclusion ι can be extended from “trees” (elements of FL) to “wheels of trees” (elements of $CW(FL)$). Given a letter $u \in T$ and an element $\gamma \in FL(T)$, we let $\text{div}_u \gamma$ be the sum of all ways of gluing the root of γ to near any one of the u -labelled leafs of γ ; each such gluing is a wheel of trees, and hence can be interpreted as an element of $CW(T)$. An example is below, and a formula-level definition follows: we first define $\sigma_u : FL(T) \rightarrow FA(T)$ by setting $\sigma_u(v) := \delta_{uv}$ for letters $v \in T$ and then setting $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$, and then we set $\text{div}_u(\gamma) := \text{tr}(u\sigma_u(\gamma))$. An alternative definition of a similar functional div is in [1, Proposition 3.20], and some further discussion is in [5, Section 3.2].



Now given $u \in T$ and $\gamma \in FL(T)$ define

$$J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}. \tag{18}$$

Note that at degree d , the integrand in the above formula is a degree d element of $CW(T)$ with coefficients that are polynomials of degree at most $d - 1$ in s . Hence the above formula is entirely algebraic. The following (difficult!) proposition contains all that we will need to know about J_u .

Proposition 5.1 *If $\alpha, \beta, \gamma \in FL$ then the following three equations hold:*

$$J_u(\text{bch}(\alpha, \beta)) = J_u(\alpha) + J_u(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha}, \tag{19}$$

$$J_u(\alpha) - J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta} = J_v(\beta) - J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha} \tag{20}$$

$$J_w(\gamma \parallel tm_w^{uv}) = (J_u(\gamma) + J_v(\gamma \parallel RC_u^\gamma) \parallel C_u^{-\gamma}) \parallel tm_w^{uv} \tag{21}$$

We postpone the proof of this proposition to Section 10.4.

Remark 5.2 J_u can be characterized as the unique functional $J_u : FL(T) \rightarrow CW(T)$ which satisfies (19) as well as the conditions $J_u(0) = 0$ and

$$\frac{d}{d\epsilon} J_u(\epsilon\gamma) \Big|_{\epsilon=0} = \text{div}_u(\gamma), \tag{22}$$

616 which in themselves are easy consequences of the definition of J_u , (18). Indeed, taking
 617 $\alpha = s\gamma$ and $\beta = \epsilon\gamma$ in (19), where s and ϵ are scalars, we find that

$$J_u((s + \epsilon)\gamma) = J_u(s\gamma) + J_u(\epsilon\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}.$$

618 Differentiating the above equation with respect to ϵ at $\epsilon = 0$ and using (22), we find that

$$\frac{d}{ds} J_u(s\gamma) = \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma},$$

619 and integrating from 0 to 1 we get (18).

620 Finally, for this section, one may easily verify that the degree 1 piece of CW is preserved
 621 by the actions of C_u^γ and RC_u^γ , and hence it is possible to reduce modulo degree 1. Namely,
 622 set $CW^r(T) := CW(T)/\text{deg } 1 = CW^{>1}(T)$, and all operations remain well defined and
 623 satisfy the same identities.

624 5.2 The MMA M

625 Let M be the collection $\{M(T; H)\}$, where

$$M(T; H) := FL(T)^H \times CW^r(T) = M_0(T; H) \times CW^r(T)$$

626 (I really mean \times , not \otimes). The collection M has MMA operations as follows:

- 627 • $t\sigma_v^u, t\eta^u$, and tm_w^{uv} are defined by the same formulae as in Section 4.2. Note that these
 628 formulae make sense on CW and on CW^r just as they do on FL .
- 629 • $h\sigma_y^x, h\eta^x$, and hm_z^{xy} are extended to act as the identity on the $CW^r(T)$ factor of
 630 $M(T; H)$.
- 631 • If $\mu_i = (\lambda_i; \omega_i) \in M(T_i; H_i)$ for $i = 1, 2$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$),
 632 set

$$\mu_1 * \mu_2 := (\lambda_1 * \lambda_2; \iota_1(\omega_1) + \iota_2(\omega_2)),$$

633 where ι_i are the obvious inclusions $\iota_i: CW^r(T_i) \rightarrow CW^r(T_1 \cup T_2)$.

- 634 • The only truly new definition is that of tha^{ux} :

$$(\lambda; \omega) \parallel tha^{ux} := (\lambda; \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x}.$$

635 Thus the “new” tha^{ux} is just the “old” tha^{ux} , with an added term of $J_u(\lambda_x)$.

- 636 • $t\epsilon_u := ((); 0)$ and $h\epsilon_x := ((x \rightarrow 0); 0)$.

637 **Theorem 5.3** *M , with the operations defined above, is a meta-monoid-action (MMA). Fur-*
 638 *thermore, if $\zeta: \mathcal{K}_0^{rhh} \rightarrow M$ is defined on the generators in the same way as ζ_0 , except*
 639 *extended by 0 to the CW^r factor, $\zeta(\rho_{ux}^\pm) := ((x \rightarrow \pm u); 0)$, then it is well-defined;*
 640 *namely, the values above satisfy the relations in Definition 3.5.*

641 *Proof* Given Theorem 4.4 and Proposition 4.5, the only non-obvious checks remaining are
 642 the “wheel parts” of the main equations defining and MMA (2)–(6) and the conjugation
 643 relation (8), and the FI relation (9). As the only interesting wheels-creation occurs with the
 644 operation tha , (2) and (3) are easy. As easily $J_u(v) = 0$ if $u \neq v$, no wheels are created
 645 by the tha action within the proof of Proposition 4.5, so that proof still holds. We are left
 646 with (4)–(6) and (8)–(9).

Let us start with the wheels part of (4). If $\mu = ((x \rightarrow \alpha, y \rightarrow \beta, \dots); \omega) \in M$, then 647

$$\mu \parallel tha^{\mu x} = ((x \rightarrow \alpha \parallel RC_u^\alpha, y \rightarrow \beta \parallel RC_u^\alpha, \dots); (\omega + J_u(\alpha)) \parallel RC_u^\alpha)$$

and hence the wheels-only part of $\mu \parallel tha^{\mu x} \parallel tha^{\nu y}$ is 648

$$\begin{aligned} \omega \parallel RC_u^\alpha \parallel RC_v^\beta \parallel RC_u^\alpha + J_u(\alpha) \parallel RC_u^\alpha \parallel RC_v^\beta \parallel RC_u^\alpha + J_v(\beta \parallel RC_u^\alpha) \parallel RC_v^\beta \parallel RC_u^\alpha \\ = [\omega + J_u(\alpha) + J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha}] \parallel RC_u^\alpha \parallel RC_v^\beta \parallel RC_u^\alpha. \end{aligned}$$

In a similar manner, the wheels-only part of $\mu \parallel tha^{\nu y} \parallel tha^{\mu x}$ is 650

$$[\omega + J_v(\beta) + J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta}] \parallel RC_v^\beta \parallel RC_u^\alpha \parallel RC_v^\beta.$$

Using (13), the operators outside the square brackets in the above two formulae are the same, and so we only need to verify that 651 652

$$\omega + J_u(\alpha) + J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha} = \omega + J_v(\beta) + J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta}.$$

But this is (20). In a similar manner, the wheels parts of (5) and (6) reduce to (21) and (19), respectively. One may also verify that no wheels appear within (8), and that wheels appear in (9) only in degree 1, which is eliminated in CW^r . \square 653 654 655

Thus, we have a tree-and-wheel valued invariant ζ defined on \mathcal{K}_0^{rbh} , and thus $\delta \parallel \zeta$ is a tree-and-wheel valued invariant of tangles and w-tangles. 656 657

As we shall see in Section 7, the wheels part ω of ζ is related to the wheels part of the Kontsevitch integral. Thus by finite-type folklore (e.g., [19]), the Abelianization of ω (obtained by declaring all the letters in $CW(T)$ to be commuting) should be closely related to the multi-variable Alexander polynomial. More on that in Section 9. I don't know what the bigger (non-commutative) part of ω measures. 658 659 660 661 662

6 Some Computational Examples 663

Part of the reason I am happy about the invariant ζ is that it is relatively easily computable. Cyclic words are easy to implement, and using the Lyndon basis (e.g. [24, Chapter 5]), free Lie algebras are easy too. Hence, I include here a demo-run of a rough implementation, written in *Mathematica*. The full source files are available at [web/]. 664 665 666 667


6.1 The Program 668


First, we load the package `FreeLie.m`, which contains a collection of programs to manipulate series in completed free Lie algebras and series of cyclic words. We tell `FreeLie.m` to show series by default only up to degree 3, and that if two (infinite) series are compared, they are to be compared by default only up to degree 5: 669 670 671

```
<< FreeLie.m
$SeriesShowDegree = 3; $SeriesCompareDegree = 5;
```




673 Merely as a test of `FreeLie.m`, we tell it to set `t1` to be `bch(u, v)`. The computer's response is to print that series to degree 3:


 `t1 = BCH[⟨u⟩, ⟨v⟩]`

 `LS [u + v, $\frac{uv}{2}$, $\frac{1}{12} \overline{uuv} + \frac{1}{12} \overline{uvv}$]`

674
675 Note that by default, Lie series are printed in “top bracket form”, which means that
676 brackets are printed above their arguments, rather than around them. Hence \overline{uuv} means
677 $[u, [u, v]]$. This practise is especially advantageous when it is used on highly nested expres-
678 sions, when it becomes difficult for the eye to match left brackets with their corresponding
679 right brackets.

680 Note also that that `FreeLie.m` utilizes *lazy evaluation*, meaning that when a Lie series
681 (or a series of cyclic words) is defined, its definition is stored but no computations take
682 place until it is printed or until its value (at a certain degree) is explicitly requested. Hence,
683 `t1` is a reference to the entire Lie series `bch(u, v)`, and not merely to the degrees 1–3 parts
684 of that series, which are printed above. Hence, when we request the value of `t1` to degree
6, the computer complies:

 `t1@{6}`

 `LS [u + v, $\frac{uv}{2}$, $\frac{1}{12} \overline{uuv} + \frac{1}{12} \overline{uvv}$, $\frac{1}{24} \overline{uuvv}$, $-\frac{1}{720} \overline{uuuuv} +$
 $\frac{1}{180} \overline{uuuvv} + \frac{1}{180} \overline{uuvvv} + \frac{1}{120} \overline{uvuvv} + \frac{1}{360} \overline{uuvuv} - \frac{1}{720} \overline{uvvvv}$,
 $-\frac{1}{1440} \overline{uuuuuv} + \frac{1}{360} \overline{uuuuvv} + \frac{1}{240} \overline{uuvuvv} + \frac{1}{720} \overline{uuuvuv} - \frac{1}{1440} \overline{uuvvvv}$]`

685
686 (It is surprisingly easy to compute `bch` to a high degree and some amusing patterns
687 emerge. See [\[web/mo\]](#) and [\[web/bch\]](#).)

688 The package `FreeLie.m` know about various free Lie algebra operations, but not about
689 our specific circumstances. Hence, we have to make some further definitions. The first
690 few are set-theoretic in nature. We define the “domain” of a function stored as a list of
691 *key* → *value* pairs to be the set of “first elements” of these pairs; meaning, the set of keys.
692 We define what it means to remove a key (and its corresponding value), and likewise for a
693 list of keys. We define what it means for two functions to be equal (their domains must be
694 equal, and for every key #, we are to have # // $f_1 = \# // f_2$). We also define how to apply a
695 Lie morphism `mor` to a function (apply it to each value), and how to compare (λ, ω) pairs
696 (in $FL(T)^H \times CW'(T)$):

```

Domain[f_List] := First /@ f;
f \ key_ := DeleteCases[f, key -> _];
f \ keys_List := Fold[#1 \ #2 &, f, keys];
f1_List ≡ f2_List := Domain[f1] == Domain[f2] && (And @@ (
    ((# /. f1) ≡ (# /. f2)) & /@ Domain[f1]));
LieMorphism[mor_] [f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M[λ1_, ω1_] ≡ M[λ2_, ω2_] := (λ1 ≡ λ2) && (ω1 ≡ ω2);
    
```

Next, we enter some free-Lie definitions that are not a part of `FreeLie.m`. Namely, we define $RC_{u,\bar{u}}^\gamma(s)$ to be the result of “stable application” of the morphism $u \rightarrow e^{\text{ad}(\gamma)}(\bar{u})$ to s (namely, apply the morphism repeatedly until things stop changing; at any fixed degree this happens after a finite number of iterations). We define RC_u^γ to be $RC_{u,\bar{u}}^\gamma // (\bar{u} \rightarrow u)$. Finally, we define J as in (18):

```

RC_u[γ_LieSeries, ub_][s_] := StableApply[LieMorphism[⟨u⟩ -> Ad[γ][⟨ub⟩]], s];
RC_u[γ_LieSeries][s_] := s // RC_u[γ, ⟨u⟩] // LieMorphism[⟨u⟩ -> ⟨u⟩];
J_u[γ_] :=
    Module[{s}, ∫_0^1 (γ // RC_u[s γ] // div_u // LieMorphism[u -> Ad[-s γ][u]]) ds];
    
```

Mostly, to introduce our notation for cyclic words, let us compute $J_v(\text{bch}(u, v))$ to degree 4. Note that when a series of wheels is printed out here, its degree 1 piece is greyed out to honour the fact that it “does not count” within ζ :

```

J_v[t1]@{4}
    
```

$$\text{CWS} \left[\overline{v}, \overline{uv}, \frac{\overline{uuv}}{2} - \frac{\overline{uvv}}{2}, \frac{\overline{uuuv}}{6} + \frac{3 \overline{uuvv}}{4} - \frac{3 \overline{uvuv}}{2} + \frac{\overline{uvvv}}{6} \right]$$

Next is a series of definitions that implement the definitions of $*$, tm , hm , and tha following Sections 4.2 and 5.2:




```

M /: M[λ1_, ω1_] * M[λ2_, ω2_] := M[λ1 ∪ λ2, ω1 + ω2];
tm[u_, v_, w_] [λ_List] := λ // LieMorphism[⟨u⟩ -> ⟨w⟩, ⟨v⟩ -> ⟨w⟩];
tm[u_, v_, w_] [M[λ_, ω_]] := LieMorphism[⟨u⟩ -> ⟨w⟩, ⟨v⟩ -> ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_] [λ_List] := Union[λ \ {x, y}, {z -> BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_] [M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
tha[u_, x_] [λ_List] := MapAt[RC_u[x /. λ], λ, {All, 2}];
tha[u_, x_] [M[λ_, ω_]] :=
    M[λ // tha[u, x], (ω + J_u[x /. λ]) // RC_u[x /. λ]];
    
```



Next, we set the values of $\zeta(t\epsilon_x)$ and $\zeta(\rho_{ux}^\pm)$, which we simply denote $t\epsilon_x$ and ρ_{ux}^\pm :



```

 he[x_] := M[{x → MakeLieSeries[0]}, MakeCWSeries[0]]
 ρ+[u_, x_] := M[{x → MakeLieSeries[⟨u⟩]}, MakeCWSeries[0]];
 ρ-[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩]}, MakeCWSeries[0]];

```

707
708 The final bit of definitions have to do with 3D tangles. We set R^+ to be the value of
709 $\zeta(\delta(\curvearrowright))$ as in the proof of Theorem 3.4, likewise for R^- , and we define `dm` by following
(7):

```

 R+[a_, b_] := ρ+[a, b] * he[a]; R-[a_, b_] := ρ-[a, b] * he[a];
 dm[a_, b_, c_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];



```

710

711 6.2 Testing Properties and Relations


712 It is always good to test both the program and the math by verifying that the operations we
713 have implemented satisfy the relations predicted by the mathematics. As a first example,
714 we verify the meta-associativity of `tm`. Hence, in line 1 below, we set `t1` to be the element
715 $t_1 = ((x \rightarrow u + v + w, y \rightarrow [u, v] + [v, w]); uvw)$ of $M(u, v, w; x, y)$. In line
716 2, we compute $t_1 \parallel tm_u^{uv}$, in line 3 we compute $t_2 := t_1 \parallel tm_u^{uv} \parallel tm_u^{uv}$ and store its value
717 in `t2`; in line 4, we compute $t_1 \parallel tm_v^{vw}$, in line 5 we compute $t_3 := t_1 \parallel tm_v^{vw} \parallel tm_u^{uv}$ and
718 store its value in `t3`, and then in line 6, we test if t_2 is equal to t_3 . The computer thinks the
answer is “True”, at least to the degree tested:

```

 Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩}};

  1 → (t1 = M[{
    x → MakeLieSeries[u + v + w], y → MakeLieSeries[b[u, v] + b[v, w]]
  }, MakeCWSeries[CW["uvw"]]]),
  2 → (t1 // tm[u, v, u]),
  3 → (t2 = t1 // tm[u, v, u] // tm[u, w, u]),
  4 → (t1 // tm[v, w, v]),
  5 → (t3 = t1 // tm[v, w, v] // tm[u, v, u]),
  6 → (t2 ≡ t3));

```

```


  1 → M[{x → LS[⟨u⟩ + ⟨v⟩ + ⟨w⟩, 0, 0], y → LS[0, ⟨u⟩ + ⟨v⟩ + ⟨w⟩, 0]}, CWS[0, 0, ⟨uvw⟩]]
  2 → M[{x → LS[2⟨u⟩ + ⟨w⟩, 0, 0], y → LS[0, ⟨u⟩ + ⟨w⟩, 0]}, CWS[0, 0, ⟨uuw⟩]]
  3 → M[{x → LS[3⟨u⟩, 0, 0], y → LS[0, 0, 0]}, CWS[0, 0, ⟨uuu⟩]]
  4 → M[{x → LS[⟨u⟩ + 2⟨v⟩, 0, 0], y → LS[0, ⟨u⟩ + ⟨v⟩, 0]}, CWS[0, 0, ⟨uvv⟩]]
  5 → M[{x → LS[3⟨v⟩, 0, 0], y → LS[0, 0, 0]}, CWS[0, 0, ⟨vvv⟩]]
  6 → True

```

719
720 The corresponding test for the meta-associativity of `hm` is a bit harder, yet produces the
721 same result. Note that we have declared `$SeriesCompareDegree` to be higher than
722 `$SeriesShowDegree`, so the “True” output below means a bit more than the visual
723 comparison of lines 3 and 5:

```

Print /@ {
  1 → (t1 = ρ* [u, x] ρ* [v, y] ρ* [w, z]),
  2 → (t1 // hm[x, y, x]),
  3 → (t2 = t1 // hm[x, y, x] // hm[x, z, x]),
  4 → (t1 // hm[y, z, y]),
  5 → (t3 = t1 // hm[y, z, y] // hm[x, y, x]),
  6 → (t2 ≡ t3)};

```

```

1 → M[{x → LS[ū, 0, 0], y → LS[v̄, 0, 0], z → LS[w̄, 0, 0]}, CWS[0, 0, 0]]
2 → M[{x → LS[ū + v̄,  $\frac{ūv̄}{2}$ ,  $\frac{1}{12} \overline{uūv̄} + \frac{1}{12} \overline{uūvv̄}$ ], z → LS[w̄, 0, 0]}, CWS[0, 0, 0]]
3 →
M[{x → LS[ū + v̄ + w̄,  $\frac{ūv̄}{2} + \frac{ūw̄}{2} + \frac{v̄w̄}{2}$ ,  $\frac{1}{12} \overline{uūv̄} + \frac{1}{12} \overline{uūw̄} + \frac{1}{3} \overline{uūvw̄} + \frac{1}{12} \overline{v̄v̄w̄} + \frac{1}{12} \overline{uūvv̄} + \frac{1}{6} \overline{uūw̄v̄} + \frac{1}{12} \overline{uūww̄} + \frac{1}{12} \overline{v̄vw̄w̄}$ ], CWS[0, 0, 0]]
4 → M[{x → LS[ū, 0, 0], y → LS[v̄ + w̄,  $\frac{v̄w̄}{2}$ ,  $\frac{1}{12} \overline{v̄v̄w̄} + \frac{1}{12} \overline{v̄vw̄w̄}$ ], CWS[0, 0, 0]]
5 →
M[{x → LS[ū + v̄ + w̄,  $\frac{ūv̄}{2} + \frac{ūw̄}{2} + \frac{v̄w̄}{2}$ ,  $\frac{1}{12} \overline{uūv̄} + \frac{1}{12} \overline{uūw̄} + \frac{1}{3} \overline{uūvw̄} + \frac{1}{12} \overline{v̄v̄w̄} + \frac{1}{12} \overline{uūvv̄} + \frac{1}{6} \overline{uūw̄v̄} + \frac{1}{12} \overline{uūww̄} + \frac{1}{12} \overline{v̄vw̄w̄}$ ], CWS[0, 0, 0]]
6 → True

```

We next test the meta-action axiom t on $((x \rightarrow u + [u, t], y \rightarrow u + [u, t]); uu + tuv)$ 724 and the meta-action axiom h on $((x \rightarrow u + [u, v], y \rightarrow v + [u, v]); uu + uvv)$:

```

Print /@ {{u = <"u">, v = <"v">, w = <"w">, t = <"t">}};
  1 → (t1 = M[{
    x → MakeLieSeries[u + b[u, t]], y → MakeLieSeries[u + b[u, t]]
  }, MakeCWSeries[CW["uu"] + CW["tuv"]]]),
  2 → (t2 = t1 // tm[u, v, w] // tha[w, x]),
  3 → (t3 = t1 // tha[u, x] // tha[v, x] // tm[u, v, w]),
  4 → (t2 ≡ t3)};

```

```

1 → M[{x → LS[ū, -tū, 0], y → LS[ū, -tū, 0]}, CWS[0, uū, tūv]]
2 → M[{x → LS[w̄, -t̄w̄, -t̄w̄w̄], y → LS[w̄, -t̄w̄, -t̄w̄w̄]}, CWS[w̄, -t̄w̄ + w̄w̄,  $\frac{3 \overline{t̄w̄w̄}}{2}$ ]]
3 → M[{x → LS[w̄, -t̄w̄, -t̄w̄w̄], y → LS[w̄, -t̄w̄, -t̄w̄w̄]}, CWS[w̄, -t̄w̄ + w̄w̄,  $\frac{3 \overline{t̄w̄w̄}}{2}$ ]]
4 → True

```

```

Print /@ {{u = <"u">, v = <"v">}};
  1 → (t1 = M[{
    x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + b[u, v]]
  }, MakeCWSeries[CW["uu"] + CW["uvv"]]]),
  2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
  3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
  4 → (t2 ≡ t3)};

```

```

1 → M[{x → LS[ū, uūv̄, 0], y → LS[v̄, uūv̄, 0]}, CWS[0, uū, uūvv̄]]
2 → M[{z → LS[ū + v̄,  $\frac{3 \overline{ūv̄}}{2}$ ,  $-\frac{17}{12} \overline{uūv̄} - \frac{17}{12} \overline{uūvv̄}$ ], CWS[ū, uū - 2 uūv̄,  $\frac{uūv̄}{2} + \frac{uūvv̄}{2}$ ]]
3 → M[{z → LS[ū + v̄,  $\frac{3 \overline{ūv̄}}{2}$ ,  $-\frac{17}{12} \overline{uūv̄} - \frac{17}{12} \overline{uūvv̄}$ ], CWS[ū, uū - 2 uūv̄,  $\frac{uūv̄}{2} + \frac{uūvv̄}{2}$ ]]
4 → True

```

And finally for this testing section, we test the conjugation relation of (8):



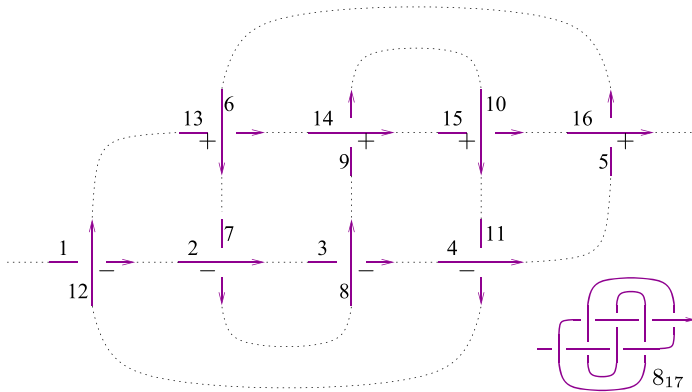
```
Print /@ {
  1 → (t1 = ρ*[u, x] ρ*[v, y] ρ*[w, z]),
  2 → (t2 = t1 // tm[v, w, v] // hm[x, y, x] // tha[u, z]),
  3 → (t3 = ρ*[v, x] ρ*[w, z] ρ*[u, y]),
  4 → (t4 = t3 // tm[v, w, v] // hm[x, y, x]),
  5 → (t2 ≡ t4)};
```

```
1 → M[{x → LS[ū, 0, 0], y → LS[v̄, 0, 0], z → LS[w̄, 0, 0]}, CWS[0, 0, 0]]
2 → M[{x → LS[ū + v̄, -ūv̄/2, 1/12 ūūv̄ + 1/12 ūv̄v̄], z → LS[v̄, 0, 0]}, CWS[0, 0, 0]]
3 → M[{x → LS[v̄, 0, 0], y → LS[ū, 0, 0], z → LS[w̄, 0, 0]}, CWS[0, 0, 0]]
4 → M[{x → LS[ū + v̄, -ūv̄/2, 1/12 ūūv̄ + 1/12 ūv̄v̄], z → LS[v̄, 0, 0]}, CWS[0, 0, 0]]
5 → True
```

725

726 6.3 Demo Run 1 — the Knot 8₁₇

727 We are ready for a more substantial computation—the invariant of the knot 8₁₇. We draw
 728 8₁₇ in the plane, with all but the neighbourhoods of the crossings dashed-out. We thus get
 729 a tangle T_1 which is the disjoint union of eight individual crossings (four positive and four
 730 negative). We number the 16 strands that appear in these eight crossings in the order of their
 eventual appearance within 8₁₇, as seen below.



731

732 The 8-crossing tangle T_1 we just got has a rather boring ζ invariant, a disjoint merge of 8
 733 ρ^{\pm} 's. We store it in $\mu 1$. Note that we used numerals as labels, and hence, in the expression
 734 below, top-bracketed numerals should be interpreted as symbols and not as integers. Note
 735 also that the program automatically converts two-digit numerical labels into alphabetical
 736 symbols, when these appear within Lie elements. Hence, in the output below, “a” is “10”,
 737 “c” is “12”, “e” is “14”, and “g” is “16”:

```

μ1 = R^- [12, 1] R^- [2, 7] R^- [8, 3] R^- [4, 11] R^+ [16, 5] R^+ [6, 13] R^+ [14, 9] R^+ [10, 15]

```

```

M [ { 1 → LS [ -c̄, 0, 0 ], 2 → LS [ 0, 0, 0 ],
      3 → LS [ -8̄, 0, 0 ], 4 → LS [ 0, 0, 0 ], 5 → LS [ c̄, 0, 0 ], 6 → LS [ 0, 0, 0 ],
      7 → LS [ -2̄, 0, 0 ], 8 → LS [ 0, 0, 0 ], 9 → LS [ c̄, 0, 0 ], 10 → LS [ 0, 0, 0 ],
      11 → LS [ -4̄, 0, 0 ], 12 → LS [ 0, 0, 0 ], 13 → LS [ 6̄, 0, 0 ],
      14 → LS [ 0, 0, 0 ], 15 → LS [ ā, 0, 0 ], 16 → LS [ 0, 0, 0 ] }, CWS [ 0, 0, 0 ] ]

```

Next is the key part of the computation. We “sew” together the strands of T_1 in order by first sewing 1 and 2 and naming the result 1, then sewing 1 and 3 and naming the result 1 once more, and so on until everything is sewn together to a single strand named 1. This is done by applying dm_1^{1k} repeatedly to μ_1 , for $k = 2, \dots, 16$, each time storing the result back again in μ_1 . Finally, we only wish to print the wheels part of the output, and this we do to degree 6:

```

Do [ μ1 = μ1 // dm [ 1, k, 1 ], { k, 2, 16 } ];
Last [ μ1 ] @ { 6 }

```

```

CWS [ 0, -11̄, 0, -31 1111̄ / 12, 0, -1351 111111̄ / 360 ]

```

Let $A(X)$ be the Alexander polynomial of 8_{17} . Namely, $A(X) = -X^{-3} + 4X^{-2} - 8X^{-1} + 11 - 8X + 4X^2 - X^3$. For comparison with the above computation, we print the series expansion of $\log A(e^x)$, also to degree 6:

```

Series [ Log [ -1/x^3 + 4/x^2 - 8/x + 11 - 8x + 4x^2 - x^3 /. x → e^x ], { x, 0, 6 } ]

```

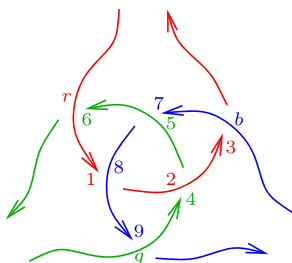
```

-x^2 - 31 x^4 / 12 - 1351 x^6 / 360 + O [ x ] ^ 7

```

6.4 Demo Run 2—the Borromean Tangle

In a similar manner, we compute the invariant of the *rgb*-coloured Borromean tangle, shown below.



750 We label the edges near the crossings as shown, using the labels $\{r, 1, 2, 3\}$ for the r
 751 component, $\{g, 4, 5, 6\}$ for the g component, and $\{b, 7, 8, 9\}$ for the b component. We let
 752 μ_2 store the invariant of the disjoint union of six independent crossings labelled as in
 753 the Borromean tangle, we concatenate the numerically labelled strands into their corre-
 754 sponding letter-labelled strands, and we then print μ_2 , which now contains the invariant
 755 we seek:

```

    ☹️  $\mu_2 = R^-[r, 6] R^+[2, 4] R^-[g, 9] R^+[5, 7] R^-[b, 3] R^+[8, 1];$   

    🖥️ (Do[ $\mu_2 = \mu_2 // \text{dm}[r, k, r], \{k, 1, 3\}$ ]; Do[ $\mu_2 = \mu_2 // \text{dm}[g, k, g], \{k, 4, 6\}$ ];  

    🖥️ Do[ $\mu_2 = \mu_2 // \text{dm}[b, k, b], \{k, 7, 9\}$ ];  $\mu_2$ )
    
```

```

    🖥️ M[ { b -> LS[ 0,  $\overline{gr}$ ,  $\frac{1}{2} \overline{ggr} + \overline{brr} + \frac{1}{2} \overline{grr}$  ],  

    g -> LS[ 0,  $-\overline{br}$ ,  $\frac{1}{2} \overline{bbr} - \overline{bgr} - \overline{brg} + \frac{1}{2} \overline{brr}$  ],  

    r -> LS[ 0,  $\overline{bg}$ ,  $\frac{1}{2} \overline{bbg} + \overline{bgr} + \frac{1}{2} \overline{bgr}$  ] }, CWS[ 0, 0, 2  $\overline{bgr}$  ] ]
    
```

756 We then print the r -head part of the tree part of the invariant to degree 5 (the g -head and
 757 b -head parts can be computed in a similar way, or deduced from the cyclic symmetry of r ,
 758 g , and b), and the wheels part to the same degree:

```

    ☹️ (r /. First[ $\mu_2$ ]) @ {5}
    🖥️
    
```

```

    🖥️ LS[ 0,  $\overline{bg}$ ,  $\frac{1}{2} \overline{bbg} + \overline{bgr} + \frac{1}{2} \overline{bgr}$ ,  

     $\frac{1}{6} \overline{bbbg} + \frac{1}{2} \overline{bbgr} + \frac{1}{2} \overline{bggr} + \frac{1}{4} \overline{bbgg} + \frac{1}{2} \overline{bgrg} + \frac{1}{6} \overline{bggg}$ ,  

     $\frac{1}{24} \overline{bbbbg} + \frac{1}{6} \overline{bbbgr} + \frac{1}{4} \overline{bbggr} + \frac{1}{12} \overline{bbbgg} + \frac{1}{4} \overline{bbgrr} +$   

     $\frac{1}{6} \overline{bgggr} + \frac{1}{4} \overline{bggrr} - \overline{bbgrg} + \frac{1}{12} \overline{bbggg} - 2 \overline{brrgg} + \frac{1}{6} \overline{bgrrr} +$   

     $\frac{1}{2} \overline{bgbgr} - \overline{bgrbr} - \frac{1}{12} \overline{bbgbg} - \frac{1}{2} \overline{bgrgr} + \frac{1}{24} \overline{bgggg}$  ]
    
```

```

    ☹️ Last[ $\mu_2$ ] @ {5}
    🖥️
    
```

```

    🖥️ CWS[ 0, 0, 2  $\overline{bgr}$ ,  $\overline{bbgr} - \overline{bgrb} + \overline{bggr} - \overline{bgrg} + \overline{bgrr} - \overline{brgr}$ ,  

     $\frac{\overline{bbbg}}{3} - \frac{\overline{bbgr}}{2} + \frac{\overline{bbggr}}{2} + \frac{\overline{bbgrg}}{2} + \frac{\overline{bbgrr}}{2} + \frac{\overline{bbrbg}}{2} - \frac{3 \overline{bbgrg}}{2} + \frac{\overline{bgbr}}{2} - \frac{3 \overline{bggbr}}{2} +$   

     $\frac{\overline{bgggr}}{3} - \frac{\overline{bggrg}}{2} + \frac{\overline{bggrr}}{2} + \frac{\overline{bgrgg}}{2} - \frac{3 \overline{bgrrg}}{2} + \frac{\overline{bgrrr}}{3} + \frac{\overline{brggr}}{2} - \frac{\overline{brgr}}{2} + \frac{\overline{brgr}}{2}$  ]
    
```

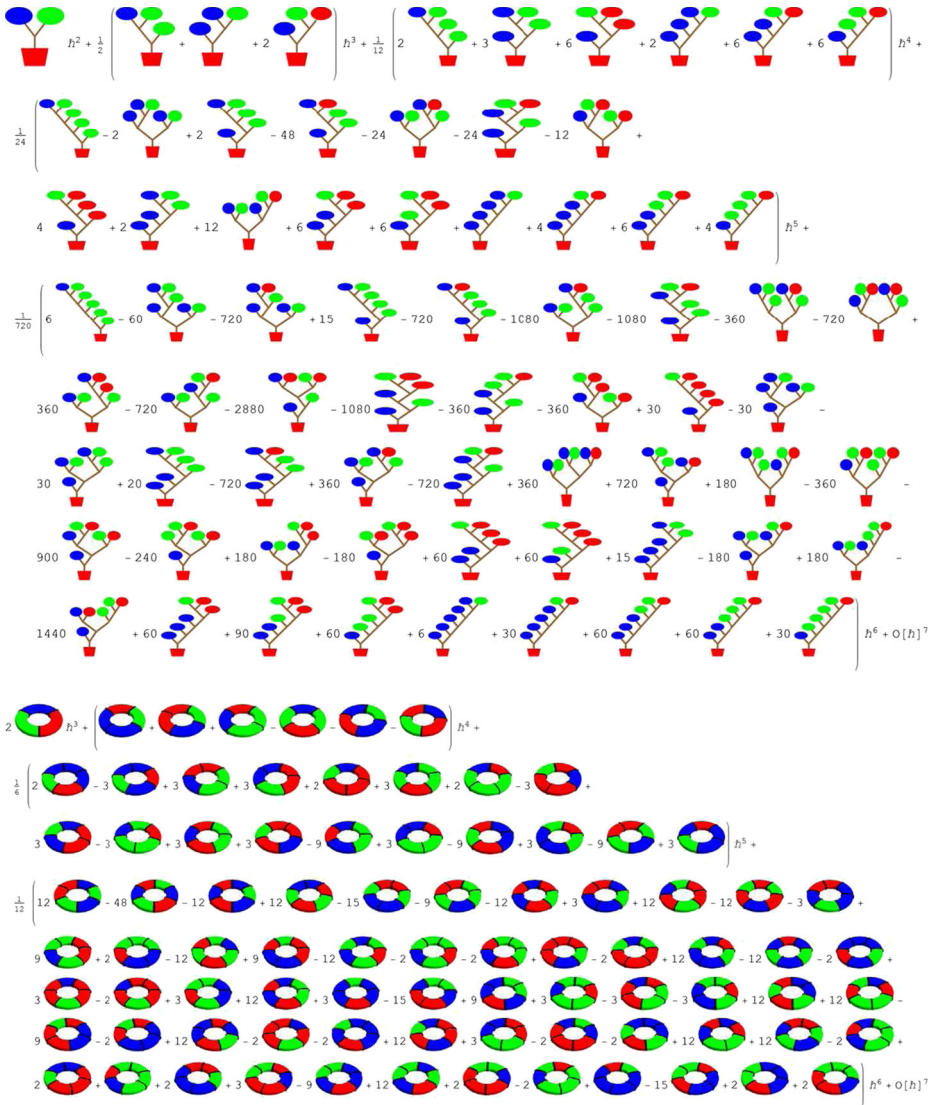



Fig. 6 The redhead part of the tree part and the wheels part of the invariant of the Borromean tangle, to degree 6

A more graphically pleasing presentation of the same values, with the degree raised to 6, appears in Fig. 6. 759
760

7 Sketch of the Relation with Finite Type Invariants 761

One way to view the invariant ζ of Section 5 is as a mysterious extension of the reasonably natural invariant ζ_0 of Section 4. Another is as a solution to a universal problem—as we shall see in this section, ζ is a universal finite type invariant of objects in \mathcal{K}_0^{rbh} . Given that \mathcal{K}_0^{rbh} 762
763
764

765 is closely related to $w\mathcal{T}$ (w-tangles), and given that much was already said on finite-type
 766 invariants of w-tangles in [5], this section will be merely a sketch, difficult to understand
 767 without reading much of [4] and sections 1–3 of [5], as well as the parts of Section 4 that
 768 concern with caps.

769 Over all, defining ζ using the language of Sections 4 and 5 is about as difficult as using
 770 finite-type invariants. Yet computing it using the language of Sections 4 and 5 is much easier
 771 while proving invariance is significantly harder.

772 7.1 A Circuit Algebra Description of \mathcal{K}_0^{rbh}

773 A w-tangle represents a collection of ribbon-knotted tubes in \mathbb{R}^4 . It follows from Theorem
 774 2.9 that every rKBH can be obtained from a w-tangle by capping some of its tubes and
 775 puncturing the rest, where puncturing a tube means “replacing it with its spine, a strand that
 776 runs along it”. Using thick red lines to denote tubes, red bullets to denote caps, and dotted
 777 blue lines to denote punctured tubes, we find that

$$\mathcal{K}_0^{rbh} = \text{CA} \left\langle \begin{array}{c} \text{Diagrams of crossings and caps} \\ \text{Reidemeister moves (including R1) and} \\ \text{and} \end{array} \right\rangle \text{R1: } \begin{array}{c} \text{Diagram of a loop} \\ = \\ \text{Diagram of a cap} \\ = \\ \text{Diagram of a circle} \end{array}$$

778 Note that punctured tubes (meanings strands or hoops) can only go under capped tubes
 779 (balloons), and that while it is allowed to slide tubes over caps, it is not allowed to slide them
 780 under caps. Further explanations and the meaning of “CA” are in [5]. The “red bullet” sub-
 781 script on the right hand side indicates that we restrict our attention to the subspace in which
 782 all red strands are eventually capped. We leave it to the reader to interpret the operations
 783 hm , tha , and tm in this language (tm is non-obvious!).

784 7.2 Arrow Diagrams for \mathcal{K}_0^{rbh}

785 As in [4, 5], one we finite-type invariants of elements on \mathcal{K}_0^{rbh} bi considering iterated dif-
 786 ferences of crossings and non-crossings (virtual crossings), and then again as in [4, 5], we
 787 find that the arrow-diagram space $\mathcal{A}^{bh}(T; H)$ corresponding to these invariants may be
 788 described schematically as follows:

$$\mathcal{A}^{bh}(T; H) = \left\langle \begin{array}{c} \text{Diagram with red tubes } T \text{ and blue tubes } H \\ \text{Relations} \end{array} \right\rangle$$

In the above, arrow tails may land only on the red “tail” strands, but arrow heads may land on either kind of strand. The “relations” are the TC and $\overrightarrow{4T}$ relations of [4, Section 2.3], the CP relation of [5, Section 4.2], and the relation $D_L = D_R = 0$, which corresponds to the R1 relation (D_L and D_R are defined in [4, Section 3]).

The operation hm acts on \mathcal{A}^{bh} by concatenating two head stands. The operation tha acts by duplicating a head strand (with the usual summation over all possible ways of reconnecting arrow-heads as in [4, Section 2.5.1.6]), changing the colour of one of the duplicates to red, and then concatenating it to the beginning of some tail strand.

We note that modulo the relations, one may eliminate all arrow-heads from all tail strands. For diagrams in which there are no arrow-heads on tail strands, the operation tm is defined by merging together two tail strands. The TC relation implies that arrow-tails on the resulting tail-strand can be ordered in any desired way.

As in [4, Section 3.5], \mathcal{A}^{bh} has an alternative model in which internal “2-in 1-out” trivalent vertices are allowed, and in which we also impose the \overrightarrow{AS} , \overrightarrow{STU} , and \overrightarrow{IHX} relations (ibid.).

7.3 The Algebra Structure on \mathcal{A}^{bh} and its Primitives

For any fixed finite sets T and H , the space $\mathcal{A}^{bh}(T; H)$ is a co-commutative bi-algebra. Its product defined using the disjoint union followed by the tm operation on all tail strands and the hm operation on all head strands, and its co-product is the “sum of all splittings” as in [4, Section 3.2]. Thus by Milnor-Moore, $\mathcal{A}^{bh}(T; H)$ is the universal enveloping algebra of its set of primitives \mathcal{P}^{bh} . The latter is the set of connected diagrams in \mathcal{A}^{bh} (modulo relations), and those, as in [5, Section 3.2], are the trees and the degree > 1 wheels. (Though note that even if $T = H = \{1, \dots, n\}$, the algebra structure on $\mathcal{A}^{bh}(T; H)$ is different from the algebra structure on the space $\mathcal{A}^w(\uparrow_n)$ of ibid.). Identifying trees with $FL(T)$ and wheels with $CW^r(T)$, we find that

$$\mathcal{P}^{bh}(T; H) \cong FL(T)^H \times CW^r(T) = M(T; H).$$

Theorem 7.1 *By taking logarithms (using formal power series and the algebra structure of \mathcal{A}^{bh}), $\mathcal{P}^{bh}(T; H)$ inherits the structure of an MMA from the group-like elements of \mathcal{A}^{bh} . Furthermore, $\mathcal{P}^{bh}(T; H)$ and $M(T; H)$ are isomorphic as MMAs.*

Sketch of the proof Once it is established that $\mathcal{P}^{bh}(T; H)$ is an MMA, that tm and hm act in the same way as on M and that the tree part of the action of tha is given using the RC operation, it follows that the wheels part of the action of tha is given by some functional J' which necessarily satisfies (19). But according to Remark 5.2, (19) and a few auxiliary conditions determine J uniquely. These conditions are easily verified for J' , and hence $J' = J$. This concludes the proof.

Note that the above theorem and the fact that $\mathcal{P}^{bh}(T; H)$ is an MMA provided an alternative proof of Proposition 5.1 which bypasses the hard computations of Section 10.4. In fact, personally, I first knew that J exists and satisfies Proposition 5.1 using the reasoning of this section, and only then did I observe using the reasoning of Remark 5.2 that J must be given by the formula in (18).

7.4 The Homomorphic Expansion Z^{bh}

As in [4, Section 3.4] and [5, Section 3.1], there is a homomorphic expansion (a universal finite type invariant with good composition properties) $Z^{bh}: \mathcal{K}_0^{rbh} \rightarrow \mathcal{A}^{bh}$ defined by

830 mapping crossings to exponentials of arrows. It is easily verified that Z^{bh} is a morphism of
 831 MMAs, and therefore it is determined by its values on the generators ρ^\pm of \mathcal{K}_0^{rbh} , which are
 832 single crossings in the language of Section 7.1. Taking logarithms we find that $\log Z^{bh} = \zeta$
 833 on the generators and hence always, and hence ζ is the logarithm of a universal finite type
 834 invariant of elements of \mathcal{K}_0^{rbh} .

835 8 The Relation with the BF Topological Quantum Field Theory

836 8.1 Tensorial Interpretation

837 Given a Lie algebra \mathfrak{g} , any element of $FL(T)$ can be interpreted as a function taking $|T|$
 838 inputs in \mathfrak{g} and producing a single output in \mathfrak{g} . Hence, putting aside issues of comple-
 839 tion and convergence, there is a map $\tau_1 : FL(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g})$, where in general,
 840 $\text{Fun}(X \rightarrow Y)$ denotes the space of functions from X to Y . To deal with completions more
 841 precisely, we pick a formal parameter \hbar , multiply the degree k part of τ_1 by \hbar^k , and get a per-
 842 fectly good $\tau = \tau_{\mathfrak{g}} : FL(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}[[\hbar]])$, where in general, $V[[\hbar]] := \mathbb{Q}[[\hbar]] \otimes V$
 843 for any vector space V . The map τ obviously extends to $\tau : FL(T)^H \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}^H[[\hbar]])$.

844 Similarly, if also \mathfrak{g} is finite dimensional, then by taking traces in the adjoint representation
 845 we get a map $\tau = \tau_{\mathfrak{g}} : CW(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathbb{Q}[[\hbar]])$. Multiplying this τ with the τ from
 846 the previous paragraph, we get $\tau = \tau_{\mathfrak{g}} : M(T; H) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}^H[[\hbar]])$. Exponenti-
 847 ating, we get

$$e^\tau : M(T; H) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes H}[[\hbar]]).$$

848 8.2 ζ and BF Theory

849 Fix a finite dimensional Lie algebra \mathfrak{g} . In [7] (see especially Section 4), Cattaneo and Rossi
 850 discuss the BF quantum field theory with fields $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ and $B \in \Omega^2(\mathbb{R}^4, \mathfrak{g}^*)$
 851 and construct an observable “ $U(A, B, \Xi)$ ” for each “long” \mathbb{R}^2 in \mathbb{R}^4 ; meaning, for each 2-
 852 sphere in S^4 with a prescribed behaviour at ∞ . We interpret these as observables defined on
 853 our “balloons”. The Cattaneo-Rossi observables are functions of a variable $\Xi \in \mathfrak{g}$, and they
 854 can be interpreted as power series in a formal parameter \hbar . Further, given the connection-
 855 field A , one may always consider its formal holonomy along a closed path (a “hoop”) and
 856 interpret it as an element in $\mathcal{U}(\mathfrak{g})[[\hbar]]$. Multiplying these hoop observables and also the
 857 Cattaneo-Rossi balloon observables, we get an observable \mathcal{O}_γ for any KBH γ , taking values
 858 in $\text{Fun}(\mathfrak{g}^T \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes H}[[\hbar]])$.

859 **Conjecture 8.1** *If γ is an rKBH, then $\langle \mathcal{O}_\gamma \rangle_{BF} = e^\tau(\zeta(\gamma))$.*

860 Of course, some interpretation work is required before Conjecture 8.1 even becomes a
 861 well-posed mathematical statement.

862 We note that the Cattaneo-Rossi observable does not depend on the ribbon property of
 863 the KBH γ . I hesitate to speculate whether this is an indication that the work presented in
 864 this paper can be extended to non-ribbon knots or an indication that somewhere within the
 865 rigorous mathematical analysis of BF theory an obstruction will arise that will force one to
 866 restrict to ribbon knots (yet I speculate that one of these possibilities holds true).

867 Most likely the work of Watanabe [28] is a proof of Conjecture 8.1 for the case of a single
 868 balloon and no hoops, and very likely, it contains all key ideas necessary for a complete
 869 proof of Conjecture 8.1.

9 The Simplest Non-Commutative Reduction and an Ultimate Alexander Invariant

870

9.1 Informal

871

Let us start with some informal words. All the fundamental operations within the definition of M , namely $[\cdot, \cdot]$, C_u^γ , RC_u^γ and div_u , act by modifying trees and wheels near their extremities—their tails and their heads (for wheels, all extremities are tails). Thus, all operations will remain well-defined and will continue to satisfy the MMA properties if we extend or reduce trees and wheels by objects or relations that are confined to their “inner” parts.

872

873

874

875

876

In this section, we discuss the “ β -quotient of M ”, an extension/reduction of M as discussed above, which is even better-computable than M . As we have seen in Section 6, objects in M , and in particular the invariant ζ , are machine-computable. Yet the dimensions of FL and of CW grow exponentially in the degree, and so does the complexity of computations in M . Objects in the β -quotient are described in terms of commutative power series, their dimensions grow polynomially in the degree, and computations in the β -quotient are polynomial time. In fact, the power series appearing with the β -quotient can be “summed”, and *non-perturbative* formulae can be given to everything in sight.

877

878

879

880

881

882

883

884

Yet ζ^β , meaning ζ reduced to the β -quotient, remains strong enough to contain the (multi-variable) Alexander polynomial. I argue that in fact, the formulae obtained for the Alexander polynomial within this β -calculus are “better” than many standard formulae for the Alexander polynomial.

885

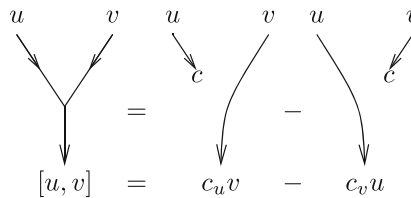
886

887

888

More on the relationship between the β -calculus and the Alexander polynomial (though nothing about its relationship with M and ζ), is in [6].

889



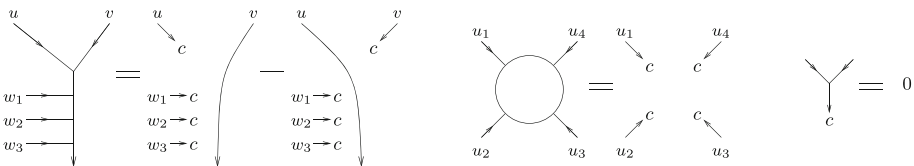
890

891

892

893

Still on the informal level, the β -quotient arises by allowing a new type of a “sink” vertex c and imposing the β -relation, shown above, on both trees and wheels. One easily sees that under this relation, trees can be shaved to single arcs union “ c -stubs”, wheels become unions of c -stubs, and c -stubs “commute with everything”:



894

895

896

897

Hence, c -stubs can be taken as generators for a commutative power series ring R (with one generator c_u for each possible tail label u), $CW(T)$ becomes a copy of the ring R , elements of $FL(T)$ becomes column vectors whose entries are in R and whose entries

898 correspond to the tail label in the remaining arc of a shaved tree, and elements of $FL(T)^H$
 899 can be regarded as $T \times H$ matrices with entries in R . Hence, in the β -quotient, the MMA
 900 M reduces to an MMA $\{\beta_0(T; H)\}$ whose elements are $T \times H$ matrices of power series,
 901 with yet an additional power series to encode the wheels part. We will introduce β_0 more
 902 formally below, and then note that it can be simplified even further (with no further loss of
 903 information) to an MMA β whose entries and operations involve rational functions, rather
 904 than power series.

905 *Remark 9.1* The β -relation arose from studying the (unique non-commutative) 2D Lie algebra
 906 $\mathfrak{g}_2 := FL(\xi_1, \xi_2)/([\xi_1, \xi_2] = \xi_2)$, as in Section 8.1. Loosely, within \mathfrak{g}_2 the β -relation
 907 is a “polynomial identity” in a sense similar to the “polynomial identities” of the theory of
 908 PI-rings [25]. For a more direct relationship between this Lie algebra and the Alexander
 909 polynomial, see [web/chic1].

910 9.2 Less Informal

911 For a finite set T let $R = R(T) := \mathbb{Q}[\{c_u\}_{u \in T}]$ denote the ring of power series with com-
 912 muting generators c_u corresponding to the elements u of T , and let $L = L(T) := R \otimes \mathbb{Q}T$
 913 be the free R -module with generators T . Turn L into a Lie algebra over R by declaring that
 914 $[u, v] = c_u v - c_v u$ for any $u, v \in T$. Let $c: L \rightarrow R$ be the R -linear extension of
 915 $u \mapsto c_u$; namely,

$$\gamma = \sum_u \gamma_u u \in L \mapsto c_\gamma := \sum_u \gamma_u c_u \in R, \tag{23}$$

916 where the γ_u 's are coefficients in R . Note that with this definition, we have
 917 $[\alpha, \beta] = c_\alpha \beta - c_\beta \alpha$ for any $\alpha, \beta \in L$. There are obvious surjections $\pi: FL \rightarrow L$ and
 918 $\pi: CW \rightarrow R$ (strictly speaking, the first of those maps has a small cokernel yet becomes
 919 a surjection once the ground ring of its domain space is extended to R).

920 The following Lemma-Definition may appear scary, yet its proof is nothing more than
 921 high school level algebra, and the messy formulae within it mostly get renormalized away
 922 by the end of this section. Hang on!

923 **Lemma-Definition 9.2** The operations $C_u, RC_u, \text{bch}, \text{div}_u,$ and J_u descend from FL/CW to
 924 L/R , and, for $\alpha, \beta, \gamma \in L$ (with $\gamma = \sum_v \gamma_v v$) they are given by

$$v \parallel C_u^{-\gamma} = v \parallel RC_u^\gamma = v \quad \text{for } u \neq v \in T, \tag{24}$$

$$\rho \parallel C_u^{-\gamma} = \rho \parallel RC_u^\gamma = \rho \quad \text{for } \rho \in R, \tag{25}$$

$$u \parallel C_u^{-\gamma} = e^{-c_\gamma} \left(u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \gamma \right) \tag{26}$$

$$= e^{-c_\gamma} \left(\left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right) u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right), \tag{27}$$

$$u \parallel RC_u^\gamma = \left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right), \tag{28}$$

$$\text{bch}(\alpha, \beta) = \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \left(\frac{e^{c_\alpha} - 1}{c_\alpha} \alpha + e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta} \beta \right), \tag{29}$$

$$\text{div}_u \gamma = c_u \gamma_u, \tag{30}$$

$$J_u(\gamma) = \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right). \tag{31}$$

Proof (Sketch) Equation (24) is obvious— C_u or RC_u conjugate or repeatedly conjugate u , but not v . Equation (25) is the statement that C_u and RC_u are R -linear, namely that they act on scalars as the identity. Informally this is the fact that 1-wheels commute with everything, and formally it follows from the fact that $\pi : FL \rightarrow L$ is a well-defined morphism of Lie algebras.

To prove (26), we need to compute $e^{-\text{ad}_\gamma}(u)$, and it is enough to carry this computation out within the 2D subspace of L spanned by u and by γ . Hence, the computation is an exercise in diagonalization—one needs to diagonalize the 2×2 matrix $\text{ad}(-\gamma)$ in order to exponentiate it. Here, are some details: set $\delta = [-\gamma, u] = c_u\gamma - c_\gamma u$. Then, clearly $\text{ad}(-\gamma)(\delta) = -c_\gamma\delta$, and hence $e^{-\text{ad}_\gamma}(\delta) = e^{-c_\gamma}\delta$. Also note that $\text{ad}(-\gamma)(\gamma) = 0$, and hence $e^{-\text{ad}_\gamma}(\gamma) = \gamma$. Thus

$$u \parallel C_u^{-\gamma} = e^{-\text{ad}_\gamma}(u) = e^{-\text{ad}_\gamma}\left(-\frac{\delta}{c_\gamma} + \frac{c_u\gamma}{c_\gamma}\right) = -\frac{e^{-c_\gamma}\delta}{c_\gamma} + \frac{c_u\gamma}{c_\gamma} = e^{-c_\gamma}\left(u + c_u\frac{e^{c_\gamma} - 1}{c_\gamma}\gamma\right).$$

Equation (27) is simply (26) rewritten using $\gamma = \sum v\gamma_u v$. To prove (28), take its right hand side and use (27) and (24) to get u back again, and hence our formula for RC_u^γ indeed inverts the formula already established for $C_u^{-\gamma}$.

Equation (29) amounts to writing the group law of a 2D Lie group in terms of its 2D Lie algebra, $L_0 := \text{span}(\alpha, \beta)$, and this is again an exercise in 2×2 matrix algebra, though a slightly harder one. We work in the adjoint representation of L_0 and aim to compare the exponential of the left hand side of (29) with the exponential of its right hand side. If a and b are scalars, let $e(a, b)$ be the matrix representing $e^{\text{ad}(a\alpha + b\beta)}$ on L_0 relative to the basis (α, β) . Then using $[\alpha, \beta] = c_\alpha\beta - c_\beta\alpha$ we find that $e(a, b) = \exp\begin{pmatrix} bc_\beta & -ac_\beta \\ -bc_\alpha & ac_\alpha \end{pmatrix}$, and we need to show that $e(1, 0) \cdot e(0, 1) = e\left(\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \frac{e^{c_\alpha} - 1}{c_\alpha}, \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta}\right)$. Lazy bums do it as follows:

```
e[a_, b_] := MatrixExp[{{ b c_β  -a c_β },
                        { -b c_α  a c_α }}];
e[1, 0] . e[0, 1] == e[ { (c_α + c_β) (e^{c_α} - 1) / (e^{c_α + c_β} - 1), c_α },
                        { (c_α + c_β) e^{c_α} (e^{c_β} - 1) / (e^{c_α + c_β} - 1), c_β } ] // Simplify
```



Equation (30) is the fact that $\text{div}_u u = c_u$, along with the R -linearity of div_u . For (31), note that using (28), the coefficient of u in $\gamma \parallel RC_u^{s\gamma}$ is $\gamma_u e^{s c_\gamma} \left(1 + c_u \gamma_u \frac{e^{s c_\gamma} - 1}{c_\gamma}\right)^{-1}$. Thus using (30) and the fact that C_u acts trivially on R ,

$$\begin{aligned} J_u(\gamma) &= \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma} = \int_0^1 ds \left(1 + c_u \gamma_u \frac{e^{s c_\gamma} - 1}{c_\gamma}\right)^{-1} c_u \gamma_u e^{s c_\gamma} \\ &= \log\left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u\right)\Big|_0^1 = \log\left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u\right). \end{aligned}$$

□ 951

952 9.3 The Reduced Invariant ζ^{β_0}

953 We now let $\beta_0(T; H)$ be the β -reduced version of $M(T; H)$. Namely, in parallel with
 954 Section 5.2 we define

$$\beta_0(T; H) := L(T)^H \times R^r(T) = R(T)^{T \times H} \times R^r(T).$$

955 In other words, elements of $\beta_0(T; H)$ are $T \times H$ matrices $A = (A_{ux})$ of power series in
 956 the variables $\{c_u\}_{u \in T}$, along with a single additional power series $\omega \in R^r$ (R^r is R modded
 957 out by its degree 1 piece) corresponding to the last factor above, which we write at the top
 958 left of A :

$$\beta_0(u, v, \dots; x, y, \dots) = \left\{ \left(\begin{array}{c|ccc} \omega & x & y & \cdots \\ u & A_{ux} & A_{uy} & \cdot \\ v & A_{vx} & A_{vy} & \cdot \\ \vdots & \cdot & \cdot & \ddots \end{array} \right) : \omega \in R^r(T), A.. \in R(T) \right\}$$

959 Continuing in parallel with Section 5.2 and using the formulae from Lemma-
 960 Definition 9.2, we turn $\{\beta_0(T; H)\}$ into an MMA with operations defined as follows (on a
 961 typical element of β_0 , which is a decorated matrix (A, ω) as above):

- 962 • $t\sigma_v^u$ acts by renaming row u to v and sending the variable c_u to c_v everywhere. $t\eta^u$ acts
 963 by removing row u and sending c_u to 0. tm_w^{uv} acts by adding row u to row v calling the
 964 result row w , and by sending c_u and c_v to c_w everywhere.
- 965 • $h\sigma_y^x$ and $h\eta^x$ are clear. To define hm_z^{xy} , let $\alpha = (A_{ux})_{u \in T}$ and $\beta = (A_{uy})_{u \in T}$ denote
 966 the columns of x and y in A , let $c_\alpha := \sum_{u \in T} A_{ux}c_u$ and $c_\beta := \sum_{u \in T} A_{uy}c_u$ in parallel
 967 with (23), and let hm_z^{xy} act by removing the x - and y -columns α and β and introducing
 968 a new column, labelled z , and containing $\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \left(\frac{e^{c_\alpha} - 1}{c_\alpha} \alpha + e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta} \beta \right)$, as in (29).
- 969 • We now describe the action of tha^{ux} on an input (A, ω) as depicted below. Let $\gamma =$
 970 $\begin{pmatrix} \gamma_u \\ \gamma_{rest} \end{pmatrix}$ be the column of x , split into the “row u ” part γ_u and the rest, γ_{rest} . Let c_γ be
 971 $\sum_{v \in T} \gamma_v c_v$ as in (23). Then tha^{ux} acts as follows:

$$\begin{array}{c|ccc} \omega & x & \cdot & y & \cdot \\ u & \gamma_u & \cdot & \alpha_u & \cdot \\ \vdots & \gamma_{rest} & \cdot & \alpha_{rest} & \cdot \end{array}$$

- 972 – As dictated by (31), ω is replaced by $\omega + \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right)$.
- 973 – As dictated by (24) and (28), every column $\alpha = \begin{pmatrix} \alpha_u \\ \alpha_{rest} \end{pmatrix}$ in A (including the
 974 column γ itself) is replaced by

$$\left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \begin{pmatrix} e^{c_\gamma} \alpha_u \\ \alpha_{rest} - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} (c_\gamma)_{rest} \end{pmatrix},$$

975 where $(c_\gamma)_{rest}$ is the column whose row v entry is $c_v \gamma_v$, for any $v \neq u$.

- 976 • The “merge” operation $*$ is $\frac{\omega_1 | H_1}{T_1 | A_1} * \frac{\omega_2 | H_2}{T_2 | A_2} := \frac{\omega_1 + \omega_2 | H_1 \ H_2}{T_1 \ T_2 | A_1 \ 0 \ 0 \ A_2}$.

- $t\epsilon_u = \frac{0|\emptyset}{u|\emptyset}$ and $h\epsilon_x = \frac{0|x}{\emptyset|\emptyset}$ (these values correspond to a matrix with an empty set of columns and a matrix with an empty set of rows, respectively).

We have concocted the definition of the MMA β_0 so that the projection $\pi : M \rightarrow \beta_0$ would be a morphism of MMAs. Hence, to completely compute $\zeta^{\beta_0} := \pi \circ \zeta$ on any rKBH (to all orders!), it is enough to note its values on the generators. These are determined by the values in Theorem 5.3: $\zeta^{\beta_0}(\rho_{ux}^\pm) = \frac{0|x}{u|\pm 1}$.

9.4 The Ultimate Alexander Invariant ζ^β .

Some repackaging is in order. Noting the ubiquity of factors of the form $\frac{e^c-1}{e^{c\alpha}-1}$ in the previous section, it makes sense to multiply any column α of the matrix A by $\frac{e^c-1}{e^{c\alpha}-1}$. Noting that row- u entries (things like γ_u) often appear multiplied by c_u , we multiply every row by its corresponding variable c_u . Doing this and rewriting the formulae of the previous section in the new variables, we find that the variables c_u only appear within exponentials of the form e^{c_u} . So, we set $t_u := e^{c_u}$ and rewrite everything in terms of the t_u 's. Finally, the only formula that touches ω is additive and has a log term. So, we replace ω with e^ω . The result is “ β -calculus”, which was described in detail in [6]. A summary version follows. In these formulae, $\alpha, \beta, \gamma,$ and δ denote entries, rows, columns, or submatrices as appropriate, and whenever α is a column, $\langle \alpha \rangle$ is the sum of its entries:

$$\beta(T; H) = \left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} \omega \text{ and the } \alpha_{ux} \text{'s are rational functions in} \\ \text{variables } t_u, \text{ one for each } u \in T. \text{ When all} \\ t_u \text{'s are set to 1, } \omega \text{ is 1 and every } \alpha_{ux} \text{ is} \\ 0. \end{array} \right\},$$

$$tm_{uv}^{uv} : \frac{\omega|H}{u|\alpha} \mapsto \left(\frac{\omega|H}{w|\alpha+\beta} \right) // (t_u, t_v \rightarrow t_w),$$

$$hm_z^{xy} : \frac{\omega|x y H}{T|\alpha \beta \gamma} \mapsto \frac{\omega|z H}{T|\alpha+\beta+\langle\alpha\rangle\beta \gamma},$$

$$tha^{ux} : \frac{\omega|x H}{u|\alpha \beta} \mapsto \frac{\omega(1+\alpha)|x H}{u|\alpha(1+\langle\gamma\rangle/(1+\alpha)) \beta(1+\langle\gamma\rangle/(1+\alpha))},$$

$$\frac{\omega_1|H_1}{T_1|A_1} * \frac{\omega_2|H_2}{T_2|A_2} := \frac{\omega_1\omega_2|H_1 H_2}{T_2|A_1 \ 0 \ A_2},$$

$$\zeta^\beta(t\epsilon_u) = \frac{1|\emptyset}{u|\emptyset}, \quad \zeta^\beta(h\epsilon_x) = \frac{1|x}{\emptyset|\emptyset}, \quad \text{and} \quad \zeta^\beta(\rho_{ux}^\pm) = \frac{1|x}{u|t_u^\pm - 1}.$$

Theorem 9.3 *If K is a u -knot regarded as a 1-component pure tangle by cutting it open, then the ω part of $\zeta^\beta(\delta(K))$ is the Alexander polynomial of K .*

I know of three winding paths that constitute a proof of the above theorem:



- 998 • Use the results of Section 7 here, of [4, Section 3.7], and of [21].
 999 • Use the results of Section 7 here, of [4, Section 3.9], and the known relation of the
 1000 Alexander polynomial with the wheels part of the Kontsevich integral (e.g. [19]).
 1001 • Use the results of [18], where formulae very similar to ours appear.
- 1002 Yet to me, the strongest evidence that Theorem 9.3 is true is that it was verified explicitly
 1003 on very many knots—see the single example in Section 6.3 here and many more in [6].
 1004 In several senses, ζ^β is an “ultimate” Alexander invariant:
- 1005 • The formulae in this section may appear complicated, yet note that if an rKBH consists
 1006 of about n balloons and hoops, its invariant is described in terms of only $O(n^2)$ poly-
 1007 nomials and each of the operations tm , hm , and tha involves only $O(n^2)$ operations on
 1008 polynomials.
- 1009 • It is defined for tangles and has a prescribed behaviour under tangle compositions (in
 1010 fact, it is defined in terms of that prescribed behaviour). This means that when ζ^β is
 1011 computed on some large knot with (say) n crossings, the computation can be broken
 1012 up into n steps of complexity $O(n^2)$ at the end of each the quantity computed is the
 1013 invariant of some topological object (a tangle), or even into $3n$ steps at the end of each
 1014 the quantity computed is the invariant of some rKBH¹⁰.
- 1015 • ζ^β contains also the multivariable Alexander polynomial and the Burau representation
 1016 (overwhelmingly verified by experiment, not written-up yet).
- 1017 • ζ^β has an easily prescribed behaviour under hoop- and balloon-doubling, and $\zeta^\beta \circ \delta$
 1018 has an easily prescribed behaviour under strand-doubling (not shown here).

1019 10 Odds and Ends

1020 10.1 Linking Numbers and Signs

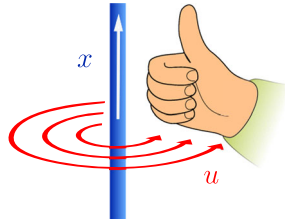
1021 If x is an oriented S^1 and u is an oriented S^2 in an oriented S^4 (or \mathbb{R}^4) and the two are disjoint,
 1022 their linking number l_{ux} is defined as follows. Pick a ball B whose oriented boundary is
 1023 u (using the “outward pointing normal” convention for orienting boundaries), and which
 1024 intersects x in finitely many transversal intersection points p_i . At any of these intersection
 1025 points p_i , the concatenation of the orientation of B at p_i (thought of a basis to the tangent
 1026 space of B at p_i) with the tangent to x at p_i is a basis of the tangent space of S^4 at p_i , and
 1027 as such it may either be positively oriented or negatively oriented. Define $\sigma(p_i) = +1$ in
 1028 the former case and $\sigma(p_i) = -1$ in the latter case. Finally, let $l_{ux} := \sum_i \sigma(p_i)$. It is a
 1029 standard fact that l_{ux} is an isotopy invariant of (u, x) .

1030 *Exercise 10.1* Verify that $l_{ux}(\rho_{ux}^\pm) = \pm 1$, where ρ_{ux}^+ and ρ_{ux}^- are the positive and negative
 1031 Hopf links as in Example 2.2. For the purpose of this exercise, the plane in which Fig. 1

¹⁰A similar statement can be made for Alexander formulae based on the Burau representation. Yet note that such formulae still end with a computation of a determinant which may take $O(n^3)$ steps. Note also that the presentation of knots as braid closures is typically inefficient—typically a braid with $O(n^2)$ crossings is necessary in order to present a knot with just n crossings.

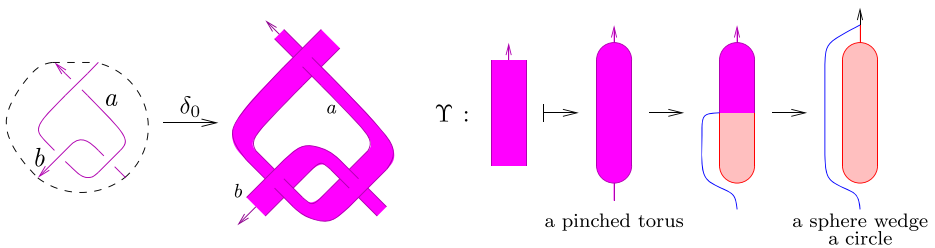
is drawn is oriented counterclockwise, the 3D space it represents has its third coordinate oriented up from the plane of the paper, and \mathbb{R}^4_{txyz} is oriented so that the t coordinate is “first”.

An efficient thumb rule for deciding the linking number signs for a balloon u and a hoop x presented using our standard notation as in Section 2.1 is the “right-hand rule” of the figure below, shown here without further explanation. The lovely figure is adopted from [Wikipedia: Right-hand_rule].



10.2 A Topological Construction of δ

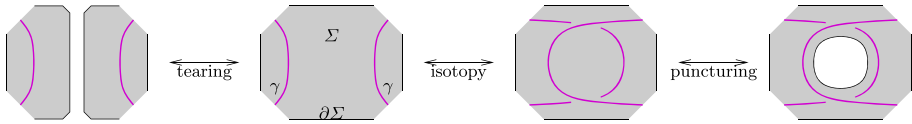
The map δ is a composition $\delta_0 // \Upsilon$ (“ δ_0 followed by Υ ”, aka $\Upsilon \circ \delta_0$. See Section 10.5.). Here, δ_0 is the standard “tubing” map δ_0 (called t' in Satoh’s [26]), though with the tubes decorated by an additional arrowhead to retain orientation information. The map Υ caps and strings both ends of all tubes to ∞ and then uses, at the level of embeddings, the fact that a pinched torus is homotopy equivalent to a sphere wedge a circle:



It is worthwhile to give a completely “topological” definition of the tubing map δ_0 , thus giving $\delta = \delta_0 // \Upsilon$ a topological interpretation. We must start with a topological interpretation of v-tangles, and even before, with v-knots, also known as virtual knots.

1050 The standard topological interpretation of v-knots (e.g. [20]) is that they are oriented
 1051 knots drawn¹¹ on an oriented surface Σ , modulo “stabilization”, which is the addition and/or
 1052 removal of empty handles (handles that do not intersect with the knot). We prefer an equiv-
 1053 alent, yet even more bare-bones approach. For us, a virtual knot is an oriented knot γ drawn
 1054 on a “virtual surface Σ for γ ”. More precisely, Σ is an oriented surface that may have a
 1055 boundary, γ is drawn on Σ , and the pair (Σ, γ) is taken modulo the following relations:

- 1056 • Isotopies of γ on Σ (meaning, in $\Sigma \times [-\epsilon, \epsilon]$).
- 1057 • Tearing and puncturing parts of Σ away from γ :



1058 (We call Σ a “virtual surface” because tearing and puncturing imply that we only care
 1059 about it in the immediate vicinity of γ).

1060 We can now define¹² a map δ_0 , defined on v-knots and taking values in ribbon tori in
 1061 \mathbb{R}^4 : given (Σ, γ) , embed Σ arbitrarily in $\mathbb{R}^3_{xyz} \subset \mathbb{R}^4$. Note that the unit normal bundle of
 1062 Σ in \mathbb{R}^4 is a trivial circle bundle and it has a distinguished trivialization, constructed using
 1063 its positive t -direction section and the orientation that gives each fibre a linking number $+1$
 1064 with the base Σ . We say that a normal vector to Σ in \mathbb{R}^4 is “near unit” if its norm is between
 1065 $1 - \epsilon$ and $1 + \epsilon$. The near-unit normal bundle of Σ has as fibre an annulus that can be
 1066 identified with $[-\epsilon, \epsilon] \times S^1$ (identifying the radial direction $[1 - \epsilon, 1 + \epsilon]$ with $[-\epsilon, \epsilon]$ in
 1067 an orientation-preserving manner), and hence the near-unit normal bundle of Σ defines an
 1068 embedding of $\Sigma \times [-\epsilon, \epsilon] \times S^1$ into \mathbb{R}^4 . On the other hand, γ is embedded in $\Sigma \times [-\epsilon, \epsilon]$
 1069 so $\gamma \times S^1$ is embedded in $\Sigma \times [-\epsilon, \epsilon] \times S^1$, and we can let $\delta_0(\Sigma, \gamma)$ be the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$

1070 which is a torus in \mathbb{R}^4 , oriented using the given orientation of γ and the standard orientation
 1071 of S^1 .

1072 We leave it to the reader to verify that $\delta_0(\Sigma, \gamma)$ is ribbon, that it is independent of the
 1073 choices made within its construction, that it is invariant under isotopies of γ and under
 1074 tearing and puncturing, that it is also invariant under the “overcrossing commute” relation
 1075 of Fig. 3, and that it is equivalent to Satoh’s tubing map.

1076 The map δ_0 has straightforward generalizations to v-links, v-tangles, framed-v-links, v-
 1077 knotted-graphs, etc.

1078 10.3 Monoids, Meta-Monoids, Monoid-Actions, and Meta-Monoid-Actions

1079 How do we think about meta-monoid-actions? Why that name? Let us start with ordinary
 1080 monoids.

¹¹Here and below, “drawn on Σ ” means “embedded in $\Sigma \times [-\epsilon, \epsilon]$ ”.

¹²Following a private discussion with Dylan Thurston.

A monoid¹³ G gives rise to a slew of spaces and maps between them: the spaces would be the spaces of sequences $G^n = \{(g_1, \dots, g_n) : g_i \in G\}$, and the maps will be the maps “that can be written using the monoid structure”—they will include, for example, the map $m_i^{ij} : G^n \rightarrow G^{n-1}$ defined as “store the product $g_i g_j$ as entry number i in G^{n-1} while erasing the original entries number i and j and re-numbering all other entries as appropriate”. In addition, there is also an obvious binary “concatenation” map $*$: $G^n \times G^m \rightarrow G^{n+m}$ and a special element $\epsilon \in G^1$ (the monoid unit).

Equivalently but switching from “numbered registers” to “named registers”, a monoid G automatically gives rise to another slew of spaces and operations. The spaces are $G^X = \{f : X \rightarrow G\} = \{(x \rightarrow g_x)_{x \in X}\}$ of functions from a finite set X to G , or as we prefer to say it, of X -indexed sequences of elements in G , or how computer scientists may say it, of associative arrays of elements of G with keys in X . The maps between such spaces would now be the obvious “register multiplication maps” $m_z^{xy} : G^{X \cup \{x, y\}} \rightarrow G^{X \cup \{z\}}$ (defined whenever $x, y, z \notin X$ and $x \neq y$), and also the obvious “delete a register” map $\eta^x : G^X \rightarrow G^{X \setminus \{x\}}$, the obvious “rename a register” map $\sigma_y^x : G^{X \cup \{x\}} \rightarrow G^{X \cup \{y\}}$, and an obvious $*$: $G^X \times G^Y \rightarrow G^{X \cup Y}$, defined whenever X and Y are disjoint. Also, there are special elements, “units”, $\epsilon_x \in G^{\{x\}}$.

This collection of spaces and maps between them (and the units) satisfies some properties. Let us highlight and briefly discuss two of those:

- (1) The “associativity property”: For any $\Omega \in G^X$,

$$\Omega \parallel m_x^{xy} \parallel m_x^{xz} = \Omega \parallel m_y^{yz} \parallel m_x^{xy}. \tag{32}$$

This property is an immediate consequence of the associativity axiom of monoid theory. Note that it is a “linear property”—its subject, Ω , appears just once on each side of the equality. Similar linear properties include $\Omega \parallel \sigma_y^x \parallel \sigma_z^y = \Omega \parallel \sigma_z^x$, $\Omega \parallel m_z^{xy} \parallel \sigma_u^z = \Omega \parallel m_u^{xy}$, etc., and there are also “multi-linear” properties like $(\Omega_1 * \Omega_2) * \Omega_3 = \Omega_1 * (\Omega_2 * \Omega_3)$, which are “linear” in each of their inputs.

- (2) If $\Omega \in G^{\{x, y\}}$, then

$$\Omega = (\Omega \parallel \eta^y) * (\Omega \parallel \eta^x) \tag{33}$$

(indeed, if $\Omega = (x \rightarrow g_x, y \rightarrow g_y)$, then $\Omega \parallel \eta^y = (x \rightarrow g_x)$ and $\Omega \parallel \eta^x = (y \rightarrow g_y)$ and so the right hand side is $(x \rightarrow g_x) * (y \rightarrow g_y)$, which is Ω back again), so an element of $G^{\{x, y\}}$ can be factored as an element of $G^{\{x\}}$ times an element of $G^{\{y\}}$. Note that Ω appears twice in the right hand side of this property, so this property is “quadratic”. In order to write this property one must be able to “make two copies of Ω ”.

10.3.2 Meta-Monoids

Definition 10.2 A meta-monoid is a collection $(G_X, m_z^{xy}, \sigma_z^x, \eta^x, *)$ of sets G_X , one for each finite set X “of labels”, and maps between them $m_z^{xy}, \sigma_z^x, \eta^x, *$ with the same domains and ranges as above, and special elements $\epsilon_x \in G_{\{x\}}$, and with the same **linear and multi-linear** properties as above.

¹³A monoid is a group sans inverses. You lose nothing if you think “group” whenever the discussion below states “monoid”.



1119 Very crucially, we do not insist on the non-linear property (33) of the above, and so we
 1120 may not have the factorization $G_{\{x,y\}} = G_{\{x\}} \times G_{\{y\}}$, and in general, it need not be the
 1121 case that $G_X = G^X$ for some monoid G . (Though of course, the case $G_X = G^X$ is an
 1122 example of a meta-monoid, which perhaps may be called a “classical meta-monoid”).

1123 Thus a meta-monoid is like a monoid in that it has sets G_X of “multi-elements” on
 1124 which almost-ordinary monoid theoretic operations are defined. Yet, the multi-elements in
 1125 G_X need not simply be lists of elements as in G^X , and instead, they may be somehow
 1126 “entangled”. A relatively simple example of a meta-monoid which isn’t a monoid is $H^{\otimes X}$
 1127 where H is a Hopf algebra¹⁴. This simple example is similar to “quantum entanglement”.
 1128 But a meta-monoid is not limited to the kind of entanglement that appears in tensor powers.
 1129 Indeed many of the examples within the main text of this paper aren’t tensor powers and
 1130 their “entanglement” is closer to that of the theory of tangles. This especially applied to the
 1131 meta-monoid $w\mathcal{T}$ of Section 3.2.

1132 *10.3.3 Monoid-Actions*

1133 A monoid-action¹⁵ of a monoid G_1 on another monoid G_2 is a single algebraic structure
 1134 MA consisting of two sets G_1 (heads) and G_2 (tails), a binary operation defined on G_1 ,
 1135 a binary operation defined on G_2 , and a mixed operation $G_1 \times G_2 \rightarrow G_2$ (denoted
 1136 $(x, u) \mapsto u^x$) which satisfy some well-known axioms, of which the most interesting are the
 1137 associativities of the first two binary operations and the two action axioms $(uv)^x = u^x v^x$
 1138 and $u^{(xy)} = (u^x)^y$.

1139 As in the case of individual monoids, a monoid-action MA gives rise to a slew of spaces
 1140 and maps between them. The spaces are $\text{MA}(T; H) := G_2^T \times G_1^H$, defined when-
 1141 ever T and H are finite sets of tail labels and head labels. The main operations¹⁶ are
 1142 $tm_w^{uv} : \text{MA}(T \cup \{u, v\}; H) \rightarrow \text{MA}(T \cup \{w\}; H)$ defined using the multiplication in G_2
 1143 (assuming $u, v, w \notin T$ and $u \neq v$), $hm_z^{xy} : \text{MA}(T; H \cup \{x, y\}) \rightarrow \text{MA}(T; H \cup \{z\})$
 1144 (assuming $x, y \notin H$ and $x \neq y$) defined using the multiplication in G_1 , and
 1145 $tha^{ux} : \text{MA}(T; H) \rightarrow \text{MA}(T; H)$ (assuming $x \in H$ and $u \in T$) defined using the
 1146 action of G_1 on G_2 . These operations have the following properties, corresponding to the
 1147 associativity of G_1 and G_2 and to the two action axioms of the previous paragraph:

$$\begin{aligned}
 hm_x^{xy} \parallel hm_x^{xz} &= hm_y^{yz} \parallel hm_x^{xy}, & tm_u^{uv} \parallel tm_u^{uw} &= tm_v^{vw} \parallel tm_u^{uv}, \\
 tm_w^{uv} \parallel tha^{wx} &= tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}, & hm_z^{xy} \parallel tha^{uz} &= tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy}.
 \end{aligned}
 \tag{34}$$

1148 There are also routine properties involving also $*$, η ’s and σ ’s as before.

1149 *10.3.4 Meta-Monoid-Actions*

1150 Finally, a meta-monoid-action is to a monoid-action like a meta-monoid is to a monoid.
 1151 Thus it is a collection

$$(M(T; H), tm_w^{uv}, hm_z^{xy}, tha^{ux}, t\sigma_w^u, h\sigma_y^x, t\eta^u, h\eta^x, *, t\epsilon_u, h\epsilon_x)$$

¹⁴Or merely an algebra.

¹⁵Think “group-action”.

¹⁶There are also $*$, $t\eta^u$, $h\eta^x$, $t\sigma_v^u$ and $h\sigma_y^x$ and units $t\epsilon_u$ and $h\epsilon_x$ as before.

of sets $M(T; H)$, one for each pair of finite sets $(T; H)$ of tail labels and head labels, and maps between them $tm_w^{uv}, hm_z^{xy}, tha^{ux}, t\sigma_v^u, h\sigma_y^x, t\eta^u, h\eta^x, *,$ and units $t\epsilon_u$ and $h\epsilon_x$, with the same domains and ranges as above and with the same **linear and multi-linear** properties as above; most importantly, the properties in (34).

Thus a meta-monoid-action is like a monoid-action in that it has sets $M(T; H)$ of multi-elements on which almost-ordinary monoid theoretic operations are defined. Yet the multi-elements in $M(T; H)$ need not simply be lists of elements as in $G_2^T \times G_1^H$, and instead they may be somehow entangled.

10.3.5 Meta-Groups / Meta-Hopf-Algebras

Clearly, the prefix meta can be added to many other types of algebraic structures, though sometimes a little care must be taken. To define a “meta-group”, for example, one may add to the definition of a meta-monoid in Section 10.3.2 a further collection of operations S^x , one for each $x \in X$, representing “invert the (meta-)element in register x ”. Except that the axiom for an inverse, $g \cdot g^{-1} = \epsilon$, is quadratic in g —one must have two copies of g in order to write the axiom, and hence it cannot be written using S^x and the operations in Section 10.3.2. Thus, in order to define a meta-group, we need to also include “meta-co-product” operations $\Delta_{yz}^x : G_{X \cup \{x\}} \rightarrow G_{X \cup \{y,z\}}$. These operations should satisfy some further axioms, much like within the definition of a Hopf algebra. The major ones are: a meta-co-associativity, a meta-compatibility with the meta-multiplication, and a meta-inverse axiom $\Omega \parallel \Delta_{yz}^x \parallel S^y \parallel m_x^{yz} = (\Omega \parallel \eta^x) * \epsilon_x$.

A strict analogy with groups would suggest another axiom: a meta-co-commutativity of Δ , namely $\Delta_{yz}^x = \Delta_{zy}^x$. Yet, experience shows that it is better to sometimes not insist on meta-co-commutativity. Perhaps the name meta-group should be used when meta-co-commutativity is assumed, and “meta-Hopf-algebra” when it isn’t.

Similarly one may extend “meta-monoid-actions” to “meta-group-actions” and/or “meta-Hopf-actions”, in which new operations $t\Delta$ and $h\Delta$ are introduced, with appropriate axioms.

Note that $v\mathcal{T}$ and $w\mathcal{T}$ have a meta-co-product, defined using “strand doubling”. It is not meta-co-commutative.

Note also that \mathcal{K}^{rbh} and \mathcal{K}_0^{rbh} have operations $h\Delta$ and $t\Delta$, defined using “hoop doubling” and “balloon doubling”. The former is meta-co-commutative while the latter is not.

Note also that M and M_0 have an operation $h\Delta_{yz}^x$ defined by cloning one Lie word, and an operation $t\Delta_{vw}^u$ defined using the substitution $u \rightarrow v + w$. Both of these operations are meta-co-commutative.

Thus ζ_0 and ζ cannot be homomorphic with respect to $t\Delta$. The discussion of trivalent vertices in [5, Section 4] can be interpreted as an analysis of the failure of ζ to be homomorphic with respect to $t\Delta$, but this will not be attempted in this paper.

10.4 Some Differentials and the Proof of Proposition 5.1

We prove Proposition 5.1, namely (19) through (21), by verifying that each of these equations holds at one point, and then by differentiating each side of each equation and showing that the derivatives are equal. While routine, this argument appears complicated because the spaces involved are infinite dimensional and the operations involved are non-commutative. In fact, even the well-known derivative of the exponential function, which appears in the definition of C_u which appears in the definitions of RC_u and of J_u , may surprise readers who are used to the commutative case $de^x = e^x dx$.



1197 Recall that FA denotes the graded completion of the free associative algebra on some
 1198 alphabet T , and that the exponential map $\exp: FL \rightarrow FA$ defined by $\gamma \mapsto \exp(\gamma) =$
 1199 $e^\gamma := \sum_{k=0}^\infty \frac{\gamma^k}{k!}$ makes sense in this completion.

1200 **Lemma 10.3** *If $\delta\gamma$ denotes an infinitesimal variation of γ , then the infinitesimal variation*
 1201 *δe^γ of e^γ is given as follows:*

$$\delta e^\gamma = e^\gamma \cdot \left(\delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma} \right) = \left(\delta\gamma \parallel \frac{e^{\text{ad}\gamma} - 1}{\text{ad}\gamma} \right) \cdot e^\gamma. \tag{35}$$

1202 Above expressions such as $\frac{e^{\text{ad}\gamma} - 1}{\text{ad}\gamma}$ are interpreted via their power series expansions,
 1203 $\frac{e^{\text{ad}\gamma} - 1}{\text{ad}\gamma} = 1 + \frac{1}{2}\text{ad}\gamma + \frac{1}{6}(\text{ad}\gamma)^2 + \dots$, and hence $\delta\gamma \parallel \frac{e^{\text{ad}\gamma} - 1}{\text{ad}\gamma} = \delta\gamma + \frac{1}{2}[\gamma, \delta\gamma] +$
 1204 $\frac{1}{6}[\gamma, [\gamma, \delta\gamma]] + \dots$. Also, the precise meaning of (35) is that for any $\delta\gamma \in FL$, the deriva-
 1205 tive $\delta e^\gamma := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (e^{\gamma + \epsilon\delta\gamma} - e^\gamma)$ is given by the right-hand-side of that equation.
 1206 Equivalently, δe^γ is the term proportional to $\delta\gamma$ in $e^{\gamma + \delta\gamma}$, where during calculations, we
 1207 may assume that “ $\delta\gamma$ is an infinitesimal”, meaning that anything quadratic or higher in $\delta\gamma$
 1208 can be regarded as equal to 0.

1209 Lemma 10.3 is rather standard (e.g. [8, Section 1.5], [22, Section 7]). Here’s a tweet:

1210 *Proof of Lemma 10.3* With an infinitesimal $\delta\gamma$, consider $F(s) := e^{-s\gamma} e^{s(\gamma + \delta\gamma)} - 1$.
 1211 Then, $F(0) = 0$ and $\frac{d}{ds} F(s) = e^{-s\gamma} (-\gamma) e^{s(\gamma + \delta\gamma)} + e^{-s\gamma} (\gamma + \delta\gamma) e^{s(\gamma + \delta\gamma)} =$
 1212 $e^{-s\gamma} \delta\gamma e^{s(\gamma + \delta\gamma)} = e^{-s\gamma} \delta\gamma e^{s\gamma} = \delta\gamma \parallel e^{-s\text{ad}\gamma}$. So $e^{-\gamma} \delta\gamma = F(1) = \int_0^1 ds \frac{d}{ds} F(s) =$
 1213 $\delta\gamma \parallel \int_0^1 ds e^{-s\text{ad}\gamma} = \delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma}$. The second part of (35) is proven in a similar manner,
 1214 starting with $G(s) := e^{s(\gamma + \delta\gamma)} e^{-s\gamma} - 1$.

1215 **Lemma 10.4** *If $\gamma = \text{bch}(\alpha, \beta)$ and $\delta\alpha, \delta\beta$, and $\delta\gamma$ are infinitesimals related by $\gamma + \delta\gamma =$
 1216 $\text{bch}(\alpha + \delta\alpha, \beta + \delta\beta)$, then*

$$\delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma} = \left(\delta\alpha \parallel \frac{1 - e^{-\text{ad}\alpha}}{\text{ad}\alpha} \parallel e^{-\text{ad}\beta} \right) + \left(\delta\beta \parallel \frac{1 - e^{-\text{ad}\beta}}{\text{ad}\beta} \right) \tag{36}$$

1217 *Proof* Use Leibniz’ law on $e^\gamma = e^\alpha e^\beta$ to get $\delta e^\gamma = (\delta e^\alpha) e^\beta + e^\alpha (\delta e^\beta)$. Now use
 1218 Lemma 10.3 three times to get

$$e^\gamma \left(\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma} \right) = e^\alpha \left(\delta\alpha \parallel \frac{1 - e^{-\text{ad}\alpha}}{\text{ad}\alpha} \right) e^\beta + e^\alpha e^\beta \left(\delta\beta \parallel \frac{1 - e^{-\text{ad}\beta}}{\text{ad}\beta} \right),$$

1219 conjugate the e^β in the first summand to the other side of the parenthesis, and cancel $e^\gamma =$
 1220 $e^\alpha e^\beta$ from both sides of the resulting equation. □

1221 Recall that C_u^γ and RC_u^γ are automorphisms of FL . We wish to study their variations
 1222 δC_u^γ and δRC_u^γ with respect to γ (these variations are “infinitesimal” automorphisms of
 1223 FL). We need a definition and a property first.

Definition 10.5 For $u \in T$ and $\gamma \in FL(T)$ let $\text{ad}_u\{\gamma\} = \text{ad}_u^\gamma : FL(T) \rightarrow FL(T)$ denote the derivation of $FL(T)$ defined by its action of the generators as follows:

$$v \parallel \text{ad}_u\{\gamma\} = v \parallel \text{ad}_u^\gamma := \begin{cases} [\gamma, u] & v = u \\ 0 & \text{otherwise.} \end{cases}$$

Property 10.6 ad_u is the infinitesimal version of both C_u and RC_u . Namely, if $\delta\gamma$ is an infinitesimal, then $C_u^{\delta\gamma} = RC_u^{\delta\gamma} = 1 + \text{ad}_u\{\delta\gamma\}$.

We omit the easy proof of this property and move on to δC_u^γ and δRC_u^γ :

Lemma 10.7 $\delta C_u^\gamma = \text{ad}_u \left\{ \delta\gamma \parallel \frac{e^{\text{ad}\gamma} - 1}{\text{ad}\gamma} \parallel RC_u^{-\gamma} \right\} \parallel C_u^\gamma$

and $\delta RC_u^\gamma = RC_u^\gamma \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma} \parallel RC_u^\gamma \right\}$.

Proof Substitute α and $\delta\beta$ into (16) and get $RC_u^{\text{bch}(\alpha, \delta\beta)} = RC_u^\alpha \parallel RC_u^{\delta\beta} \parallel RC_u^\alpha$, and hence using Property 10.6 for the infinitesimal $\delta\beta \parallel RC_u^\alpha$ and Lemma 10.4 with $\delta\alpha = \beta = 0$ on $\text{bch}(\alpha, \delta\beta)$,

$$RC_u^{\alpha + (\delta\beta \parallel \frac{\text{ad}\alpha}{1 - e^{-\text{ad}\alpha}})} = RC_u^\alpha + RC_u^\alpha \parallel \text{ad}_u\{\delta\beta \parallel RC_u^\alpha\}.$$

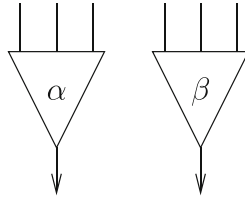
Now, replacing $\alpha \rightarrow \gamma$ and $\delta\beta \rightarrow \delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma}$, we get the equation for δRC_u^γ . The equation for δC_u^γ now follows by taking the variation of $C_u^\gamma \parallel RC_u^{-\gamma} = Id$. \square

Our next task is to compute $\delta J_u(\gamma)$. Yet before we can do that, we need to know one of the two properties of div_u that matter for us (besides its linearity):

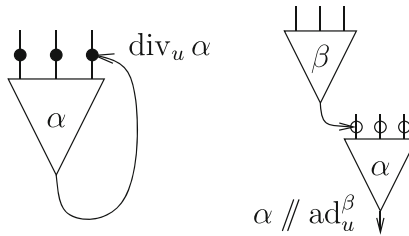
Proposition 10.8 For any $u, v \in T$ and any $\alpha, \beta \in FL$ and with δ_{uv} denoting the Kronecker delta function, the following ‘‘cocycle condition’’ holds: (compare with [1, Proposition 3.20])

$$\underbrace{(\text{div}_u \alpha) \parallel \text{ad}_v^\beta}_A - \underbrace{(\text{div}_v \beta) \parallel \text{ad}_u^\alpha}_B = \underbrace{\delta_{uv} \text{div}_u[\alpha, \beta]}_C + \underbrace{\text{div}_u(\alpha \parallel \text{ad}_v^\beta)}_D - \underbrace{\text{div}_v(\beta \parallel \text{ad}_u^\alpha)}_E. \quad (37)$$

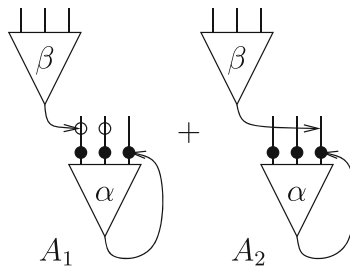
Proof Start with the case where $u = v$. We draw each contribution to each of the terms above and note that all of these contributions cancel, but we must first explain our drawing conventions. We draw α and β as the ‘‘logic gates’’ appearing below. Each is really a linear combination, but (37) is bilinear so this doesn’t matter. Each is really a tree, but the proof does not use this so we don’t display this. Each may have many tail-legs labelled by other elements of T , but we care only about the legs labelled $u = v$ and so we display only those, and without real loss of generality, we draw it as if α and β each have exactly three such tails.



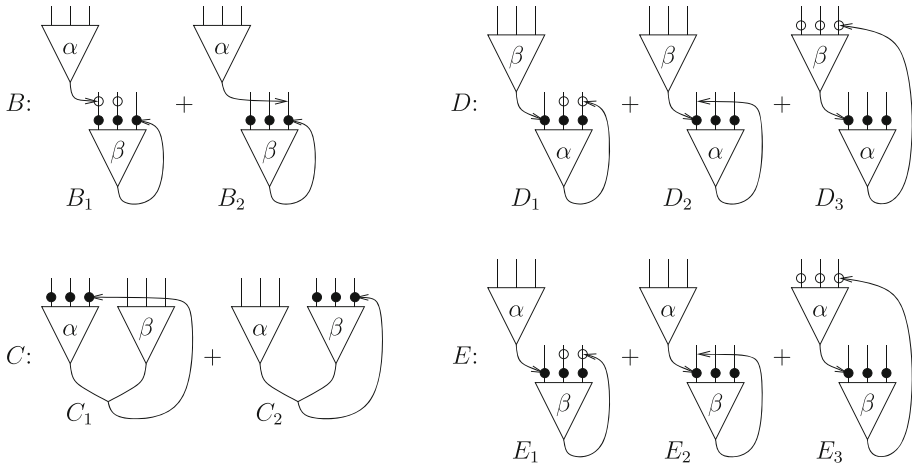
1250 Objects such as $\text{div}_u \alpha$ and $\alpha // \text{ad}_u^\beta$ are obtained from α and β by connecting the head
 1251 of one near its own tails, or near the other's tails, in all possible ways. We draw just one
 1252 summand from each sum, yet we indicate the other possible summands in each case by
 1253 marking the other places where the relevant head could go with filled circles (●) or empty
 1254 circles (○) (the filling of the circles has no algebraic meaning; it is there only to separate
 1255 summations in cases where two summations appear in the same formula). I hope the pictures
 below explain this better than the words.



1256 We illustrate our next convention with the pictorial representation of term A of (37),
 1257 $(\text{div}_u \alpha) // \text{ad}_u^\beta$, shown below. Namely, when the two relevant summations dictate that two
 1258 heads may fall on the same arc, we split the sum into the generic part, A_1 below, in
 1259 which the two heads do not fall on the same arc, and the exceptional part, A_2 below,
 1260 in which the two heads do indeed fall on the same arc. The last convention is that ●
 1261 indicates the first summation, and ○, the second. Hence in A_1 , the α head may fall in
 1262 three places, and after that, the β head may only fall on one of the remaining rele-
 1263 vant tails, whereas in A_1 , the α is again free, but the β head must fall on the same
 1264 arc.

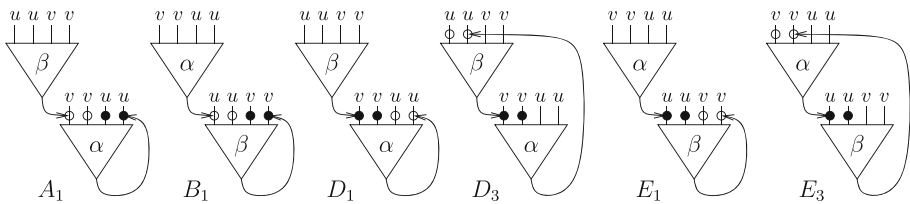


With all these conventions in place and with term A as above, we depict terms B – E :



Clearly, $A_1 = D_1$, $B_1 = E_1$, and $D_3 = E_3$ (the last equality is the only place in this paper that we need the cyclic property of cyclic words). Also, by the Jacobi identity, $A_2 - D_2 = C_1$ and $E_2 - B_2 = C_2$. So altogether, $A - B = C + D - E$.

The case where $u \neq v$ is similar, except we have to separate between u and v tails, the terms analogous to A_2 , B_2 , D_2 and E_2 cannot occur, and $C = 0$:



Clearly, $A - B = D - E$. □ 1272

For completeness and for use within the proof of (21), here's the remaining property of div we need to know, presented without its easy proof: 1273
1274

Proposition 10.9 For any $\gamma \in FL$, $\gamma \parallel t_w^{uv} \parallel \text{div}_w = \gamma \parallel \text{div}_u \parallel t_w^{uv} + \gamma \parallel \text{div}_v \parallel t_w^{uv}$. □ 1275

Proposition 10.10 $\delta J_u(\gamma) = \delta\gamma \parallel \frac{1 - e^{-\text{ady}}}{\text{ady}} \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma}$. 1276

1277 *Proof* Let $I_s := \gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma}$ denote the integrand in the definition of J_u . Then
 1278 under $\gamma \rightarrow \gamma + \delta\gamma$, using Leibniz, the linearity of div_u , and both parts of Lemma 10.7, we
 1279 have

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad + \gamma \parallel RC_u^{s\gamma} \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \right\} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \right\} \parallel C_u^{-s\gamma}. \end{aligned}$$

1280 Taking the last two terms above as D and A of (37), with $\alpha = \gamma \parallel RC_u^{s\gamma}$ and $\beta = \delta\gamma \parallel$
 1281 $\frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma}$, and using $[\alpha, \beta] = [\gamma, \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma}] \parallel RC_u^{s\gamma} = \delta\gamma \parallel (1-e^{-\text{ads}\gamma}) \parallel RC_u^{s\gamma}$,
 1282 we get

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad + \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \parallel \text{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel \text{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel (1 - e^{-\text{ads}\gamma}) \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma}, \end{aligned}$$

1283 and so, by combining the first and the last terms above,

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel e^{-\text{ads}\gamma} \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad + \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \parallel \text{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel \frac{1-e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel \text{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel C_u^{-s\gamma}, \end{aligned}$$

1284 and hence, once again using Lemma 10.7 to differentiate $RC_u^{s\gamma}$ and $C_u^{-s\gamma}$ (except that things
 1285 are now simpler because $s\gamma$ and $\delta(s\gamma) = \frac{d}{ds}(s\gamma) = \gamma$ commute), we get

$$\delta I_s = \frac{d}{ds} \left(\delta\gamma \parallel \frac{1 - e^{-\text{ads}\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma} \right).$$

1286 Integrating with respect to the variable s and using the fundamental theorem of calculus, we
 1287 are done. □

1288 *Proof of Equation (19).* We fix α and show that (19) holds for every β . For this it is enough
 1289 to show that (19) holds for $\beta = 0$ (it trivially does), and that the derivatives of both sides of
 1290 (19) in the radial direction are equal, for any given β . Namely, it is enough to verify that the
 1291 variations of the two sides of (19) under $\beta \rightarrow \beta + \delta\beta$ are equal, where $\delta\beta$ is proportional
 1292 to β . Indeed, using the chain rule, Lemma 10.4, Proposition 10.10, the fact that β commutes
 1293 with $\delta\beta$, and with $\gamma := \text{bch}(\alpha, \beta)$,

$$\begin{aligned} \delta LHS &= \left(\delta\beta \parallel \frac{1-e^{-\text{ad}\beta}}{\text{ad}\beta} \parallel \frac{\text{ad}\gamma}{1-e^{-\text{ad}\gamma}} \right) \parallel \frac{1-e^{-\text{ad}\gamma}}{\text{ad}\gamma} \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma} \\ &= \delta\beta \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma}. \end{aligned}$$

1294 Similarly, using Proposition 10.10 and the fact that $\beta \parallel RC_u^\alpha$ commutes with $\delta\beta \parallel RC_u^\alpha$,

$$\delta RHS = \delta\beta \parallel RC_u^\alpha \parallel RC_u^\beta \parallel RC_u^\alpha \parallel \text{div}_u \parallel C_u^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha} = \delta\beta \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma},$$

1295 where in the last equality, we have used (16) to combine the RC s and its inverse to combine
 1296 the C s. □

Proof of Equation (20). Equation (20) clearly holds when $\alpha = 0$, so as before, it is enough to prove it after taking the radial derivative with respect to α . So we need (ouch!) 1297
1298

$$\begin{aligned} & \alpha \parallel RC_u^\alpha \parallel \operatorname{div}_u \parallel C_u^{-\alpha} - \alpha \parallel RC_v^\beta \parallel RC_u^\alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ &= -\beta \parallel RC_u^\alpha \parallel \operatorname{ad}_u^\alpha \parallel RC_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta \parallel RC_u^\alpha)}}{\operatorname{ad}(\beta \parallel RC_u^\alpha)} \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_v \parallel C_v^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha} \\ & \qquad \qquad \qquad - \beta \parallel RC_u^\alpha \parallel J_v \parallel \operatorname{ad}_u^{-\alpha} \parallel RC_u^\alpha \parallel C_u^{-\alpha}. \end{aligned}$$

This we simplify using (13) and (14), cancel the $C_u^{-\alpha}$ on the right, and get 1299

$$\begin{aligned} & \alpha \parallel RC_u^\alpha \parallel \operatorname{div}_u - \alpha \parallel RC_u^\alpha \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_u \parallel C_v^{-\beta} \parallel RC_u^\alpha \\ & \stackrel{?}{=} -\beta \parallel RC_u^\alpha \parallel \operatorname{ad}_u^\alpha \parallel RC_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta \parallel RC_u^\alpha)}}{\operatorname{ad}(\beta \parallel RC_u^\alpha)} \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_v \parallel C_v^{-\beta} \parallel RC_u^\alpha \\ & \qquad \qquad \qquad - \beta \parallel RC_u^\alpha \parallel J_v \parallel \operatorname{ad}_u^{-\alpha} \parallel RC_u^\alpha. \end{aligned}$$

We note that above α and β only appear within the combinations $\alpha \parallel RC_u^\alpha$ and $\beta \parallel RC_u^\alpha$, so we rename $\alpha \parallel RC_u^\alpha \rightarrow \alpha$ and $\beta \parallel RC_u^\alpha \rightarrow \beta$: 1300
1301

$$\begin{aligned} & \alpha \parallel \operatorname{div}_u - \alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_v^{-\beta} \\ & \stackrel{?}{=} -\beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} - \beta \parallel J_v \parallel \operatorname{ad}_u^{-\alpha}. \quad (38) \end{aligned}$$

Equation (38) still contains a J_v in it, so in order to prove it, we have to differentiate once again. So note that it holds at $\beta = 0$, multiply by -1 , and take the radial variation with respect to β (note that $\frac{d}{ds} \frac{1 - e^{-\operatorname{ad}(s\beta)}}{\operatorname{ad}(s\beta)} \Big|_{s=1} = \frac{e^{-\operatorname{ad}(\beta)}(1 + \operatorname{ad}(\beta) - e^{\operatorname{ad}(\beta)})}{\operatorname{ad}(\beta)}$): 1302
1303
1304

$$\begin{aligned} & \alpha \parallel RC_v^\beta \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_v^{-\beta} - \alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \stackrel{?}{=} \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{e^{-\operatorname{ad}(\beta)}(1 + \operatorname{ad}(\beta) - e^{\operatorname{ad}(\beta)})}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel \operatorname{ad}_v^{-\beta} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \quad + \beta \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel CC_v^{-\beta} \parallel \operatorname{ad}_u^{-\alpha}. \quad (39) \end{aligned}$$

We massage three independent parts of the above desired equality at the same time: 1305

- The div and the ad on the left hand side make terms D and A of (37), with $\alpha \parallel RC_v^\beta \rightarrow \alpha$ and $\beta \parallel RC_v^\beta \rightarrow \beta$. We replace them by terms A and E . 1306
1307
- We combine the first two terms of the right hand side using $\frac{1 - e^{-a}}{a} + \frac{e^{-a}(1 + a - e^a)}{a} = e^{-a}$. 1308
1309
- In (14), $C_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} = C_v^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha}$, take an infinitesimal α and use Property 10.6 and Lemma 10.7 to get 1310
1311

$$\operatorname{ad}_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} = \operatorname{ad}_v^{-\beta} \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta} + C_v^{-\beta} \parallel \operatorname{ad}_u^{-\alpha}. \quad (40)$$

The last of that matches the last of (39), so we can replace the last of (39) with the start of (40). 1312
1313

1314 All of this done, (39) becomes the lowest point of this paper:

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \quad + \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \quad - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta}. \end{aligned}$$

1315 Next, we cancel the $C_v^{-\beta}$ at the right of every term, and a pair of repeating terms to get

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \\ & \quad + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel RC_v^\beta \parallel \text{div}_v \\ & \quad - \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^\beta \parallel RC_v^\beta \\ & \quad - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta. \end{aligned}$$

1316 The two middle terms above differ only in the order of ad_v and div_v . So we apply (37)
1317 again and get

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \\ & \quad + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \\ & \quad + \left[\beta \parallel RC_v^\beta, \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \right] \parallel \text{div}_v - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta. \end{aligned}$$

1318
1319 In the above, the two terms that do not end in div_v cancel each other. We then remove the
1320 div_v at the end of all remaining terms, thus making our quest only harder. Finally, we note
1321 that RC_v^β is a Lie algebra morphism, so we can pull it out of the bracket in the penultimate
1322 term, getting

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \\ & \quad + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta + \left[\beta, \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \right] \parallel RC_v^\beta. \end{aligned}$$

1323 The bracketing with β in the last term above cancels the $\text{ad}(\beta)$ denominator there, and
1324 then that term combines with the first term of the right hand side to yield

$$\beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta.$$

We make our task harder again, 1325

$$RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \text{ad}_u^\alpha \parallel RC_v^\beta + RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta$$

and then we both pre-compose and post-compose with the isomorphism $C_v^{-\beta}$, getting 1326

$$\text{ad}_u^\alpha \parallel RC_v^\beta \parallel C_v^{-\beta} \stackrel{?}{=} C_v^{-\beta} \parallel \text{ad}_u^\alpha + \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta}.$$

The above is (40), with α replaced by $-\alpha$, and hence it holds true. □ 1327

Proof of Equation (21) As before, the equation clearly holds at $\gamma = 0$, so we take its radial derivative. That of the left hand side is 1328
1329

$$\gamma \parallel tm_w^{uv} \parallel RC_w^\gamma \parallel tm_w^{uv} \parallel \text{div}_w \parallel C_w^{-\gamma} \parallel tm_w^{uv}.$$

Using (15) and then Proposition 10.9, this becomes 1330

$$\gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel (\text{div}_u + \text{div}_v) \parallel tm_w^{uv} \parallel C_w^{-\gamma} \parallel tm_w^{uv}.$$

Now using the reverse of (15), proven by reading the horizontal arrows within its proof backwards, this becomes 1331
1332

$$\gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel (\text{div}_u + \text{div}_v) \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel tm_w^{uv}.$$

On the other hand, the radial variation of the right hand side of (21) is 1333

$$\begin{aligned} \gamma \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma} \parallel tm_w^{uv} + \gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_v \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel tm_w^{uv} \\ + \gamma \parallel RC_u^\gamma \parallel \text{ad}_u^\gamma \parallel RC_u^\gamma \parallel \frac{1 - e^{-\text{ad}(\gamma \parallel RC_u^\gamma)}}{\text{ad}(\gamma \parallel RC_u^\gamma)} \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_v \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel t_w^{uv} \\ + \gamma \parallel RC_u^\gamma \parallel J_v \parallel \text{ad}_u^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel t_w^{uv}. \end{aligned}$$

Equating the last two formulae while eliminating the common term (the second term in each) and removing all trailing $C_u^{-\gamma} \parallel t_w^{uv}$'s (thus making the quest harder), we need to show that 1334
1336
1337

$$\begin{aligned} \gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_v^{-\gamma} \parallel RC_u^\gamma = \gamma \parallel RC_u^\gamma \parallel \text{div}_u \\ + \gamma \parallel RC_u^\gamma \parallel \text{ad}_u^\gamma \parallel RC_u^\gamma \parallel \frac{1 - e^{-\text{ad}(\gamma \parallel RC_u^\gamma)}}{\text{ad}(\gamma \parallel RC_u^\gamma)} \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_v \parallel C_v^{-\gamma} \parallel RC_u^\gamma \\ + \gamma \parallel RC_u^\gamma \parallel J_v \parallel \text{ad}_u^{-\gamma} \parallel RC_u^\gamma. \end{aligned}$$

Nicely enough, the above is (38) with $\alpha = \beta = \gamma \parallel RC_u^\gamma$. □ 1338

10.5 Notational Conventions and Glossary 1340

For $n \in \mathbb{N}$ let \underline{n} denote some fixed set with n elements, say $\{1, 2, \dots, n\}$. 1341

Often, within this paper, we use postfix notation for operator evaluations, so $f(x)$ may also be denoted $x \parallel f$. Even better, we use $f \parallel g$ for “composition done right”, meaning 1342
1343

$f \parallel g = g \circ f$, meaning that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $X \xrightarrow{f \parallel g} Z$ rather than the uglier (though 1344

equally correct) $X \xrightarrow{g \circ f} Z$. We hope that this notation will be adopted by others, to be used 1345

alongside and eventually instead of $g \circ f$, much as we hope that τ will be used alongside 1346

and eventually instead of the presently popular $\pi := \tau/2$. In \LaTeX , $\parallel = \backslash\text{slash} \in$ 1347
1348

stmaryrd.sty.



1349 In the few paragraphs that follow, X is an arbitrary set. Though within this paper such
 1350 X 's will usually be finite, and their elements will thought of as labels. Hence, if $f \in G^X$ is
 1351 a function $f: X \rightarrow G$ where G is some other set, we think of f as a collection of elements
 1352 of G labelled by the elements of X . We often write f_x to denote $f(x)$.

1353 If $f \in G^X$ and $x \in X$, we let $f \setminus x$ denote the restricted function $f|_{X \setminus x}$ in which x is
 1354 removed from the domain of f . In other words, $f \setminus x$ is “the collection f , with the element
 1355 labelled x removed”. We often neglect to state the condition $x \in X$. Thus, when writing
 1356 $f \setminus x$ we implicitly assume that $x \in X$.

1357 Likewise, we write $f \setminus \{x, y\}$ for “ f with x and y removed from its domain” and as before
 1358 this includes the implicit assumption that $\{x, y\} \subset X$.

1359 If $f_1: X_1 \rightarrow G$ and $f_2: X_2 \rightarrow G$ and X_1 and X_2 are disjoint, we denote by $f \cup g$ the
 1360 obvious “union function” with domain $X_1 \cup X_2$ and range G . In fact, whenever we write
 1361 $f \cup g$, we make the implicit assumption that the domains of f_1 and f_2 are disjoint.

1362 In the spirit of “associative arrays” as they appear in various computer languages, we use
 1363 the notation $(x \rightarrow a, y \rightarrow b, \dots)$ for “inline function definition”. Thus, $()$ is the empty
 1364 function, and if $f = (x \rightarrow a, y \rightarrow b)$, then the domain of f is $\{x, y\}$ and $f_x = a$ and
 1365 $f_y = b$.

1366 We denote by σ_y^x the operation that renames the key x in an associative array to y .
 1367 Namely, if $f \in G^X$, $x \notin X$, and $y \notin X \setminus x$, then

$$\sigma_y^x f = (f \setminus x) \cup (y \rightarrow f_x).$$

1368 **Acknowledgments** I wish to thank the people of the Knot at Lunch seminar in Toronto (Oleg Chterental,
 1369 Karene Chu, Zsuzsanna Dancso, Peter Lee, Stephen Morgan, James Mracek, David Penneys, Peter Samuel-
 1370 son, and Sam Selmani) for hearing this several times and for making comments and suggestions. Further
 1371 thanks to Zsuzsanna Dancso for pushing me to write this, to Martin Bridson for telling me about LOT groups
 1372 and to Marcel Bökstedt, Eckhard Meinrenken, and Jim Stasheff for further comments. This work was par-
 1373 tially supported by NSERC grant RGPIN 262178, by ITGP (Interactions of Low-Dimensional Topology and
 1374 Geometry with Mathematical Physics), an ESF RNP, and by the Newton Institute in Cambridge, UK.

1375 **Glossary of Notations** (Greek letters, then Latin, then symbols)

1376	α, β, γ	Free Lie series	Sec. 4
1377	$\alpha, \beta, \gamma, \delta$	Matrix parts	Sec. 9.4
1378	β	A repackaging of β	Sec. 9.4
1379	β_0	A reduction of M	Sec. 9.3
1380	δ	A map $u\mathcal{T}/v\mathcal{T}/w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$	Sec. 2.2
1381	$\delta\alpha, \delta\beta, \delta\gamma$	Infinitesimal free Lie series	Sec. 10.4
1382	ϵ_a	Units	Sec. 3.2
1383	Π	The MMA “of groups”	Sec. 3.4
1384	π	The fundamental invariant	Sec. 2.3
1385	π	The projection $\mathcal{K}_0^{rbh} \rightarrow \mathcal{K}^{rbh}$	Prop. 3.6
1386	ρ_{ux}^\pm	\pm -Hopf links in 4D	Ex. 2.2
1387	σ_y^x	Re-labelling	Sec. 10.5
1388	τ	Tensorial interpretation map	Sec. 8.1
1389	ω	The wheels part of M/ζ	Sec. 5
1390	ω	The scalar part in β/β_0	Sec. 9.3
1391	Υ	Capping and sliding	Sec. 10.2
1392	ζ	The main invariant	Sec. 5

ζ_0	The tree-level invariant	Sec. 4	1393
ζ^β	A β -valued invariant	Sec. 9.4	1394
ζ^{β_0}	A β_0 -valued invariant	Sec. 9.3	1395
A	The matrix part in β/β_0	Sec. 9.3	1396
a, b, c	Strand labels	Sec. 2.2	1397
$\text{ad}_u^\gamma, \text{ad}_u\{\gamma\}$	Derivations of FL	Def. 105	1398
\mathcal{A}^{bh}	Space of arrow diagrams	Sec. 7.2	1399
bch	Baker-Campbell-Hausdorff	Sec. 4.2	1400
C_u^γ	Conjugating a generator	Sec. 4.2	1401
CA	Circuit algebra	Sec. 7.1	1402
CW	Cyclic words	Sec. 5.1	1403
CW^r	CW mod degree 1	Sec. 5.1	1404
c	A “sink” vertex	Sec. 9.1	1405
c_u	A “ c -stub”	Sec. 9.1	1406
div_u	The “divergence” $FL \rightarrow CW$	Sec. 5.1	1407
dm_c^{ab}	Double/diagonal multiplication	Sec. 3.2	1408
FA	Free associative algebra	Sec. 5.1	1409
FL	Free Lie algebra	Sec. 4.2	1410
$\text{Fun}(X \rightarrow Y)$	Functions $X \rightarrow Y$	Sec. 8.1	1411
H	Set of head/hoop labels	Sec. 2	1412
$h \in_x$	Units	Ex. 2.2, Sec. 4.2,5.2	1413
$h\eta$	Head delete	Sec. 3,4,2,5.2	1414
hm_z^{xy}	Head multiply	Sec. 3,4,2,5.2	1415
$h\sigma_y^x$	Head re-label	Sec. 3,4,2,5.2	1416
J_u	The “spice” $FL \rightarrow CW$	Sec. 5.1	1417
\mathcal{K}^{rbh}	All rKBHs	Def. 2.1	1418
\mathcal{K}_0^{rbh}	Conjectured version of \mathcal{K}^{rbh}	Sec. 3.3	1419
l_{ux}	4D linking numbers	Sec. 10.1	1420
l_x	Longitudes	Sec. 2.3	1421
M	The “main” MMA	Sec. 5.2	1422
M_0	The MMA of trees	Sec. 4.2	1423
MMA	Meta-monoid-action	Def. 3.2, Sec. 10.3.4	1424
m_u	Meridians	Sec. 2.3	1425
m_c^{ab}	Strand concatenation	Sec. 3.2	1426
OC	Overcrossings commute	Fig. 3	1427
\mathcal{P}^{bh}	Primitives of \mathcal{A}^{bh}	Sec. 7.3	1428
R	Ring of c -stubs	Sec. 9.2	1429
R^r	R mod degree 1	Sec. 9.3	1430
R1,R1',R2,R3	Reidemeister moves	Sec. 2.2, 7.1	1431
RC_u^γ	Repeated C_u^γ / reverse $C_u^{-\gamma}$	Sec. 4.2	1432
rKBH	Ribbon knotted balloons&hoops	Def. 2.1	1433
S	Set of strand labels	Sec. 2.2	1434
T	Set of tail / balloon labels	Sec. 2	1435
$t \in^u$	Units	Ex. 2.2, Sec. 4.2,5.2	1436
tha^{ux}	Tail by head action	Sec. 3,4,2,5.2	1437
$t\eta^u$	Tail delete	Sec. 3,4,2,5.2	1438
tm_w^{uv}	Tail multiply	Sec. 3,4,2,5.2	1439
$t\sigma_y^x$	Tail re-label	Sec. 3,4,2,5.2	1440

1441	t, x, y, z	Coordinates	Sec. 2
1442	UC	Undercrossings commute	Fig. 3
1443	u-tangle	A usual tangle	Sec. 2.2
1444	$u\mathcal{T}$	All u-tangles	Sec. 2.2
1445	u, v, w	Tail / balloon labels	Sec. 2
1446	v-tangle	A virtual tangle	Sec. 2.4
1447	$v\mathcal{T}$	All v-tangles	Sec. 2.4
1448	w-tangle	A virtual tangle mod OC	Sec. 2.4
1449	$w\mathcal{T}$	All w-tangles	Sec. 2.4
1450	x, y, z	Head / hoop labels	Sec. 2
1451	Z^{bh}	An \mathcal{A}^{bh} -valued expansion	Sec. 7.4
1452	*	Merge operation	Sec. 3,4,2,5,2
1453	//	Composition done right	Sec. 10.5
1454	$x // f$	Postfix evaluation	Sec. 10.5
1455	$f \setminus x$	Entry removal	Sec. 10.5
1456	$x \rightarrow a$	Inline function definition	Sec. 10.5
1457	\overline{uv}	“Top bracket form”	Sec. 6
1458	$\overbrace{[L1R]uv}$	A cyclic word	Sec. 6

1459 References

1. Alekseev, A., Torossian, C.: The Kashiwara-Vergne conjecture and Drinfel'd's associators. *Ann. Math.* **175**, 415–463 (2012). arXiv:0802.4300
2. Bar-Natan, D.: On the assiliev knot invariants. *Topology* **34**, 423–472 (1995)
3. Bar-Natan, D.: Finite type invariants of w-knotted objects IV: some computations, in preparation
4. Bar-Natan, D., Dancso, Z.: Finite type invariants of W-knotted objects I: braids, knots and the alexander polynomial. arXiv:1405.1956. <http://www.math.toronto.edu/drorbn/LOP.html#WKO1>
5. Bar-Natan, D., Dancso, Z.: Finite type invariants of W-knotted objects II: tangles and the kashiwara-vergne problem. arXiv:1405.1955. <http://www.math.toronto.edu/drorbn/LOP.html#WKO2>
6. Bar-Natan, D., Selmani, S.: Meta-monoids, meta-bicrossed products, and the Alexander polynomial. *J. Knot Theory Ramifications* **22–10** (2013). arXiv:1302.5689
7. Cattaneo, A.S., Rossi, C.A.: Wilson surfaces and higher dimensional knot invariants. *Commun. Math. Phys.* **256-3**, 513–537 (2005). arXiv:math-ph/0210037
8. Duistermaat, J.J., Kolk, J.A.C.: *Lie groups*. Springer (1999)
9. Habegger, N., Lin, X.-S.: The classification of links up to link-homotopy. *J. Am. Math. Soc.* **3**, 389–419 (1990)
10. Habegger, N., Masbaum, G.: The Kontsevich integral and Milnor's invariants. *Topology* **39-6**, 1253–1289 (2000)
11. Habiro, K., Kanenobu, T., Shima, A.: Finite type invariants of ribbon 2-knots. In: Nencka, H. (ed.) *Low dimensional topology Cont. Math.* vol. 233 pp. 187–196 (1999)
12. Habiro, K., Shima, A.: Finite type invariants of ribbon 2-knots, II. *Topol Appl* **111-3**, 265–287 (2001)
13. Howie, J.: On the asphericity of ribbon disc complements. *Trans. Am. Math. Soc.* **289-1**, 281–302 (1985)
14. Howie, J.: Minimal seifert manifolds for higher ribbon knots. *Geom. Topol. Monogr.* **1**, 261–293. arXiv:math/9810185
15. Kanenobu, T., Shima, A.: Two filtrations of ribbon 2-knots. *Topol. Appl.* **121**, 143–168 (2002)
16. Kauffman, L.H.: Virtual knot theory. *European J. Comb.* **20**, 663–690 (1999). arXiv:math.GT/9811028
17. Kawauchi, A.: *A survey of knot theory*. Basel, Birkhäuser (1996)
18. Kirk, P., Livingston, C., Wang, Z.: The gassner representation for string links. *Comm. Cont. Math.* **3**, 87–136 (2001). arXiv:math/9806035
19. Krieger, A.: The lines of the Kontsevich integral and Rozansky's rationality conjecture. arXiv:math/0005284
20. Kuperberg, G.: What is a virtual link? *Algebr. Geom. Topol.* **3**, 587–591 (2003). arXiv:math.GT/0208039
21. Lee, P.: Proof of a conjectured formula for the Alexander invariant. arXiv:1209.0668

22. Meinrenken, E.: Lie groups and lie algebras. Lecture notes, University of Toronto (2010). <http://www.math.toronto.edu/mein/teaching/lie.pdf> 1492
23. Milnor, J.W.: Link groups. *Annals Math.* **59**, 177–195 (1954) 1493
24. Reutenauer, C.: Free lie algebras. Clarendon, Oxford (1993) 1494
25. Rowen, L.H.: Polynomial identities in ring theory. Academic, New York (1980) 1495
26. Satoh, S.: Virtual knot presentations of ribbon torus knots. *J. of Knot Theory and its Ramifications* **9-4**, 531–542 (2000) 1496
27. Watanabe, T.: Clasper-moves among ribbon 2-knots characterizing their finite type invariants. *J. of Knot Theory and its Ramifications* **15-9**, 1163–1199 (2006) 1497
28. Watanabe, T.: Configuration space integrals for long n -Knots, the Alexander polynomial and knot space cohomology. *Alg. Geom. Top.* **7**, 47–92 (2007). arXiv:[math/0609742](https://arxiv.org/abs/math/0609742) 1498
29. Bar-Natan, D.: Balloons and hoops and their universal finite type invariant, BF Theory, and an ultimate Alexander invariant (self-reference). Web resources at <http://www.math.toronto.edu/~drorbn/papers/KBH/> 1500
30. Winter, B.: The classification of spun torus knots, vol. 18-9. arXiv:[0711.1638](https://arxiv.org/abs/0711.1638) (2009) 1503
31. Winter, B.: On codimension two ribbon embeddings. arXiv:[0904.0684](https://arxiv.org/abs/0904.0684) 1504