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# Balloons and Hoops and their Universal Finite-Type Invariant, BF Theory, and an Ultimate Alexander Invariant

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**Abstract** Balloons are 2D spheres. Hoops are 1D loops. Knotted balloons and hoops (KBH) in 4-space behave much like the first and second homotopy groups of a topological space—hoops can be composed as in $\pi_1$, balloons as in $\pi_2$, and hoops "act" on balloons as $\pi_1$ acts on $\pi_2$. We observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops. We give an ansatz for a tree and wheel (that is, free Lie and cyclic word)-valued invariant $\zeta$ of (ribbon) KBHs in terms of the said compositions and action and we explain its relationship with finite-type invariants. We speculate that $\zeta$ is a complete evaluation of the BF topological quantum field theory in 4D. We show that a certain "reduction and repackaging" of $\zeta$ is an "ultimate Alexander invariant" that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least wasteful in a computational sense. If you believe in categorification, that should be a wonderful playground.

**Keywords** 2-knots · Tangles · Virtual knots · w-tangles · Ribbon knots · Finite type invariants · BF theory · Alexander polynomial · Meta-groups · Meta-monoids

**Mathematics Subject Classification (2010)** 57M25

Web resources for this paper are available at [Web/]:=http://www.math.toronto.edu/~drorbn/papers/KBH/, including an electronic version, source files, computer programs, lecture handouts and lecture videos; *Throughout this paper, we follow the notational conventions and notations outlined in Section* 10.5.

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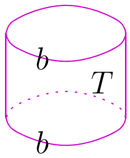
## 1 Introduction

**Riddle 1.1** The set of homotopy classes of maps of a tube $T = S^1 \times [0, 1]$ into a based topological space $(X, b)$ which map the rim $\partial T = S^1 \times \{0, 1\}$ of $T$ to the basepoint $b$ is a group with an obvious "stacking" composition; we call that group $\pi_T(X)$. Homotopy theorists often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$ but seldom, if ever, do they study $\pi_T(X) = [T, X]$. Why?




The solution of this riddle is on page 13. Whatever it may be, the moral is that it is better to study the homotopy classes of circles and spheres in $X$ rather than the homotopy classes of tubes in $X$, and by morphological transfer, it is better to study isotopy classes of embeddings of circles and spheres into some ambient space than isotopy classes of embeddings of tubes into the same space.

In [4, 5], Zsuzsanna Dancso and I studied the finite-type knot theory of ribbon tubes in $\mathbb{R}^4$ and found it to be closely related to deep results by Alekseev and Torossian [1] on the Kashiwara-Vergne conjecture and Drinfel'd's associators. At some point, we needed a computational tool with which to make and to verify conjectures.

This paper started in being that computational tool. After a lengthy search, I found a language in which all the operations and equations needed for [4, 5] could be expressed and computed. Upon reflection, it turned out that the key to that language was to work with knotted balloons and hoops, meaning spheres and circles, rather than with knotted tubes.

Then, I realized that there may be independent interest in that computational tool. For (ribbon) knotted balloons and hoops in $\mathbb{R}^4$ ($\mathcal{K}^{\text{rbh}}$, Section 2) in themselves form a lovely algebraic structure (a meta-monoid-action (MMA), Section 3), and the "tool" is really a well-behaved invariant $\zeta$. More precisely, $\zeta$ is a "homomorphism $\zeta$ of the MMA $\mathcal{K}_0^{\text{rbh}}$ to the MMA $M$ of trees and wheels" (trees in Section 4 and wheels in Section 5). Here, $\mathcal{K}_0^{\text{rbh}}$ is a variant of $\mathcal{K}^{\text{rbh}}$ defined using generators and relations (Definition 3.5). Assuming a sorely missing Reidemeister theory for ribbon-knotted tubes in $\mathbb{R}^4$ (Conjecture 3.7), $\mathcal{K}_0^{\text{rbh}}$ is actually equal to $\mathcal{K}^{\text{rbh}}$.

The invariant $\zeta$ has a rather concise definition that uses only basic operations written in the language of free Lie algebras. In fact, a nearly complete definition appears within Fig. 4, with lesser extras in Figs. 5 and 1. These definitions are relatively easy to implement on a computer, and as that was my original goal, the implementation along with some computational examples is described in Section 6. Further computations, more closely related to [1] and to [4, 5], will be described in [3].

In Section 7, we sketch a conceptual interpretation of $\zeta$. Namely, we sketch the statement and the proof of the following theorem:

**Theorem 2.7** *The invariant $\zeta$ is (the logarithm of) a universal finite type invariant of the objects in $\mathcal{K}_0^{rbh}$ (assuming Conjecture 3.7, of ribbon-knotted balloons and hoops in $\mathbb{R}^4$).*

While the formulae defining $\zeta$ are reasonably simple, the proof that they work using only notions from the language of free Lie algebras involves some painful computations—the more reasonable parts of the proof are embedded within Sections 4 and 5, and the less reasonable parts are postponed to Section 10.4. An added benefit of the results of Section 7 is that they constitute an alternative construction of $\zeta$ and an alternative proof of its invariance—the construction requires more words than the free Lie construction, yet the proof of invariance becomes simpler and more conceptual.

In Section 8, we discuss the relationship of $\zeta$ with the BF topological quantum field theory, and in Section 9, we explain how a certain reduction of $\zeta$ becomes a system of formulae for the (multivariable) Alexander polynomial which, in some senses, is better than any previously available formula.

Section 10 is for "odds and ends"—things worth saying, yet those that are better postponed to the end. This includes the details of some definitions and proofs, some words about our conventions, and an attempt at explaining how I think about "meta" structures.

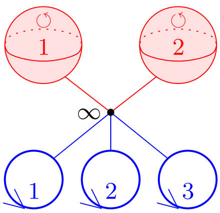
*Remark 1.3* Nothing of substance places this paper in $\mathbb{R}^4$. Everything works just as well in $\mathbb{R}^d$ for any $d \geq 4$, with "balloons" meaning ($d$-2)-dimensional spheres and "hoops" always meaning 1-dimensional circles. We have only specialized to $d = 4$ only for reasons of concreteness.

## 2 The Objects

### 2.1 Ribbon-Knotted Balloons and Hoops

This paper is about ribbon-knotted balloons ($S^2$s) and hoops (or loops, or $S^1$s) in $\mathbb{R}^4$ or, equivalently, in $S^4$. Throughout this paper, $T$ and $H$ will denote finite[1] (not necessarily disjoint) sets of "labels", where the labels in $T$ label the balloons (though for reasons that will become clear later, they are also called "tail labels" and the things they label are sometimes called "tails"), and the labels in $H$ label the hoops (though they are sometimes called "head labels" and they sometimes label "heads").

**Definition 2.1** A $(T; H)$-labelled ribbon-knotted balloons and hoops (rKBH) is a ribbon[2] up-to-isotopy embedding into $\mathbb{R}^4$ or into $S^4$ of $|T|$-oriented 2-spheres labelled by the elements of $T$ (the balloons), of $|H|$-oriented circles labelled by the elements of $H$ (the hoops), and of $|T| + |H|$ strings (namely, intervals) connecting the $|T|$ balloons and the $|H|$ hoops to some fixed base point, often denoted $\infty$. Thus a $(\underline{2}; \underline{3})$-labelled[3] rKBH, for example, is a ribbon up-to-isotopy embedding into $\mathbb{R}^4$ or into $S^4$ of the space drawn below. Let $\mathcal{K}^{\text{rbh}}(T; H)$ denote the set of all $(T; H)$-labelled rKBHs.




[1]The bulk of the paper easily generalizes to the case where $H$ (not $T$!) is infinite, though nothing is gained by allowing $H$ to be infinite.

[2]The adjective "ribbon" will be explained in Definition 2.4.

[3]See "notational conventions", Section 10.5.

Recall that 1D objects cannot be knotted in 4D space. Hence, the hoops in an rKBH are not in themselves knotted, and hence an rKBH may be viewed as a knotting of the $|T|$ balloons in it, along with a choice of $|H|$ elements of the fundamental group of the complement of the balloons. Likewise, the $|T| + |H|$ strings in an rKBH only matter as homotopy classes of paths in the complement of the balloons. In particular, they can be modified arbitrarily in the vicinity of $\infty$, and hence they don't even need to be specified near $\infty$—it is enough that we know that they emerge from a small neighbourhood of $\infty$ (small enough so as to not intersect the balloons) and that each reaches its balloon or its hoop.

Conveniently, we often pick our base point to be the point $\infty$ of the formula $S^4 = \mathbb{R}^4 \cup \{\infty\}$ and hence, we can draw rKBHs in $\mathbb{R}^4$ (meaning, of course, that we draw in $\mathbb{R}^2$ and adopt conventions on how to lift these drawings to $\mathbb{R}^4$).

We will usually reserve the labels $x$, $y$ and $z$ for hoops; the labels $u$, $v$ and $w$ for balloons and the labels $a$, $b$ and $c$ for things that could be either balloons or hoops. With almost no risk of ambiguity, we also use $x$, $y$ and $z$, along also with $t$, to denote the coordinates of $\mathbb{R}^4$. Thus, $\mathbb{R}^2_{xy}$ is the $xy$ plane within $\mathbb{R}^4$, $\mathbb{R}^3_{txy}$ is the hyperplane perpendicular to the $z$-axis and $\mathbb{R}^4_{txyz}$ is just another name for $\mathbb{R}^4$.

Examples 2.2 and 2.3 are more than just examples, for they introduce much notation that we use later on.

*Example 2.2* The first four examples of rKBHs are the "four generators" shown in Fig. 1:

- $h\epsilon_x$ is an element of $\mathcal{K}^{\text{rbh}}(;x)$ (more precisely, $\mathcal{K}^{\text{rbh}}(\emptyset;\{x\})$). It has a single hoop extending from near $\infty$ and back to near $\infty$, and as indicated above, we didn't bother to indicate how it closes near $\infty$ and how it is connected to $\infty$ with an extra piece of string. Clearly, $h\epsilon_x$ is the "unknotted hoop".
- $t\epsilon_u$ is an element of $\mathcal{K}^{\text{rbh}}(u;)$. As a picture in $\mathbb{R}^3_{xyz}$, it looks like a simplified tennis racket, consisting of a handle, a rim, and a net. To interpret a tennis racket in $\mathbb{R}^4$, we embed $\mathbb{R}^3_{xyz}$ into $\mathbb{R}^4_{txyz}$ as the hyperplane $[t = 0]$, and inside it, we place the handle and the rim as they were placed in $\mathbb{R}^3_{xyz}$. We also make two copies of the net, the "upper" copy and the "lower" copy. We place the upper copy so that its boundary is the rim and so that its interior is pushed into the $[t > 0]$ half-space (relative to the pictured $[t = 0]$ placement) by an amount proportional to the distance from the boundary. Similarly, we place the lower copy, except we push it into the $[t < 0]$ half space. Thus, the two nets along with the rim make a 2-sphere in $\mathbb{R}^4$, which is connected to $\infty$ using the handle. Clearly, $t\epsilon_u$ is the "unknotted balloon" (see below). We orient $t\epsilon_u$ by adopting the conventions that surfaces drawn in the plane are oriented counterclockwise

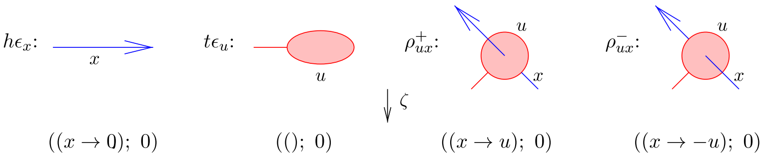
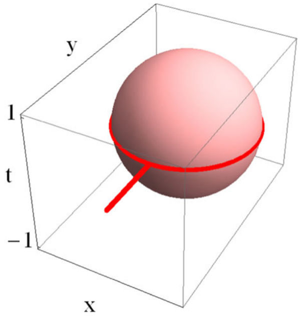



**Fig. 1** The four generators $h\epsilon_x$, $t\epsilon_u$, $\rho^+_{ux}$ and $\rho^-_{ux}$, drawn in $\mathbb{R}^3_{xyz}$ ($\rho^\pm_{ux}$ differ in the direction in which $x$ pierces $u$—from below at $\rho^+_{ux}$ and from above at $\rho^-_{ux}$). The lower part of the figure previews the values of the main invariant $\zeta$ discussed in this paper on these generators. More later, in Section 5

(unless otherwise noted) and that when pushed to 4D, the upper copy retains the original orientation while the lower copy reverses it.

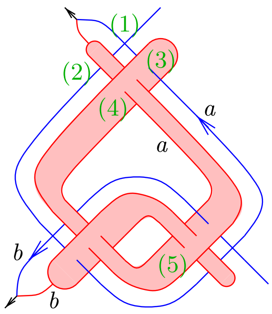



Warning: the vertical direction here is the "time" coordinate $t$, so this is an $\mathbb{R}^3_{txy}$ picture.

- $\rho^+_{ux}$ is an element of $\mathcal{K}^{\text{rbh}}(u; x)$. It is the 4D analogue of the "positive Hopf link". Its picture in Fig. 1 should be interpreted in much the same way as the previous two—what is displayed should be interpreted as a 3D picture using standard conventions (what's hidden is "below"), and then it should be placed within the $[t = 0]$ copy of $\mathbb{R}^3_{xyz}$ in $\mathbb{R}^4$. This done, the racket's net should be split into two copies, one to be pushed to $[t > 0]$ and the other to $[t < 0]$. In $\mathbb{R}^3_{xyz}$, it appears as if the hoop $x$ intersects the balloon $u$ right in the middle. Yet in $\mathbb{R}^4$, our picture represents a legitimate knot as the hoop is embedded in $[t = 0]$, the nets are pushed to $[t \neq 0]$, and the apparent intersection is eliminated.
- $\rho^-_{ux}$ is the "negative Hopf link". It is constructed out of its picture in exactly the same way as $\rho^+_{ux}$. We postpone to Section 10.1 the explanation of why $\rho^+_{ux}$ is "positive" and $\rho^-_{ux}$ is "negative".

*Example 2.3* Below is a somewhat more sophisticated example of an rKBH with two balloons labelled $a$ and $b$ and two hoops labelled with the same labels (hence it is an element of $\mathcal{K}^{\text{rbh}}(a, b; a, b)$). It should be interpreted using the same conventions as in the previous example, though some further comments are in order:

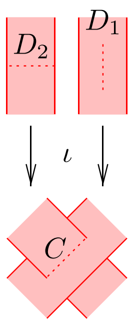
- The "crossing" marked (1) below is between two hoops and in 4D it matters not if it is an overcrossing or an undercrossing. Hence, we did not bother to indicate which of the two it is. A similar comment applies in two other places.
- Likewise, crossing (2) is between a 1D strand and a thin tube, and its sense is immaterial. For no real reason, we've drawn the strand "under" the tube, but had we drawn it "over", it would be the same rKBH. A similar comment applies in two other places.
- Crossing (3) is "real" and is similar to $\rho^-$ in the previous example. Two other crossings in the picture are similar to $\rho^+$.




- Crossing (4) was not seen before, though its 4D meaning should be clear from our interpretation rules: nets are pushed up (or down) along the $t$ coordinate by an amount proportional to the distance from the boundary. Hence, the wider net in crossing (4) gets pushed more than the narrower one, and hence, in 4D, they do not intersect even though their projections to 3D do intersect, as the figure indicates. A similar comment applies in two other places.
- Our example can be simplified a bit using isotopies. Most notably, crossing (5) can be eliminated by pulling the narrow "\" finger up and out of the wider "/" membrane. Yet note that a similar feat cannot be achieved near (3) and (4). Over there, the wider "/" finger cannot be pulled down and away from the narrower "\" membrane and strand without a singularity along the way.

We can now complete Definition 2.1 by providing the the definition of "ribbon embedding".

**Definition 2.4** We say that an embedding of a collection of 2-spheres $S_i$ into $\mathbb{R}^4$ (or into $S^4$) is a "ribbon" if it can be extended to an immersion $\iota$ of a collection of 3-balls $B_i$ whose boundaries are the $S_i$s, so that the singular set $\Sigma \subset \mathbb{R}^4$ of $\iota$ consists of transverse self-intersections, and so that each connected component $C$ of $\Sigma$ is a "ribbon singularity": $\iota^{-1}(C)$ consists of two closed disks $D_1$ and $D_2$, with $D_1$ embedded in the interior of one of the $B_i$ and with $D_2$ embedded with its interior in the interior of some $B_j$ and with its boundary in $\partial B_j = S_j$. A dimensionally reduced illustration is below. The ribbon condition does not place any restriction on the hoops of an rKBH.

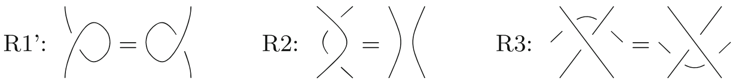



It is easy to verify that all the examples above are ribbon, and that all the operations we define below preserve the ribbon condition.

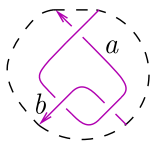
There is much literature about ribbon knots in $\mathbb{R}^4$. See, e.g. [4, 5, 11, 12, 15, 26, 27].

### 2.2 Usual Tangles and the Map $\delta$

For the purposes of this paper, a "usual tangle",[4] or a "u-tangle", is a "framed pure labelled tangle in a disk". In detail, it is a piece of an oriented knot diagram drawn in a disk, having no closed components and with its components labelled by the elements of some set $S$, with all regarded modulo the Reidemeister moves R1', R2 and R3:




The set of all tangles with components labelled by $S$ is denoted as $u\mathcal{T}(S)$. An example of a member of $u\mathcal{T}(a, b)$ is below. Note that our u-tangles do not have a specific "up" direction so they do not form a category, and that the condition "no closed components" prevents them from being a planar algebra. In fact, $u\mathcal{T}$ carries almost no interesting algebraic structure. Yet it contains knots (as 1-component tangles) and more generally, by restricting to a subset, it contains "pure tangles" or "string links" [9]. And in the next section, $u\mathcal{T}$ will be generalized to $v\mathcal{T}$ and to $w\mathcal{T}$, which do carry much interesting structure.

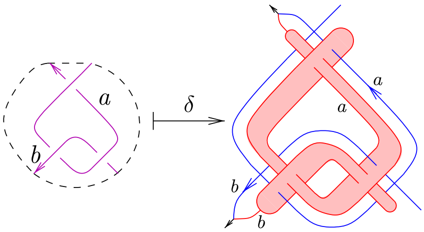



There is a map $\delta\colon u\mathcal{T}(S) \to \mathcal{K}^{\text{rbh}}(S; S)$. The picture should precede the words, and it appears as the left half of Fig. 2.

In words, if $T \in u\mathcal{T}(S)$, to make $\delta(T)$ we convert each strand $s \in S$ of $T$ into a pair of parallel entities: a copy of $s$ on the right and a band on the left ($T$ is a planar diagram and $s$ is oriented, so "left" and "right" make sense). We cap the resulting band near its beginning and near its end, connecting the cap at its end to $\infty$ (namely, to outside the picture) with an extra piece of string—so that when the bands are pushed to 4D in the usual way, they become balloons with strings. Finally, near the crossings of $T$ we apply the following (sign-preserving) local rules:


[4]Better English would be "ordinary tangle", but I want the short form to be "u-tangle", which fits better with the "v-tangles" and "w-tangles" that arise later in this paper.

**Fig. 2** A $T_0 \mapsto \delta(T_0)$ example, and its invariant $\zeta$ of Section 5 (computed to degree 3)

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T₀ = R⁻[3, a] R⁺[b, 2] R⁺[1, 4];
T₀ // dm[2, 1, 1] // dm[4, b, b] // dm[1, a, a] //
  dm[3, a, a]
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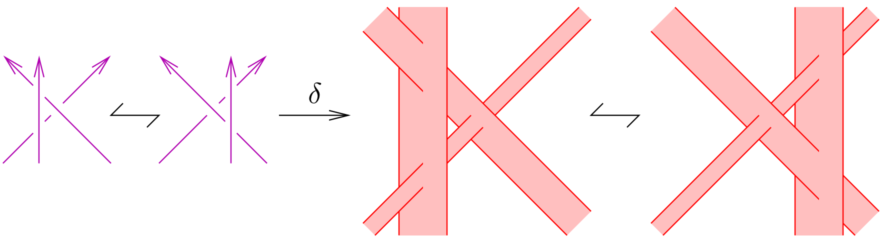
$$\mathrm{M}\Big[\Big\{a \to \mathrm{LS}\Big[-\overline{a} + \overline{b},\ \frac{3\,\overline{a b}}{2},\ \frac{13}{12}\,\overline{a\,\overline{a b}} - \frac{13}{12}\,\overline{\overline{a b}\,b}\Big],$$
$$b \to \mathrm{LS}\Big[\overline{a},\ 0,\ -\overline{a\,\overline{a b}}\Big]\Big\},\ \mathrm{CWS}\Big[-\widehat{a},\ -\widehat{ab},\ -\frac{\widehat{aab}}{2} - \frac{\widehat{abb}}{2}\Big]\Big]$$

**Proposition 2.5** *The map $\delta$ is well defined.*

*Proof* We need to check that the Reidemeister moves in $u\mathcal{T}$ are carried to isotopies in $\mathcal{K}^{\mathrm{rbh}}$. We'll only display the "band part" of the third Reidemeister move, as everything else is similar or easier:




The fact that the two "band diagrams" above are isotopic before "inflation" to $\mathbb{R}^4$, and hence also after, is visually obvious. □

## 2.3 The Fundamental Invariant and the Near-Injectivity of $\delta$

The "Fundamental invariant" $\pi(K)$ of $K \in \mathcal{K}^{\mathrm{rbh}}(u_i; x_j)$ is the triple $(\pi_1(K^c); m; l)$, where within this triple:

- The first entry is the fundamental group of the complement of the balloons of $K$, with basepoint taken to be at $\infty$.
- The second entry $m$ is the function $m\colon T \to \pi_1(K^c)$ which assigns to a balloon $u \in T$ its "base meridian" $m_u$—the path obtained by travelling along the string of $u$ from $\infty$

to near the balloon, then Hopf-linking with the balloon $u$ once in the positive direction much like in the generator $\rho^+$ of Fig. 1, and then travelling back to the basepoint again along the string of $u$.

- The third entry $l$ is the function $l\colon H \to \pi_1(K^c)$ which assigns to hoop $x \in H$ its longitude $l_x$—it is simply the hoop $x$ itself regarded as an element of $\pi_1(K^c)$.

Thus, for example, with $\langle\alpha\rangle$ denoting the group generated by a single element $\alpha$ and following the "notational conventions" of Section 10.5 for "inline functions",

$$\pi(h\epsilon_x) = (1;\ ();\ (x \to 1)), \qquad \pi(t\epsilon_u) = (\langle\alpha\rangle;\ (u \to \alpha);\ ())$$
$$\text{and} \qquad \pi(\rho^{\pm}_{ux}) = (\langle\alpha\rangle;\ (u \to \alpha);\ (x \to \alpha^{\pm 1})).$$

We leave the following proposition as an exercise for the reader:

**Proposition 2.6** *If $T$ is an $\underline{n}$-labelled u-tangle, then $\pi(\delta(T))$ is the fundamental group of the complement of $T$ (within a 3D space!), followed by the list of meridians of $T$ (placed near the outgoing ends of the components of $T$), followed by the list of longitudes of $T$.*

It is well known (e.g. [17, Theorem 6.1.7]) that knots are determined by the fundamental group of their complements, along with their "peripheral systems", namely their meridians and longitudes regarded as elements of the fundamental groups of their complements. Thus we have the following:

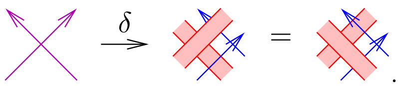
**Theorem 2.7** *When restricted to long knots (which are the same as knots), $\delta$ is injective.*

*Remark 2.8* A similar map studied by Winter [30] is (sometimes) 2 to 1, as it retains less orientation information.

I expect that $\delta$ is also injective on arbitrary tangles and that experts in geometric topology would consider this trivial, but this result would be outside of my tiny puddle.

## 2.4 The Extension to v/w-Tangles and the Near-Surjectivity of $\delta$

The map $\delta$ can be extended to "virtual crossings" [16] using the local assignment




(1)

In a few more words, u-tangles can be extended to "v-tangles" by allowing virtual crossings as on the left hand side of Eq. 1, and then modding out by the "virtual Reidemeister moves" and the "mixed move"/"detour move" of [16].[5] One may then observe, as in Fig. 3, that $\delta$ respects those moves as well as the overcrossings commute relation (yet not the undercrossings commute relation). Hence, $\delta$ descends to the space $w\mathcal{T}$ of w-tangles, which are the quotient of v-tangles by the overcrossings commute relation.

A topological-flavoured construction of $\delta$ appears in Section 10.2.

[5] In [16], the mixed/detour move was yet unnamed, and was simply "move (c) of Fig. 2".

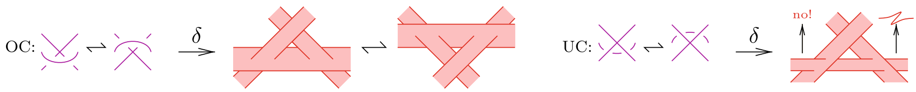



**Fig. 3** The "overcrossing commute" (OC) relation and the gist of the proof that it is respected by $\delta$, and the "undercrossing commute" (UC) relation and the gist of the reason why it is not respected by $\delta$

The newly extended $\delta\colon w\mathcal{T} \to \mathcal{K}^{\text{rbh}}$ cannot possibly be surjective, for the rKBHs in its image always have an equal number of balloons as hoops, with the same labels. Yet, if we allow the deletion of components, $\delta$ becomes surjective:

**Theorem 2.9** *For any KTG K, there is some w-tangle T so that K is obtained from $\delta(T)$ by the deletion of some of its components.*

*Proof* (Sketch) This is a variant of Theorem 3.1 of Satoh's [26]. Clearly, every knotting of 2-spheres in $\mathbb{R}^4$ can be obtained from a knotting of tubes by capping those tubes. Satoh shows that any knotting of tubes is in the image of a map he calls "tube", which is identical to our $\delta$ except that our $\delta$ also includes the capping (good) and an extra hoop component for each balloon (harmless as they can be deleted). Finally, to get the hoops of $K$, simply put them in as extra strands in $T$, and then delete the spurious balloons that $\delta$ would produce next to each hoop. □

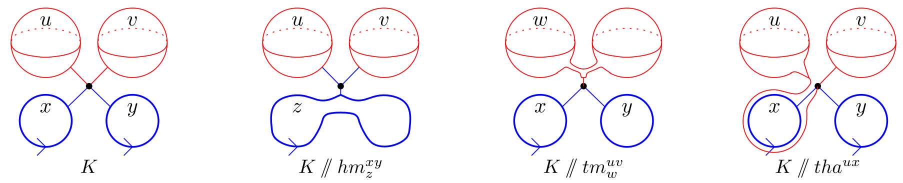
# 3 The Operations

## 3.1 The Meta-Monoid-Action

Loosely speaking, an rKBH $K$ is a map of several $S^1$s and several $S^2$s into some ambient space. The former (the hoops of $K$) resemble elements of $\pi_1$, and the latter (the balloons of $K$) resemble elements of $\pi_2$. In general, in homotopy theory, $\pi_1$ and $\pi_2$ are groups, and further, there is an action of $\pi_1$ on $\pi_2$. Thus, we find that on $\mathcal{K}^{\text{rbh}}$, there are operations that resemble the group multiplication of $\pi_1$, and the group multiplication of $\pi_2$, and the action of $\pi_1$ on $\pi_2$.

Let us describe these operations more carefully. Let $K \in \mathcal{K}^{\text{rbh}}(T; H)$.

- Analogously to the product in $\pi_1$, there is the operation of "concatenating two hoops". Specifically, if $x$ and $y$ are two distinct labels in $H$ and $z$ is a label not in $H$ (except possibly equal to $x$ or to $y$), we let[6] $K \,/\!\!/\, hm_z^{xy}$ be $K$ with the $x$ and $y$ hoops removed and replaced with a single hoop labelled $z$ that traces the path of them both. See Fig. 4.
- Analogously to the homotopy-theoretic product of $\pi_2$, there is the operation of "merging two balloons". Specifically, if $u$ and $v$ are two distinct labels in $T$ and $w$ is a label not in $T$ (except possibly equal to $u$ or to $v$), we let $K \,/\!\!/\, tm_w^{uv}$ be $K$ with the $u$ and $v$ balloons removed and replaced by a single two-lobed balloon (topologically, still a sphere!) labelled $w$ which spans them both. See Fig. 4.
- Analogously to the homotopy-theoretic action of $\pi_1$ on $\pi_2$, there is the operation $tha^{ux}$ (tail by head action on $u$ by $x$ ) of re-routing the string of the balloon $u$ to go along the hoop $x$, as illustrated in Fig. 4. In balloon-theoretic language, after the isotopy which pulls the neck of $u$ along its string, this is the operation of "tying the balloon",




$\downarrow \zeta$

$(T; H) = (u, v; x, y)$, $\mu = (\lambda; \omega)$ in $FL(T)^H \times CW^r(T)$,

$\lambda \mapsto \lambda \backslash \{x, y\} \cup (z \to \mathrm{bch}(\lambda_x, \lambda_y))$, $\omega$ untouched,

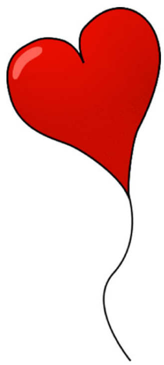
$\mu \mapsto \mu /\!/ (u, v \to w)$,

$\mu \mapsto (\lambda;\, \omega + J_u(\lambda_x)) /\!/ RC_u^{\lambda_x}$,

with $C_u^\gamma = (u \mapsto e^{\mathrm{ad}\,\gamma}(u))$, $RC_u^\gamma = (C_u^{-\gamma})^{-1}$, and $J_u(\gamma) = \int_0^1 ds\ \mathrm{div}_u(\gamma /\!/ RC_u^{s\gamma}) /\!/ C_u^{-s\gamma}$.

**Fig. 4** An rKBH $K$ and the three basic unary operators applied to it. We use schematic notation; $K$ may have plenty more components, and it may actually be knotted. The lower part of the figure is a summary of the main invariant $\zeta$ defined in this paper. See Section 5

commonly performed to prevent the leakage of air (though admittedly, this will fail in 4D).



In addition, $\mathcal{K}^{\mathrm{rbh}}$ affords the further unary operations $t\eta^u$ (when $u \in T$) of "puncturing" the balloon $u$ (implying, deleting it) and $h\eta^x$ (when $x \in H$) of "cutting" the hoop $x$ (implying, deleting it). These two operations were already used in the statement and proof of Theorem 2.9.

In addition, $\mathcal{K}^{\mathrm{rbh}}$ affords the binary operation $*$ of "connected sum", sketched in Fig. 5 (along with its $\zeta$ formulae of Section 5) Whenever we have disjoint label sets $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$, it is an operation $\mathcal{K}^{\mathrm{rbh}}(T_1; H_1) \times \mathcal{K}^{\mathrm{rbh}}(T_2; H_2) \to \mathcal{K}^{\mathrm{rbh}}(T_1 \cup T_2; H_1 \cup H_2)$. We often suppress the $*$ symbol and write $K_1K_2$ for $K_1 * K_2$. $\mathcal{K}^{\mathrm{rbh}}(T_1; H_1) \times \mathcal{K}^{\mathrm{rbh}}(T_2; H_2) \to \mathcal{K}^{\mathrm{rbh}}(T_1 \cup T_2; H_1 \cup H_2)$. We often suppress the $*$ symbol and write $K_1K_2$ for $K_1 * K_2$.

Finally, there are re-labelling operations $h\sigma^a_b$ and $t\sigma^a_b$ on $\mathcal{K}^{\mathrm{rbh}}$, which take a label $a$ (either a head or a tail) and rename it $b$ (provided $b$ is "new").

[6] See "notational conventions", Section 10.5.

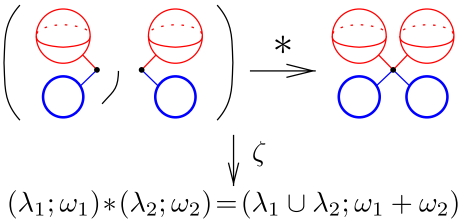



**Fig. 5** Connected sums

**Proposition 3.1** *The operations* $*$, $t\sigma^u_v$, $h\sigma^x_y$, $t\eta^u$, $h\eta^x$, $hm^{xy}_z$, $tm^{uv}_w$ *and* $tha^{ux}$ *and the special elements* $t\epsilon_u$ *and* $h\epsilon_x$ *have the following properties:*

- *If the labels involved are distinct, the unary operations all commute with each other.*
- *The re-labelling operations have some obvious properties and interactions:* $\sigma^a_b /\!\!/ \sigma^b_c = \sigma^a_c$, $hm^{xy}_x /\!\!/ h\sigma^x_z = hm^{xy}_z$, *etc., and similarly for the deletion operations* $\eta^a$.
- $*$ *is commutative and associative; where it makes sense, it bi-commutes with the unary operations* ($(K_1 /\!\!/ hm^{xy}_z) * K_2 = (K_1 * K_2) /\!\!/ hm^{xy}_z$, *etc.).*
- $t\epsilon_u$ *and* $h\epsilon_x$ *are "units":*

$$(K * t\epsilon_u) /\!\!/ tm^{uv}_w = K /\!\!/ t\sigma^v_w, \qquad (K * t\epsilon_u) /\!\!/ tm^{vu}_w = K /\!\!/ t\sigma^v_w,$$
$$(K * h\epsilon_x) /\!\!/ hm^{xy}_z = K /\!\!/ h\sigma^y_z, \qquad (K * h\epsilon_x) /\!\!/ hm^{yx}_z = K /\!\!/ h\sigma^y_z.$$

- *Meta-associativity of hm, similar to the associativity in* $\pi_1$*:*

$$hm^{xy}_x /\!\!/ hm^{xz}_x = hm^{yz}_y /\!\!/ hm^{xy}_x. \tag{2}$$

- *Meta-associativity of tm, similar to the associativity in* $\pi_2$*:*

$$tm^{uv}_u /\!\!/ tm^{uw}_u = tm^{vw}_v /\!\!/ tm^{uv}_u. \tag{3}$$

- *Meta-actions commute. The following is a special case of the first property above, yet it deserves special mention because later in this paper it will be the only such commutativity that is non-obvious to verify:*

$$tha^{ux} /\!\!/ tha^{vy} = tha^{vy} /\!\!/ tha^{ux}. \tag{4}$$

- *Meta-action axiom t, similar to* $(uv)^x = u^x v^x$*:*

$$tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w. \tag{5}$$

- *Meta-action axiom h, similar to* $u^{xy} = (u^x)^y$*:*

$$hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z. \tag{6}$$

*Proof* The first four properties say almost nothing and we did not even specify them in full.[7] The remaining four deserve attention, especially in the light of the fact that the verification of their analogues later in this paper will be non-trivial. Yet in the current context, their verification is straightforward. □

Later, we will seek to construct invariants of rKBHs by specifying their values on some generators and by specifying their behaviour under our list of operations. Thus, it is convenient to introduce a name for the algebraic structure of which $\mathcal{K}^{\mathrm{rbh}}$ is an instance:

[7]We feel that the clarity of this paper is enhanced by this omission.

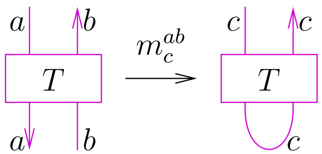
**Definition 3.2** A meta-monoid-action (MMA) $M$ is a collections of sets $M(T; H)$, one for each pair of finite sets of labels $T$ and $H$, along with partially defined operations[8] $*$, $t\sigma^u_v$, $h\sigma^x_y$, $t\eta^u$, $h\eta^x$, $hm^{xy}_z$, $tm^{uv}_w$ and $tha^{ux}$, and with special elements $t\epsilon_u \in M(\{u\}; \emptyset)$ and $h\epsilon_x \in M(\emptyset; \{x\})$, which together satisfy the properties in Proposition 3.1.

For the rationale behind the name "meta-monoid-action" see Section 10.3. In Section 10.3.5, we note that $\mathcal{K}^{\mathrm{rbh}}$ in fact has the further structure making it a meta-group-action (or more precisely, a meta-Hopf-algebra-action).

## 3.2 The Meta-Monoid of Tangles and the Homomorphism $\delta$

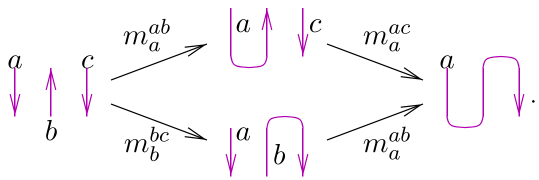
Our aim in this section is to show that the map $\delta\colon w\mathcal{T} \to \mathcal{K}^{\mathrm{rbh}}$ of Sections 2.2 and 2.4, which maps w-tangles to knotted balloons and hoops, is a "homomorphism". But first, we have to discuss the relevant algebraic structures on $w\mathcal{T}$ and on $\mathcal{K}^{\mathrm{rbh}}$.

$wT$ is a "meta-monoid" (see Section 10.3.2). Namely, for any finite set $S$ of "strand labels" $w\mathcal{T}(S)$ is a set, and whenever we have a set $S$ of labels and three labels $a \neq b$ and $c$ not in it, we have the operation $m^{ab}_c\colon w\mathcal{T}(S \cup \{a, b\}) \to w\mathcal{T}(S \cup \{c\})$ of "concatenating strand $a$ with strand $b$ and calling the resulting strand $c$". See the picture below and note that while on $u\mathcal{T}$, the operation $m^{ab}_c$ would be defined only if the head of $a$ happens to be adjacent to the tail of $b$; on $v\mathcal{T}$ and on $w\mathcal{T}$, this operation is always defined as the head of $a$ can always be brought near the tail of $b$ by adding some virtual crossings, if necessary. $w\mathcal{T}$ trivially also carries the rest of the necessary structure to form a meta-monoid—namely, strand relabelling operations $\sigma^a_b$, strand deletion operations $\eta^a$, and a disjoint union operation $*$, and units $\epsilon_a$ (tangles with a single unknotted strand labelled $a$).




It is easy to verify the associativity property (compare with (32) of Section 10.3.1):

$$m^{ab}_a \,/\!\!/\, m^{ac}_a = m^{bc}_b \,/\!\!/\, m^{ab}_a :$$




It is also easy to verify that if a tangle $T \in w\mathcal{T}(a, b)$ is non-split, then $T \neq (T \,/\!\!/\, \eta^b) * (T \,/\!\!/\, \eta^a)$, so in the sense of Section 10.3.2, $w\mathcal{T}$ is non-classical.

[8] $tm^{uv}_w$, for example, is defined on $M(T; H)$ exactly when $u, v \in T$ yet $w \notin T \setminus \{u, v\}$. All other operations behave similarly.

**Sulotion of Ridle 1.1** $\pi_T \cong \pi_1 \ltimes \pi_2$ (a semi-direct product!), so if you know all about $\pi_1$ and $\pi_2$ (and the action of $\pi_1$ and $\pi_2$), you know all about $\pi_T$.

$\mathcal{K}^{\mathrm{rbh}}$ is an analogue of both $\pi_1$ and $\pi_2$. In homotopy theory, multiplication on that part of $\mathcal{K}^{\mathrm{rbh}}$ in which the balloons and the hoops are matched together. More precisely, given a finite set of labels $S$, let $\mathcal{K}^{b=h}(S) := \mathcal{K}^{\mathrm{rbh}}(S; S)$ be the set of rKBHs whose balloons and whose hoops are both labelled with labels in $S$. Then define $dm^{ab}_c \colon \mathcal{K}^{b=h}(S \cup \{a, b\}) \to \mathcal{K}^{b=h}(S \cup \{c\})$ (the prefix $d$ is for "diagonal" or "double") by

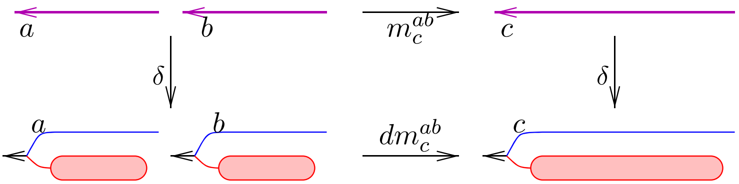
$$dm^{ab}_c = tha^{ab} \,/\!/\, tm^{ab}_c \,/\!/\, hm^{ab}_c. \tag{7}$$

It is a routine exercise to verify that the properties (2)–(6) of $hm$, $tm$ and $tha$ imply that dm is meta-associative:

$$dm^{ab}_a \,/\!/\, dm^{ac}_a = dm^{bc}_b \,/\!/\, dm^{ab}_a.$$

Thus, $dm$ (along with diagonal $\eta$'s and $\sigma$'s and an unmodified $*$) puts a meta-monoid structure on $\mathcal{K}^{b=h}$.

**Proposition 3.3** *$\delta \colon w\mathcal{T} \to \mathcal{K}^{b=h}$ is a meta-monoid homomorphism. (A rough picture is below: in the picture $a$ and $b$ are strands within the same tangle, and they may be knotted with each other and with possible further components of that tangle).*




## 3.3 Generators and Relations for $\mathcal{K}^{\mathrm{rbh}}$

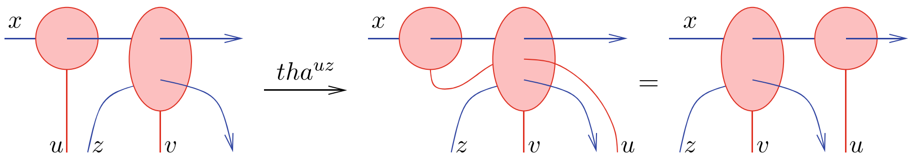
It is always good to know that a certain algebraic structure is finitely presented. If we had a complete set of generators and relations for $\mathcal{K}^{\mathrm{rbh}}$, for example, we could define a "homomorphic invariant" of rKBHs by picking some target MMA $\mathcal{M}$ (Definition 3.2), declaring the values of the invariant on the generators, and verifying that the relations are satisfied. Hence, it's good to know the following:

**Theorem 3.4** *The MMA $\mathcal{K}^{rbh}$ is generated (as an MMA) by the four rKBHs $h\epsilon_x$, $t\epsilon_u$, $\rho^+_{ux}$ and $\rho^-_{ux}$ of* Fig. 1.

*Proof* By Theorem 2.9 and the fact that the MMA operations include component deletions $t\eta^u$ and $h\eta^x$, it follows that $\mathcal{K}^{\mathrm{rbh}}$ is generated by the image of $\delta$. By the previous proposition and the fact (7) that dm can be written in terms of the MMA operations of $\mathcal{K}^{\text{rbh}}$, it follows that $\mathcal{K}^{\text{rbh}}$ is generated by the $\delta$-images of the generators of $w\mathcal{T}$. But the generators of $w\mathcal{T}$ are the virtual crossing $\underset{a\ b}{\times}$ and the right-handed and left-handed crossings $\underset{a\ b}{\nwarrow\!\!\!\!\nearrow}$ and $\underset{a\ b}{\nwarrow\!\!\!\!\nearrow}$; and so, the theorem follows from the following easily verified assertions:

$$\left(\underset{a\ b}{\times}\right) = t\epsilon_a h\epsilon_a t\epsilon_b h\epsilon_b,\ \delta\left(\underset{a\ b}{\nwarrow\!\!\!\!\nearrow}\right) = \rho^+_{ab} t\epsilon_b h\epsilon_a, \text{ and } \delta\left(\underset{a\ b}{\nwarrow\!\!\!\!\nearrow}\right) = \rho^-_{ba} t\epsilon_a h\epsilon_b. \qquad \square$$

We now turn to the study of relations. Our first is the hardest and most significant, the "Conjugation Relation", whose name is inspired by the group theoretic relation $vu^v = uv$ (here, $u^v$ denotes group conjugation, $u^v = v^{-1}uv$). Consider the following equality:




Easily, the rKBH on the very left is $\rho^+_{ux}(\rho^+_{vy}\rho^+_{wz} /\!/ tm^{vw}_v) /\!/ hm^{xy}_x$ and the one on the very right is $(\rho^+_{vx}\rho^+_{wz} /\!/ tm^{vw}_v)\rho^+_{uy} /\!/ hm^{xy}_x$, and so

$$\rho^+_{ux}\rho^+_{vy}\rho^+_{wz} /\!/ tm^{vw}_v /\!/ hm^{xy}_x /\!/ tha^{uz} = \rho^+_{vx}\rho^+_{wz}\rho^+_{uy} /\!/ tm^{vw}_v /\!/ hm^{xy}_x. \tag{8}$$

**Definition 3.2** Let $\mathcal{K}_0^{\text{rbh}}$ be the MMA freely generated by symbols $\rho^\pm_{ux} \in \mathcal{K}_0^{\text{rbh}}(u; x)$, modulo the following relations:

- Relabelling: $\rho^\pm_{ux} /\!/ h\sigma^x_y /\!/ t\sigma^u_v = \rho^\pm_{vy}$.
- Cutting and puncturing: $\rho^\pm_{ux} /\!/ h\eta^x = t\epsilon_u$ and $\rho^\pm_{ux} /\!/ t\eta^u = h\epsilon_x$.
- Inverses: $\rho^+_{ux}\rho^-_{vy} /\!/ tm^{uv}_w /\!/ hm^{xy}_z = t\epsilon_w h\epsilon_z$.
- Conjugation relations: for any $s_{1,2} \in \{\pm\}$,
$$\rho^{s_1}_{ux}\rho^{s_2}_{vy}\rho^{s_2}_{wz} /\!/ tm^{vw}_v /\!/ hm^{xy}_x /\!/ tha^{uz} = \rho^{s_2}_{vx}\rho^{s_2}_{wz}\rho^{s_1}_{uy} /\!/ tm^{vw}_v /\!/ hm^{xy}_x.$$
- Tail commutativity: on any inputs, $tm^{uv}_w = tm^{vu}_w$.
- Framing independence:
$$\rho^\pm_{ux} /\!/ tha^{ux} = \rho^\pm_{ux}. \tag{9}$$

The following proposition, whose proof we leave as an exercise, says that $\mathcal{K}_0^{\text{rbh}}$ is a pretty good approximation to $\mathcal{K}^{\text{rbh}}$:

**Proposition 3.3** *The obvious maps $\pi = \mathcal{K}_0^{rbh} \to \mathcal{K}^{rbh}$ and $\delta = w\mathcal{T} \to \mathcal{K}_0^{rbh}$ are well defined.*

**Conjecture 3.7** *The projection $\pi : \mathcal{K}_0^{rbh} \to \mathcal{K}^{rbh}$ is an isomorphism.*

We expect that there should be a Reidemeister-style combinatorial calculus of ribbon knots in $\mathbb{R}^4$. The above conjecture is that the definition of $\mathcal{K}_0^{\text{rbh}}$ is such a calculus. We expect that given any such calculus, the proof of the conjecture should be easy. In particular, the above conjecture is equivalent to the statement that the stated relations in the definition of

$w\mathcal{T}$ generate the relations in the kernel of Satoh's Tube map $\delta_0$ (see Section 10.2), and this is equivalent to the conjecture whose proof was attempted at [31]. Though I understood by private communication with B. Winter that [31] is presently flawed.

In the absence of a combinatorial description of $\mathcal{K}^{\text{rbh}}$, we replace it by $\mathcal{K}_0^{\text{rbh}}$ throughout the rest of this paper. Hence, we construct invariants of elements of $\mathcal{K}_0^{\text{rbh}}$ instead of invariants of genuine rKBHs. Yet note that the map $\delta = w\mathcal{T} \to \mathcal{K}_0^{\text{rbh}}$ is well-defined, so our invariants are always good enough to yield invariants of tangles and virtual tangles.

### 3.4 Example: The Fundamental Invariant

The fundamental invariant $\pi$ of Section 2.3 is defined in a direct manner on $\mathcal{K}^{\text{rbh}}$ and does not need to suffer from the difficulties of the previous section. Yet, it can also serve as an example for our approach for defining invariants on $\mathcal{K}_0^{\text{rbh}}$ using generators and relations.

**Definition 3.8** Let $\Pi(T; H)$ denote the set of all triples $(G; m; l)$ of a group $G$ along with functions $m \in G^T$ and $l \in G^H$, regarded modulo group isomorphisms with their obvious action on $m$ and $l$.[9] Define MMA operations $(*, t\sigma^u_v, h\sigma^x_y, t\eta^u, h\eta^x, tm^{uv}_w, hm^{xy}_z, tha^{ux})$ on $\Pi = \{\Pi(T; H)\}$ and units $t\epsilon_u$ and $h\epsilon_x$ as follows:

- $*$ is the operation of taking the free product $G_1 * G_2$ of groups and concatenating the lists of heads and tails:

$$(G_1; m_1; l_1) * (G_2; m_2; l_2) := (G_1 * G_2; m_1 \cup m_2; l_1 \cup l_2).$$

- $t\sigma^a_b$ / $h\sigma^a_b$ relabels an element labelled $a$ to be labelled $b$.
- $t\eta^u$ / $h\eta^x$ removes the element labelled $u$ / $x$.
- $tm^{uv}_w$ "combines" $u$ and $v$ to make $w$. Precisely, it replaces the input group $G$ with $G' = G/\langle m_u = m_v\rangle$, removes the tail labels $u$ and $v$, and introduces a new tail, the element $m_u = m_v$ of $G'$ and labels it $w$:

$$tm^{uv}_w(G; m; l) := (G/\langle m_u = m_v\rangle; (m\backslash\{u, v\}) \cup (w \to m_u); l).$$

- $hm^{xy}_z$ replaces two elements in $l$ by their product:

$$hm^{xy}_z(G; m; l) := (G, m, (l\backslash\{x, y\}) \cup (z \to l_x l_y).$$

- The best way to understand the action of $tha^{ux}$ is as "the thing that makes the fundamental invariant $\pi$ a homomorphism, given the geometric interpretation of $tha^{ux}$ on $\mathcal{K}^{\text{rbh}}$ in Section 3.1". In formulae, this becomes

$$tha^{ux}(G; m; l) := (G * \langle\alpha\rangle/\langle m_u = l_x \alpha l_x^{-1}\rangle; (m\backslash u) \cup (u \to \alpha), l),$$

  where $\alpha$ is some new element that is added to $G$.
- $t\epsilon_u = (\langle\alpha\rangle; (u \to \alpha); ())$ and $h\epsilon_x = (1; (); (x \to 1))$.

We state the following without its easy topological proof:

**Proposition 3.9** *$\pi : \mathcal{K}^{rbh} \to \Pi$ is a homomorphism of MMAs.*

A consequence is that $\pi$ can be computed on any rKBH starting from its values on the generators of $\mathcal{K}^{\text{rbh}}$ as listed in Section 2.3 and then using the operations of Definition 3.8.

[9] I ignore set-theoretic difficulties. If you insist, you may restrict to countable groups or to finitely presented groups.

*Comment 3.10* The fundamental groups of ribbon 2-knots are "labelled-oriented tree" (LOT) groups in the sense of Howie [13, 14]. Howie's definition has an obvious extension to labelled-oriented forests (LOF), yielding a class of groups that may be called "LOF groups". One may show that the the fundamental groups of complements of rKBHs are always LOF groups. One may also show that the subset $\Pi^{\mathrm{LOF}}$ of $\Pi$ in which the group component $G$ is an LOF group is a sub-MMA of $\Pi$. Therefore $\pi = \mathcal{K}^{\mathrm{rbh}} \to \Pi^{\mathrm{LOF}}$ is also a homomorphism of MMAs; I expect it to be an isomorphism or very close to an isomorphism. Thus, much of the rest of this paper can be read as a "theory of homomorphic (in the MMA sense) invariants of LOF groups". I don't know how much it may extend to a similar theory of homomorphic invariants of bigger classes of groups.

## 4 The Free Lie Invariant

In this section, we construct $\zeta_0$, the "tree" part to our main tree-and-wheel-valued invariant $\zeta$, by following the scheme of Section 3.3. Yet, before we succeed, it is useful to aim a bit higher and fail, and thus appreciate that even $\zeta_0$ is not entirely trivial.

### 4.1 A Free Group Failure

If the balloon part of an rKBH $K$ is unknotted, the fundamental group $\pi_1(K^c)$ of its complement is the free group generated by the meridians $(m_u)_{u\in T}$. The hoops of $K$ are then elements in that group and hence, they can be written as words $(w_x)_{x\in H}$ in the $m_u$'s and their inverses. Perhaps we can make an MMA $\mathcal{W}$ out of lists $(w_x)$ of free words in letters $m_u^{\pm 1}$ and use it to define a homomorphic invariant $W = \mathcal{K}^{\mathrm{rbh}} \to \mathcal{W}$? All we need, it seems, is to trace how MMA operations on $K$ affect the corresponding list $(w_x)$ of words.

The beginning is promising. $*$ acts on pairs of lists of words by taking the union of those lists. $hm_z^{xy}$ acts on a list of words by replacing $w_x$ and $w_y$ by their concatenation, now labelled $z$. $tm_r^{pq}$ acts on $\bar{w} = (w_x)$ by replacing every occurrence of the letter $m_p$ and every occurrence of the letter $m_q$ in $\bar{w}$ by a single new letter, $m_r$.

The problem is with $tha^{ux}$. Imitating the topology, $tha^{ux}$ should act on $\bar{w} = (w_y)$ by replacing every occurrence of $m_u$ in $\bar{w}$ with $w_x\alpha w_x^{-1}$, where $\alpha$ is a new letter, destined to replace $m_u$. But $w_x$ may also contain instances of $m_u$, so after the replacement, $m_u \mapsto \alpha^{w_x}$ is performed; it should be performed again to get rid of the $m_u$'s that appear in the "conjugator" $w_x$. But new $m_u$'s are then created, and the replacement should be carried out yet again.... The process clearly does not stop, and our attempt failed.

Yet, not all is lost. The latter and latter's replacements occur within conjugators of conjugators, deeper and deeper into the lower central series of the free groups involved. Thus, if we replace free groups by some completion thereof in which deep members of the lower central series are "small", the process becomes convergent. This is essentially what will be done in the next section.

### 4.2 A Free Lie Algebra Success

Given a set $T$, let $\mathrm{FL}(T)$ denote the graded completion of the free Lie algebra on the generators in $T$ (sometimes we will write "FL" for "$\mathrm{FL}(T)$ for some set $T$"). We define a meta-monoid-action $M_0$ as follows. For any finite set $T$ of "tail labels" and any finite set $H$ or "head labels", we let

$$M_0(T; H) := \mathrm{FL}(T)^H$$

be the set of $H$-labelled arrays of elements of $\mathrm{FL}(T)$. On $M_0 := \{M_0(T; H)\}$, we define operations as follows, starting from the trivial and culminating with the most interesting, $tha^{ax}$. All of our definitions are directly motivated by the "failure" of the previous section; in establishing the correspondence between the definitions below and the ones above, one should interpret $\lambda = (\lambda_x) \in M_0(T; H)$ as "a list of logarithms of a list of words $(w_x)$".

- $h\sigma^x_y$ is simply $\sigma^x_y$ as explained in the conventions section, Section 10.5.
- $t\sigma^u_v$ is induced by the map $\mathrm{FL}(T) \to \mathrm{FL}((T\setminus u) \cup \{v\})$ in which the generator $u$ is mapped to the generator $v$.
- $t\eta$ acts by setting one of the tail variables to 0, and $h\eta$ acts by dropping an array element. Thus, for $\lambda \in M_0(T; H)$,

$$\lambda \,/\!\!/\, t\eta^u = \lambda \,/\!\!/\, (u \mapsto 0) \qquad \text{and} \qquad \lambda \,/\!\!/\, h\eta^x = \eta\setminus x.$$

- If $\lambda_1 \in M_0(T_1; H_1)$ and $\lambda_2 \in M_0(T_2; H_2)$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$), then

$$\lambda_1 * \lambda_2 := (\lambda_1 \,/\!\!/\, \iota_1) \cup (\lambda_2 \,/\!\!/\, \iota_2)$$

  where $\iota_i$ are the natural embeddings $\iota_i \colon \mathrm{FL}(T_i) \hookrightarrow \mathrm{FL}(T_1 \cup T_2)$, for $i = 1, 2$.
- If $\lambda \in M_0(T; H)$ then

$$\lambda \,/\!\!/\, tm^{uv}_w := \lambda \,/\!\!/\, (u, v \mapsto w),$$

  where $(u, v \mapsto w)$ denotes the morphism $\mathrm{FL}(T) \to \mathrm{FL}(T\setminus\{u, v\} \cup \{w\})$ defined by mapping the generators $u$ and $v$ to the generator $w$.
- If $\lambda \in M_0(T; H)$ then

$$\lambda \,/\!\!/\, hm^{xy}_z := \lambda\setminus\{x, y\} \cup (z \to \mathrm{bch}(\lambda_x, \lambda_y)),$$

  where bch stands for the Baker-Campbell-Hausdorff formula:

$$\mathrm{bch}(a, b) := \log(e^a e^b) = a + b + \frac{1}{2}[a, b] + \dots.$$

- If $\lambda \in M_0(T; H)$ then

$$\lambda \,/\!\!/\, tha^{ux} := \lambda \,/\!\!/\, (C_u^{-\lambda_x})^{-1} = \lambda \,/\!\!/\, RC_u^{\lambda_x} \tag{10}$$

  In the above formula, $C_u^{-\lambda_x}$ denotes the automorphism of $FL(T)$ defined by mapping the generator $u$ to its "conjugate" $e^{-\lambda_x} u e^{\lambda_x}$. More precisely, $u$ is mapped to $e^{-\mathrm{ad}\lambda_x}(u)$, where ad denotes the adjoint action, and $e^{\mathrm{ad}}$ is taken in the formal sense. Thus

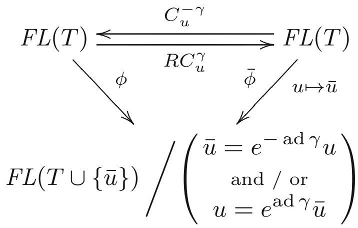
$$C_u^{-\lambda_x} \colon u \mapsto e^{-\mathrm{ad}\lambda_x}(u) = u - [\lambda_x, u] + \frac{1}{2}[\lambda_x, [\lambda_x, u]] - \dots. \tag{11}$$

  Also in (10), $RC_u^{\lambda_x} := (C_u^{-\lambda_x})^{-1}$ denotes the inverse of the automorphism $C_u^{-\lambda_x}$.
- $t\epsilon_u = ()$ and $h\epsilon_x = (x \to 0)$.

*Warning 4.1* When $\gamma \in \mathrm{FL}$, the inverse of $C_u^{-\gamma}$ may *not* be $C_u^{\gamma}$. If $\gamma$ does not contain the generator $u$, then indeed $C_u^{-\gamma} \,/\!\!/\, C_u^{\gamma} = I$. But in general, applying $C_u^{-\gamma}$ creates many new $u$s, within the $\gamma$s that appear in the right hand side of (11), and the new $u$s are then conjugated by $C_u^{\gamma}$ instead of being left in place. Yet $C_u^{-\gamma}$ is invertible, so we simply name its inverse $RC_u^{\gamma}$.

The name "*RC*" stands either for "reverse conjugation" or for "repeated conjugation". The rationale for the latter naming is that if $\alpha \in \mathrm{FL}(T)$ and $\bar{u}$ is a name for a new "temporary" free-Lie generator, then $RC_u^\gamma(\alpha)$ is the result of applying the transformation $u \mapsto e^{\mathrm{ad}\gamma}(\bar{u})$ repeatedly to $\alpha$ until it stabilizes (at any fixed degree, this will happen after a finite number of iterations), followed by the eventual renaming $\bar{u} \mapsto u$.

*Comment 4.2* Some further insight into $RC_u^\gamma$ can be obtained by studying the triangle below. The space at the bottom of the triangle is the quotient of the free Lie algebra on $T \cup \{\bar{u}\}$ (where $\bar{u}$ is a new temporary generator) by either of the two relations shown there; these two relations are, of course, equivalent. The map $\phi$ is induced from the obvious inclusion of $\mathrm{FL}(T)$ into $\mathrm{FL}(T \cup \{\bar{u}\})$, and in the presence of the relation $\bar{u} = e^{-\mathrm{ad}\gamma}u$, it is clearly an isomorphism. The map $\bar{\phi}$ is likewise induced from the renaming of $u \mapsto \bar{u}$. It, too, is an isomorphism, but slightly less trivially—indeed, using the relation $u = e^{\mathrm{ad}\gamma}\bar{u}$ *repeatedly*, any element in $\mathrm{FL}(T \cup \{\bar{u}\})$ can be written in form that does not include $u$, and hence is in the image of $\bar{\phi}$. It is clear that $C_u^{-\gamma} = \bar{\phi} /\!\!/ \phi^{-1}$. Hence, $RC_u^\gamma = \phi /\!\!/ \bar{\phi}^{-1}$, and as $\bar{\phi}^{-1}$ is described in terms of repeated applications of the relation $u = e^{\mathrm{ad}\gamma}\bar{u}$, it is clear that $RC_u^\gamma$ indeed involves repeated conjugation as asserted in the previous paragraph.




*Warning 4.3* Equation (10) does *not* say that $tha^{ux} = RC_u^{\lambda_x}$ as abstract operations, only that they are equal when evaluated on $\lambda$. In general, it is not the case that $\mu /\!\!/ tha^{ux} = \mu /\!\!/ RC_u^{\lambda_x}$ for arbitrary $\mu$—the latter equality is only guaranteed if $\mu_x = \lambda_x$.

As another example of the difference, the operations $hm_z^{xy}$ and $tha^{ux}$ do not commute—in fact, the composition $hm_z^{xy} /\!\!/ tha^{ux}$ does not even make sense, for by the time $tha^{ux}$ is evaluated, its input does not have an entry labelled $x$. Yet, the commutativity

$$\lambda /\!\!/ hm_z^{xy} /\!\!/ RC_u^{\lambda_x} = \lambda /\!\!/ RC_u^{\lambda_x} /\!\!/ hm_z^{xy} \tag{12}$$

makes perfect sense and holds true, for the operation $hm_z^{xy}$ only involves the heads/roots of trees, while $RC_u^{\lambda_x}$ only involves their tails/leafs.

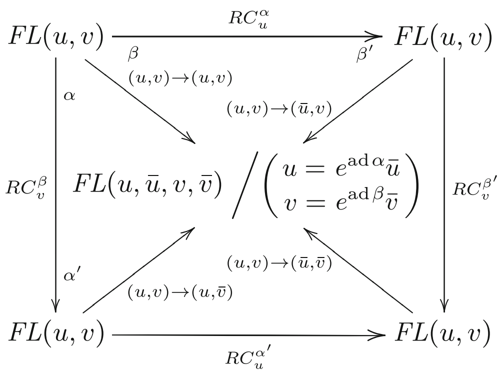
**Theorem 4.4** *$M_0$, with the operations defined above, is a meta-monoid-action (MMA).*

*Proof* Most MMA axioms are trivial to verify. The most important ones are the ones in (2) through (6). Of these, the meta-associativity of *hm* follows from the associativity of the bch formula, $\mathrm{bch}(\mathrm{bch}(\lambda_x, \lambda_y), \lambda_z) = \mathrm{bch}(\lambda_x, \mathrm{bch}(\lambda_y, \lambda_z))$, the meta-associativity of *tm* is trivial, and it remains to prove that meta-actions commute ((4); all other required commutativities are easy) and the the meta-action axiom *t* (5) and *h* (6).

*Meta-actions commute* Expanding (4) using the above definitions and denoting $\alpha := \lambda_x$, $\beta = \lambda_y$, $\alpha' := \alpha \,/\!\!/\, RC_v^\beta$, and $\beta' := \beta \,/\!\!/\, RC_u^\alpha$, we see that we need to prove the identity

$$RC_u^\alpha \,/\!\!/\, RC_v^{\beta'} = RC_v^\beta \,/\!\!/\, RC_u^{\alpha'}. \tag{13}$$

Consider the commutative diagram below. In it, $\mathrm{FL}(u, v)$ means "the (completed) free Lie algebra with generators $u$ and $v$, and some additional fixed collection of generators", and likewise, for $\mathrm{FL}(u, \bar{u}, v, \bar{v})$. The diagonal arrows are all substitution homomorphisms as indicated, and they are all isomorphisms. We put the elements $\alpha$ and $\beta$ in the upper-left space, and by comparing with the diagram in Comment 4.2, we see that the upper horizontal map is $RC_u^\alpha$ and the left vertical map is $RC_v^\beta$. Therefore, $\beta'$ is the image of $\beta$ in the top left space, and $\alpha'$ is the image of $\alpha$ in the bottom left space. Therefore, again, using the diagram in Comment 4.2, the right vertical map is $RC_v^{\beta'}$ and the lower horizontal map is $RC_u^{\alpha'}$, and (13) follows from the commutativity of the external square in the diagram below.



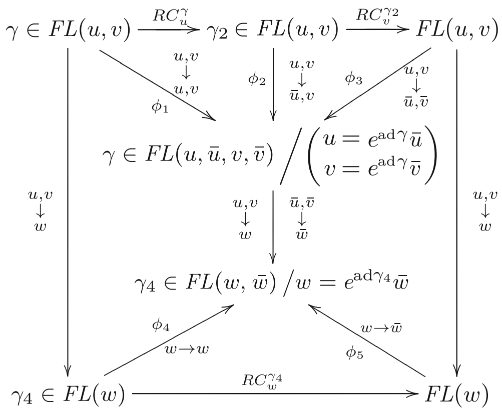

For later use, we record the fact that by reading all the horizontal and vertical arrows backwards, the above argument also proves the identity

$$C_u^{-\alpha /\!\!/ RC_v^\beta} \,/\!\!/\, C_v^{-\beta} = C_v^{-\beta /\!\!/ RC_u^\alpha} \,/\!\!/\, C_u^{-\alpha}. \tag{14}$$

*Meta-action axiom t.* Expanding (5) and denoting $\gamma := \lambda_x$, we need to prove the identity

$$tm_w^{uv} \,/\!\!/\, RC_w^{\gamma /\!\!/ t_w^{uv}} = RC_u^\gamma \,/\!\!/\, RC_v^{\gamma /\!\!/ RC_u^\gamma} \,/\!\!/\, tm_w^{uv}. \tag{15}$$

Consider the diagram below. In it, the vertical and diagonal arrows are all substitution homomorphisms as indicated. The horizontal arrows are $RC$ maps as indicated. The element $\gamma$ lives in the upper left corner of the diagram, but equally makes sense in the upper of the central spaces. We denote its image via $RC_u^\gamma$ by $\gamma_2$, and think of it as an element of the middle space in the top row. Likewise, $\gamma_4 := \gamma \,/\!\!/\, tm_w^{uv}$ lives in both the bottom left space and the bottom of the two middle spaces.




It requires a minimal effort to show that the map at the very centre of the diagram is well defined. The commutativity of the triangles in the diagram follows from Comment 4.2, and the commutativity of the trapezoids is obvious. Hence, the diagram is overall commutative. Reading it from the top left to the bottom right along the left and the bottom edges gives the left hand side of (15), and along the top and the right edges gives the right hand side.

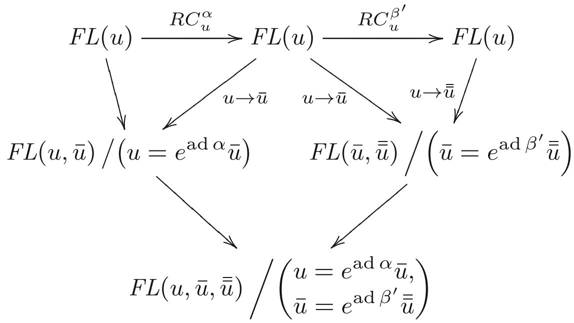
*Meta-action axiom h* Expanding (6), we need to prove

$$\lambda \,/\!\!/\, hm_z^{xy} \,/\!\!/\, RC_u^{\mathrm{bch}(\lambda_x,\lambda_y)} = \lambda \,/\!\!/\, RC_u^{\lambda_x} \,/\!\!/\, RC_u^{\lambda_y /\!\!/ RC_u^{\lambda_x}} \,/\!\!/\, hm_z^{xy}.$$

Using commutativities as in (12) and denoting $\alpha = \lambda_x$ and $\beta = \lambda_y$, we can cancel the $hm_z^{xy}$'s, and we are left with

$$RC_u^{\mathrm{bch}(\alpha,\beta)} \stackrel{?}{=} RC_u^{\alpha} \,/\!\!/\, RC_u^{\beta'}, \qquad \text{where} \qquad \beta' := \beta \,/\!\!/\, RC_u^{\alpha}. \tag{16}$$

This last equality follows from a careful inspection of the following commutative diagram:




(17)

Indeed, by the definition of $RC_u^\alpha$, we have $\beta' = \beta$ modulo and the relation $u = e^{\mathrm{ad}\alpha}\bar u$. So, in the bottom space, $u = e^{\mathrm{ad}\alpha}\bar u = e^{\mathrm{ad}\alpha}e^{\mathrm{ad}\beta'}\bar{\bar u} = e^{\mathrm{ad}\alpha}e^{\mathrm{ad}\beta}\bar{\bar u} = e^{\mathrm{bch}(\mathrm{ad}\alpha,\mathrm{ad}\beta)}\bar{\bar u} = e^{\mathrm{ad}\ \mathrm{bch}(\alpha,\beta)}\bar{\bar u}$. Hence, if we concentrate on the three corners of (17), we see the diagram below, whose top row is both $RC_u^\alpha \,/\!\!/\, RC_u^{\beta'}$ and the definition of $RC_u^{\mathrm{bch}(\alpha,\beta)}$.

$$\begin{array}{ccc} FL(u) & \dashrightarrow & FL(u) \\ & \searrow \quad \swarrow{\scriptstyle u\to\bar{\bar{u}}} & \\ & FL(u,\bar{\bar{u}}) \Big/ \left(u = e^{\operatorname{ad}\operatorname{bch}(\alpha,\beta)}\bar{\bar{u}}\right) & \end{array}$$

□

It remains to construct $\zeta_0\colon \mathcal{K}_0^{\mathrm{rbh}} \to M_0$ by proclaiming its values on the generators:

$$\zeta_0(t\epsilon_u) := (), \qquad \zeta_0(h\epsilon_x) := (x \to 0), \qquad \text{and} \qquad \zeta_0(\rho^{\pm}_{ux}) := (x \to \pm u).$$

**Proposition 4.5** *$\zeta_0$ is well defined; namely, the values above satisfy the relations in Definition 3.5.*

*Proof* We only verify the conjugation relation (8), as all other relations are easy. On the left, we have

$$\begin{aligned} \rho^+_{ux}\rho^+_{vy}\rho^+_{wz} &\xrightarrow{\zeta_0} (x\to u,\ y\to v,\ z\to w) \xrightarrow{tm^{vw}_v} (x\to u,\ y\to v,\ z\to v) \\ &\xrightarrow{hm^{xy}_x} (x\to \operatorname{bch}(u,v),\ z\to v) \xrightarrow{tha^{uz}} (x\to \operatorname{bch}(e^{\operatorname{ad} v}(u),v),\ z\to v), \end{aligned}$$

while on the right it is

$$\rho^+_{vx}\rho^+_{wz}\rho^+_{uy} \xrightarrow{\zeta_0} (x\to v,\ y\to u,\ z\to w) \xrightarrow{tm^{vw}_v /\!\!/ hm^{xy}_x} (x\to \operatorname{bch}(v,u),\ z\to v),$$

and the equality follows because $\operatorname{bch}(e^{\operatorname{ad} v}(u), v) = \log(e^v e^u e^{-v}\cdot e^v) = \operatorname{bch}(v,u)$. □

As we shall see in Section 7, $\zeta_0$ is related to the tree part of the Kontsevich integral. Thus, by finite-type folklore [2, 10], when evaluated on string links (i.e., pure tangles) $\zeta_0$ should be equivalent to the collection of all Milnor $\mu$ invariants [23]. No proof of this fact will be provided here.

## 5 The Wheel-Valued Spice and the Invariant $\zeta$

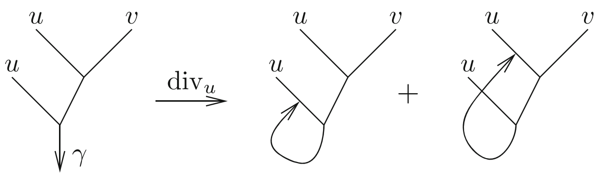
This is perhaps the most important section of this paper. In it, we construct the wheel part of the full trees-and-wheels MMA $M$ and the full tree-and-wheels invariant $\zeta\colon \mathcal{K}^{\mathrm{rbh}} \to M$.

### 5.1 Cyclic Words, $\operatorname{div}_u$, and $J_u$

The target MMA, $M$, of the extended invariant $\zeta$ is an extension of $M_0$ by "wheels", or equally well, by "cyclic words", and the main difference between $M$ and $M_0$ is the addition of a wheel-valued "spice" term $J_u(\lambda_x)$ to the meta-action $tha^{ux}$. We first need the "infinitesimal version" $\operatorname{div}_u$ of $J_u$.

Recall that if $T$ is a set (normally, of tail labels), we denote by $\mathrm{FL}(T)$ the graded completion of the free Lie algebra on the generators in $T$. Similarly, we denote by $\mathrm{FA}(T)$ the graded completion of the free associative algebra on the generators in $T$, and by $\mathrm{CW}(T)$ the graded completion of the vector space of cyclic words on $T$, namely, $\mathrm{CW}(T) := \mathrm{FA}(T)/\{uw = wu\colon u\in T, w\in \mathrm{FA}(T)\}$. Note that the last is a vector space quotient—we mod out by the vector-space span of $\{uw = wu\}$, and not by the ideal generated by that set. Hence, *CW* is not an algebra and not "commutative"; merely, the words in it are invariant under cyclic permutations of their letters. We often call the elements of $CW$ "wheels". Denote by tr the projection $\mathrm{tr}: \mathrm{FA} \to \mathrm{CW}$ and by $\iota$ the standard inclusion $\iota\colon \mathrm{FL}(T) \to \mathrm{FA}(T)$ ($\iota$ is defined to be the identity on letters in $T$, and is then extended to the rest of FL using $\iota([\lambda_1, \lambda_2]) := \iota(\lambda_1)\iota(\lambda_2) - \iota(\lambda_2)\iota(\lambda_1)$). Note that operations defined by "letter substitutions" make sense on FA and on CW. In particular, the operation $RC_u^\gamma$ of Section 4.2 makes sense on FA and on CW.

The inclusion $\iota$ can be extended from "trees" (elements of $FL$) to "wheels of trees" (elements of CW(FL)). Given a letter $u \in T$ and an element $\gamma \in \mathrm{FL(T)}$, we let $\mathrm{div}_u\gamma$ be the sum of all ways of gluing the root of $\gamma$ to near any one of the $u$-labelled leafs of $\gamma$; each such gluing is a wheel of trees, and hence can be interpreted as an element of $\mathrm{CW}(T)$. An example is below, and a formula-level definition follows: we first define $\sigma_u\colon \mathrm{FL}(T) \to \mathrm{FA}(T)$ by setting $\sigma_u(v) := \delta_{uv}$ for letters $v \in T$ and then setting $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$, and then we set $\mathrm{div}_u(\gamma) := \mathrm{tr}(u\sigma_u(\gamma))$. An alternative definition of a similar functional div is in [1, Proposition 3.20], and some further discussion is in [5, Section 3.2].




Now given $u \in T$ and $\gamma \in \mathrm{FL}(T)$ define

$$J_u(\gamma) := \int_0^1 ds\, \mathrm{div}_u\big(\gamma /\!/ RC_u^{s\gamma}\big) /\!/ C_u^{-s\gamma}. \tag{18}$$

Note that at degree $d$, the integrand in the above formula is a degree $d$ element of $\mathrm{CW}(T)$ with coefficients that are polynomials of degree at most $d-1$ in $s$. Hence the above formula is entirely algebraic. The following (difficult!) proposition contains all that we will need to know about $J_u$.

**Proposition 5.1** *If $\alpha, \beta, \gamma \in FL$ then the following three equations hold:*

$$J_u(\mathrm{bch}(\alpha, \beta)) = J_u(\alpha) + J_u(\beta /\!/ RC_u^\alpha) /\!/ C_u^{-\alpha}, \tag{19}$$

$$J_u(\alpha) - J_u(\alpha /\!/ RC_v^\beta) /\!/ C_v^{-\beta} = J_v(\beta) - J_v(\beta /\!/ RC_u^\alpha) /\!/ C_u^{-\alpha} \tag{20}$$

$$J_w(\gamma /\!/ tm_w^{uv}) = \Big(J_u(\gamma) + J_v(\gamma /\!/ RC_u^\gamma) /\!/ C_u^{-\gamma}\Big) /\!/ tm_w^{uv} \tag{21}$$

We postpone the proof of this proposition to Section 10.4.

*Remark 5.2* $J_u$ can be characterized as the unique functional $J_u\colon \mathrm{FL}(T) \to \mathrm{CW}(T)$ which satisfies (19) as well as the conditions $J_u(0) = 0$ and

$$\frac{d}{d\epsilon} J_u(\epsilon\gamma)\Big|_{\epsilon=0} = \mathrm{div}_u(\gamma), \tag{22}$$

which in themselves are easy consequences of the definition of $J_u$, (18). Indeed, taking $\alpha = s\gamma$ and $\beta = \epsilon\gamma$ in (19), where $s$ and $\epsilon$ are scalars, we find that

$$J_u((s+\epsilon)\gamma) = J_u(s\gamma) + J_u(\epsilon\gamma /\!\!/ RC_u^{s\gamma}) /\!\!/ C_u^{-s\gamma}.$$

Differentiating the above equation with respect to $\epsilon$ at $\epsilon = 0$ and using (22), we find that

$$\frac{d}{ds} J_u(s\gamma) = \operatorname{div}_u(\gamma /\!\!/ RC_u^{s\gamma}) /\!\!/ C_u^{-s\gamma},$$

and integrating from 0 to 1 we get (18).

Finally, for this section, one may easily verify that the degree 1 piece of $CW$ is preserved by the actions of $C_u^\gamma$ and $RC_u^\gamma$, and hence it is possible to reduce modulo degree 1. Namely, set $\mathrm{CW}^r(T) := \mathrm{CW}(T)/\deg 1 = \mathrm{CW}^{>1}(T)$, and all operations remain well defined and satisfy the same identities.

### 5.2 The MMA $M$

Let $M$ be the collection $\{M(T; H)\}$, where

$$M(T; H) := \mathrm{FL}(T)^H \times \mathrm{CW}^r(T) = M_0(T; H) \times \mathrm{CW}^r(T)$$

(I really mean $\times$, not $\otimes$). The collection $M$ has MMA operations as follows:

- $t\sigma_v^u$, $t\eta^u$, and $tm_w^{uv}$ are defined by the same formulae as in Section 4.2. Note that these formulae make sense on CW and on $\mathrm{CW}^r$ just as they do on FL.
- $h\sigma_y^x$, $h\eta^x$, and $hm_z^{xy}$ are extended to act as the identity on the $\mathrm{CW}^r(T)$ factor of $M(T; H)$.
- If $\mu_i = (\lambda_i; \omega_i) \in M(T_i; H_i)$ for $i = 1, 2$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$), set

  $$\mu_1 * \mu_2 := (\lambda_1 * \lambda_2;\ \iota_1(\omega_1) + \iota_2(\omega_2)),$$

  where $\iota_i$ are the obvious inclusions $\iota_i \colon \mathrm{CW}^r(T_i) \to \mathrm{CW}^r(T_1 \cup T_2)$.
- The only truly new definition is that of $tha^{ux}$:

  $$(\lambda; \omega) /\!\!/ tha^{ux} := (\lambda;\ \omega + J_u(\lambda_x)) /\!\!/ RC_u^{\lambda_x}.$$

  Thus the "new" $tha^{ux}$ is just the "old" $tha^{ux}$, with an added term of $J_u(\lambda_x)$.
- $t\epsilon_u := ((); 0)$ and $h\epsilon_x := ((x \to 0); 0)$.

**Theorem 5.3** *$M$, with the operations defined above, is a meta-monoid-action (MMA). Furthermore, if $\zeta \colon \mathcal{K}_0^{rbh} \to M$ is defined on the generators in the same way as $\zeta_0$, except extended by 0 to the $\mathrm{CW}^r$ factor, $\zeta(\rho_{ux}^{\pm}) := ((x \to \pm u);\ 0)$, then it is well-defined; namely, the values above satisfy the relations in Definition 3.5.*

*Proof* Given Theorem 4.4 and Proposition 4.5, the only non-obvious checks remaining are the "wheel parts" of the main equations defining and MMA (2)–(6) and the conjugation relation (8), and the FI relation (9). As the only interesting wheels-creation occurs with the operation *tha*, (2) and (3) are easy. As easily $J_u(v) = 0$ if $u \neq v$, no wheels are created by the *tha* action within the proof of Proposition 4.5, so that proof still holds. We are left with (4)–(6) and (8)–(9).

Let us start with the wheels part of (4). If $\mu = ((x \to \alpha, y \to \beta, \ldots); \omega) \in M$, then

$$\mu \,/\!\!/\, tha^{ux} = ((x \to \alpha \,/\!\!/\, RC_u^\alpha,\ y \to \beta \,/\!\!/\, RC_u^\alpha, \ldots); (\omega + J_u(\alpha)) \,/\!\!/\, RC_u^\alpha)$$

and hence the wheels-only part of $\mu \,/\!\!/\, tha^{ux} \,/\!\!/\, tha^{vy}$ is

$$\begin{aligned} &\omega \,/\!\!/\, RC_u^\alpha \,/\!\!/\, RC_v^{\beta /\!\!/ RC_u^\alpha} + J_u(\alpha) \,/\!\!/\, RC_u^\alpha \,/\!\!/\, RC_v^{\beta /\!\!/ RC_u^\alpha} + J_v(\beta \,/\!\!/\, RC_u^\alpha) \,/\!\!/\, RC_v^{\beta /\!\!/ RC_u^\alpha} \\ &= \left[\omega + J_u(\alpha) + J_v(\beta \,/\!\!/\, RC_u^\alpha) \,/\!\!/\, C_u^{-\alpha}\right] \,/\!\!/\, RC_u^\alpha \,/\!\!/\, RC_v^{\beta /\!\!/ RC_u^\alpha}. \end{aligned}$$

In a similar manner, the wheels-only part of $\mu \,/\!\!/\, tha^{vy} \,/\!\!/\, tha^{ux}$ is

$$\left[\omega + J_v(\beta) + J_u(\alpha \,/\!\!/\, RC_v^\beta) \,/\!\!/\, C_v^{-\beta}\right] \,/\!\!/\, RC_v^\beta \,/\!\!/\, RC_u^{\beta /\!\!/ RC_v^\beta}.$$

Using (13), the operators outside the square brackets in the above two formulae are the same, and so we only need to verify that

$$\omega + J_u(\alpha) + J_v(\beta \,/\!\!/\, RC_u^\alpha) \,/\!\!/\, C_u^{-\alpha} = \omega + J_v(\beta) + J_u(\alpha \,/\!\!/\, RC_v^\beta) \,/\!\!/\, C_v^{-\beta}.$$

But this is (20). In a similar manner, the wheels parts of (5) and (6) reduce to (21) and (19), respectively. One may also verify that no wheels appear within (8), and that wheels appear in (9) only in degree 1, which is eliminated in $CW^r$. □

Thus, we have a tree-and-wheel valued invariant $\zeta$ defined on $\mathcal{K}_0^{\mathrm{rbh}}$, and thus $\delta \,/\!\!/\, \zeta$ is a tree-and-wheel valued invariant of tangles and w-tangles.

As we shall see in Section 7, the wheels part $\omega$ of $\zeta$ is related to the wheels part of the Kontsevitch integral. Thus by finite-type folklore (e.g., [19]), the Abelianization of $\omega$ (obtained by declaring all the letters in $CW(T)$ to be commuting) should be closely related to the multi-variable Alexander polynomial. More on that in Section 9. I don't know what the bigger (non-commutative) part of $\omega$ measures.

## 6 Some Computational Examples

Part of the reason I am happy about the invariant $\zeta$ is that it is relatively easily computable. Cyclic words are easy to implement, and using the Lyndon basis (e.g. [24, Chapter 5]), free Lie algebras are easy too. Hence, I include here a demo-run of a rough implementation, written in *Mathematica*. The full source files are available at [web/].

### 6.1 The Program

First, we load the package `FreeLie.m`, which contains a collection of programs to manipulate series in completed free Lie algebras and series of cyclic words. We tell `FreeLie.m` to show series by default only up to degree 3, and that if two (infinite) series are compared, they are to be compared by default only up to degree 5:

```
<< FreeLie.m
$SeriesShowDegree = 3; $SeriesCompareDegree = 5;
```

Merely as a test of `FreeLie.m`, we tell it to set `t1` to be bch$(u, v)$. The computer's response is to print that series to degree 3:

```
t1 = BCH[⟨u⟩, ⟨v⟩]
```

$$\mathrm{LS}\left[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v}\right]$$

Note that by default, Lie series are printed in "top bracket form", which means that brackets are printed above their arguments, rather than around them. Hence $\overline{u\overline{uv}}$ means $[u, [u, v]]$. This practise is especially advantageous when it is used on highly nested expressions, when it becomes difficult for the eye to match left brackets with the their corresponding right brackets.

Note also that that `FreeLie.m` utilizes *lazy evaluation*, meaning that when a Lie series (or a series of cyclic words) is defined, its definition is stored but no computations take place until it is printed or until its value (at a certain degree) is explicitly requested. Hence, `t1` is a reference to the entire Lie series bch$(u, v)$, and not merely to the degrees 1–3 parts of that series, which are printed above. Hence, when we request the value of `t1` to degree 6, the computer complies:

```
t1@{6}
```

$$\mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v},\ \frac{1}{24}\overline{u\overline{\overline{uv}v}},\ -\frac{1}{720}\overline{u\overline{u\overline{u\overline{uv}}}}+\frac{1}{180}\overline{u\overline{u\overline{\overline{uv}v}}}+\frac{1}{180}\overline{u\overline{\overline{\overline{uv}v}v}}+\frac{1}{120}\overline{\overline{uv}\,\overline{\overline{uv}v}}+\frac{1}{360}\overline{\overline{u\overline{uv}}\,\overline{uv}}-\frac{1}{720}\overline{\overline{\overline{\overline{uv}v}v}v},$$
$$-\frac{\overline{u\overline{u\overline{u\overline{\overline{uv}v}}}}}{1440}+\frac{1}{360}\overline{u\overline{u\overline{\overline{\overline{uv}v}v}}}+\frac{1}{240}\overline{u\overline{\overline{uv}\,\overline{\overline{uv}v}}}+\frac{1}{720}\overline{u\overline{\overline{u\overline{uv}}\,\overline{uv}}}-\frac{\overline{u\overline{\overline{\overline{\overline{uv}v}v}v}}}{1440}\Big]$$

(It is surprisingly easy to compute bch to a high degree and some amusing patterns emerge. See [web/mo] and [web/bch].)

The package `FreeLie.m` know about various free Lie algebra operations, but not about our specific circumstances. Hence, we have to make some further definitions. The first few are set-theoretic in nature. We define the "domain" of a function stored as a list of *key* → *value* pairs to be the set of "first elements" of these pairs; meaning, the set of keys. We define what it means to remove a key (and its corresponding value), and likewise for a list of keys. We define what it means for two functions to be equal (their domains must be equal, and for every key #, we are to have $\# /\!\!/ f_1 = \# /\!\!/ f_2$). We also define how to apply a Lie morphism `mor` to a function (apply it to each value), and how to compare $(\lambda, \omega)$ pairs (in $\mathrm{FL}(T)^H \times \mathrm{CW}^r(T)$):

```
Domain[f_List] := First /@ f;
f_ \ key_ := DeleteCases[f, key → _];
f_ \ keys_List := Fold[#1 \ #2 &, f, keys];
f1_List ≡ f2_List := Domain[f1] === Domain[f2] && (And @@ (
      ((# /. f1) ≡ (# /. f2)) & /@ Domain[f1]));
LieMorphism[mor_][f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M[λ1_, ω1_] ≡ M[λ2_, ω2_] := (λ1 ≡ λ2) && (ω1 ≡ ω2);
```

Next, we enter some free-Lie definitions that are not a part of `FreeLie.m`. Namely, we define $RC^{\gamma}_{u,\bar{u}}(s)$ to be the result of "stable application" of the morphism $u \to e^{\mathrm{ad}(\gamma)}(\bar{u})$ to $s$ (namely, apply the morphism repeatedly until things stop changing; at any fixed degree this happens after a finite number of iterations). We define $RC^{\gamma}_{u}$ to be $RC^{\gamma}_{u,\bar{u}} /\!\!/ (\bar{u} \to u)$. Finally, we define $J$ as in (18):

```
RCu_[γ_LieSeries, ub_][s_] := StableApply[LieMorphism[⟨u⟩ → Ad[γ][⟨ub⟩]], s];
RCu_[γ_LieSeries][s_] := s // RCu[γ, ⟨υ⟩] // LieMorphism[⟨υ⟩ → ⟨u⟩];
Ju_[γ_] :=
  Module[{s}, ∫_0^1 (γ // RCu[s γ] // divu // LieMorphism[u → Ad[-s γ][u]]) ds];
```

Mostly, to introduce our notation for cyclic words, let us compute $J_v(\mathrm{bch}(u, v))$ to degree 4. Note that when a series of wheels is printed out here, its degree 1 piece is greyed out to honour the fact that it "does not count" within $\zeta$:

```
Jv[t1]@{4}
```

$$\mathrm{CWS}\left[\overline{v},\ \overline{uv},\ \frac{\overline{uuv}}{2} - \frac{\overline{uvv}}{2},\ \frac{\overline{uuuv}}{6} + \frac{3\,\overline{uuvv}}{4} - \frac{3\,\overline{uvuv}}{2} + \frac{\overline{uvvv}}{6}\right]$$

Next is a series of definitions that implement the definitions of $*$, *tm*, *hm*, and *tha* following Sections 4.2 and 5.2:

```
M /: M[λ1_, ω1_] * M[λ2_, ω2_] := M[λ1 ⋃ λ2, ω1 + ω2];
tm[u_, v_, w_][λ_List] := λ // LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
tm[u_, v_, w_][M[λ_, ω_]] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_][λ_List] := Union[λ \ {x, y}, {z → BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_][M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
tha[u_, x_][λ_List] := MapAt[RCu[x /. λ], λ, {All, 2}];
tha[u_, x_][M[λ_, ω_]] :=
  M[λ // tha[u, x], (ω + Ju[x /. λ]) // RCu[x /. λ]];
```

Next, we set the values of $\zeta(t\epsilon_x)$ and $\zeta(\rho^{\pm}_{ux})$, which we simply denote $t\epsilon_x$ and $\rho^{\pm}_{ux}$:

```
hϵ[x_] := M[{x → MakeLieSeries[0]}, MakeCWSeries[0]]
ρ⁺[u_, x_] := M[{x → MakeLieSeries[⟨u⟩]}, MakeCWSeries[0]];
ρ⁻[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩]}, MakeCWSeries[0]];
```

The final bit of definitions have to do with 3D tangles. We set $R^+$ to be the value of $\zeta(\delta(\nearrow\!\!\!\!\!\nwarrow))$ as in the proof of Theorem 3.4, likewise for $R^-$, and we define `dm` by following (7):

```
R⁺[a_, b_] := ρ⁺[a, b] ** hϵ[a]; R⁻[a_, b_] := ρ⁻[a, b] ** hϵ[a];
dm[a_, b_, c_][μ_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];
```

## 6.2 Testing Properties and Relations

It is always good to test both the program and the math by verifying that the operations we have implemented satisfy the relations predicted by the mathematics. As a first example, we verify the meta-associativity of tm. Hence, in line 1 below, we set `t1` to be the element $t_1 = ((x \to u + v + w, y \to [u, v] + [v, w]); uvw)$ of $M(u, v, w; x, y)$. In line 2, we compute $t_1 \mathbin{/\!\!/} tm^{uv}_u$, in line 3 we compute $t_2 := t_1 \mathbin{/\!\!/} tm^{uv}_u \mathbin{/\!\!/} tm^{uw}_u$ and store its value in `t2`; in line 4, we compute $t_1 \mathbin{/\!\!/} tm^{vw}_v$, in line 5 we compute $t_3 := t_1 \mathbin{/\!\!/} tm^{vw}_v \mathbin{/\!\!/} tm^{uv}_u$ and store its value in `t3`, and then in line 6, we test if $t_2$ is equal to $t_3$. The computer thinks the answer is “`True`”, at least to the degree tested:

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩};
   1 → (t1 = M[{
         x → MakeLieSeries[u + v + w], y → MakeLieSeries[b[u, v] + b[v, w]]
       }, MakeCWSeries[CW["uvw"]]]),
   2 → (t1 // tm[u, v, u]),
   3 → (t2 = t1 // tm[u, v, u] // tm[u, w, u]),
   4 → (t1 // tm[v, w, v]),
   5 → (t3 = t1 // tm[v, w, v] // tm[u, v, u]),
   6 → (t2 ≡ t3)};
```

```
1 → M[{x → LS[u̅ + v̅ + w̅, 0, 0], y → LS[0, uv + vw, 0]}, CWS[0, 0, uvw]]
2 → M[{x → LS[2 u̅ + w̅, 0, 0], y → LS[0, uw, 0]}, CWS[0, 0, uuw]]
3 → M[{x → LS[3 u̅, 0, 0], y → LS[0, 0, 0]}, CWS[0, 0, uuu]]
4 → M[{x → LS[u̅ + 2 v̅, 0, 0], y → LS[0, uv, 0]}, CWS[0, 0, uvv]]
5 → M[{x → LS[3 u̅, 0, 0], y → LS[0, 0, 0]}, CWS[0, 0, uuu]]
6 → True
```

The corresponding test for the meta-associativity of *hm* is a bit harder, yet produces the same result. Note that we have declared `$SeriesCompareDegree` to be higher than `$SeriesShowDegree`, so the “`True`” output below means a bit more than the visual comparison of lines 3 and 5:

```
Print /@ {
   1 → (t1 = ρ⁺[u, x] ρ⁺[v, y] ρ⁺[w, z]),
   2 → (t1 // hm[x, y, x]),
   3 → (t2 = t1 // hm[x, y, x] // hm[x, z, x]),
   4 → (t1 // hm[y, z, y]),
   5 → (t3 = t1 // hm[y, z, y] // hm[x, y, x]),
   6 → (t2 ≡ t3)};
```

$1 \to \mathrm{M}[\{x \to \mathrm{LS}[\overline{u}, 0, 0], y \to \mathrm{LS}[\overline{v}, 0, 0], z \to \mathrm{LS}[\overline{w}, 0, 0]\}, \mathrm{CWS}[0, 0, 0]]$

$2 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u} + \overline{v}, \frac{\overline{uv}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{\overline{uv}v}\right], z \to \mathrm{LS}[\overline{w}, 0, 0]\right\}, \mathrm{CWS}[0, 0, 0]\right]$

$3 \to$
$\mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u} + \overline{v} + \overline{w}, \frac{\overline{uv}}{2} + \frac{\overline{uw}}{2} + \frac{\overline{vw}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{u\overline{uw}} + \frac{1}{3}\overline{u\overline{vw}} + \frac{1}{12}\overline{v\overline{vw}} + \frac{1}{12}\overline{\overline{uv}v} + \frac{1}{6}\overline{\overline{uw}v} + \frac{1}{12}\overline{\overline{uw}w} + \frac{1}{12}\overline{\overline{vw}w}\right]\right\}, \mathrm{CWS}[0, 0, 0]\right]$

$4 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}[\overline{u}, 0, 0], y \to \mathrm{LS}\left[\overline{v} + \overline{w}, \frac{\overline{vw}}{2}, \frac{1}{12}\overline{v\overline{vw}} + \frac{1}{12}\overline{\overline{vw}w}\right]\right\}, \mathrm{CWS}[0, 0, 0]\right]$

$5 \to$
$\mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u} + \overline{v} + \overline{w}, \frac{\overline{uv}}{2} + \frac{\overline{uw}}{2} + \frac{\overline{vw}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{u\overline{uw}} + \frac{1}{3}\overline{u\overline{vw}} + \frac{1}{12}\overline{v\overline{vw}} + \frac{1}{12}\overline{\overline{uv}v} + \frac{1}{6}\overline{\overline{uw}v} + \frac{1}{12}\overline{\overline{uw}w} + \frac{1}{12}\overline{\overline{vw}w}\right]\right\}, \mathrm{CWS}[0, 0, 0]\right]$

$6 \to \mathrm{True}$

We next test the meta-action axiom $t$ on $((x \to u + [u, t], y \to u + [u, t]); uu + tuv)$ 724
and the meta-action axiom $h$ on $((x \to u + [u, v], y \to v + [u, v]); uu + uvv)$:

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩, t = ⟨"t"⟩};
   1 → (t1 = M[{
        x → MakeLieSeries[u + b[u, t]], y → MakeLieSeries[u + b[u, t]]
       }, MakeCWSeries[CW["uu"] + CW["tuv"]]]),
   2 → (t2 = t1 // tm[u, v, w] // tha[w, x]),
   3 → (t3 = t1 // tha[u, x] // tha[v, x] // tm[u, v, w]),
   4 → (t2 ≡ t3)};
```

$1 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u}, -\overline{tu}, 0\right], y \to \mathrm{LS}\left[\overline{u}, -\overline{tu}, 0\right]\right\}, \mathrm{CWS}\left[0, \overline{uu}, \overline{tuv}\right]\right]$

$2 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{w}, -\overline{tw}, -\overline{t\overline{ww}}\right], y \to \mathrm{LS}\left[\overline{w}, -\overline{tw}, -\overline{t\overline{ww}}\right]\right\}, \mathrm{CWS}\left[\overline{w}, -\overline{tw} + \overline{ww}, \frac{3\,\overline{tww}}{2}\right]\right]$

$3 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{w}, -\overline{tw}, -\overline{t\overline{ww}}\right], y \to \mathrm{LS}\left[\overline{w}, -\overline{tw}, -\overline{t\overline{ww}}\right]\right\}, \mathrm{CWS}\left[\overline{w}, -\overline{tw} + \overline{ww}, \frac{3\,\overline{tww}}{2}\right]\right]$

$4 \to \mathrm{True}$

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩};
   1 → (t1 = M[{
        x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + b[u, v]]
       }, MakeCWSeries[CW["uu"] + CW["uvv"]]]),
   2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
   3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
   4 → (t2 ≡ t3)};
```

$1 \to \mathrm{M}[\{x \to \mathrm{LS}[\overline{u}, \overline{uv}, 0], y \to \mathrm{LS}[\overline{v}, \overline{uv}, 0]\}, \mathrm{CWS}[0, \overline{uu}, \overline{uvv}]]$

$2 \to \mathrm{M}\left[\left\{z \to \mathrm{LS}\left[\overline{u} + \overline{v}, \frac{3\,\overline{uv}}{2}, -\frac{17}{12}\overline{u\overline{uv}} - \frac{17}{12}\overline{\overline{uv}v}\right]\right\}, \mathrm{CWS}\left[\overline{u}, \overline{uu} - 2\,\overline{uv}, \frac{\overline{uuv}}{2} + \frac{\overline{uvv}}{2}\right]\right]$

$3 \to \mathrm{M}\left[\left\{z \to \mathrm{LS}\left[\overline{u} + \overline{v}, \frac{3\,\overline{uv}}{2}, -\frac{17}{12}\overline{u\overline{uv}} - \frac{17}{12}\overline{\overline{uv}v}\right]\right\}, \mathrm{CWS}\left[\overline{u}, \overline{uu} - 2\,\overline{uv}, \frac{\overline{uuv}}{2} + \frac{\overline{uvv}}{2}\right]\right]$

$4 \to \mathrm{True}$

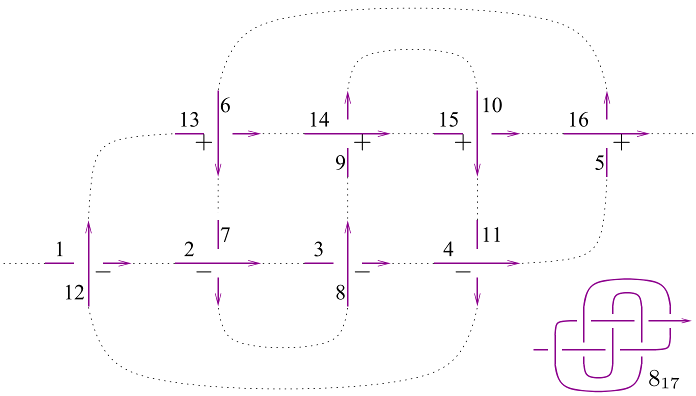
And finally for this testing section, we test the conjugation relation of (8):

```
Print /@ {
   1 → (t1 = ρ⁺[u, x] ρ⁺[v, y] ρ⁺[w, z]),
   2 → (t2 = t1 // tm[v, w, v] // hm[x, y, x] // tha[u, z]),
   3 → (t3 = ρ⁺[v, x] ρ⁺[w, z] ρ⁺[u, y]),
   4 → (t4 = t3 // tm[v, w, v] // hm[x, y, x]),
   5 → (t2 ≡ t4)};
```

$$1 \to \mathrm{M}[\{x \to \mathrm{LS}[\overline{u}, 0, 0], y \to \mathrm{LS}[\overline{v}, 0, 0], z \to \mathrm{LS}[\overline{w}, 0, 0]\}, \mathrm{CWS}[0, 0, 0]]$$
$$2 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u} + \overline{v}, -\frac{\overline{uv}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{\overline{uv}v}\right], z \to \mathrm{LS}[\overline{v}, 0, 0]\right\}, \mathrm{CWS}[0, 0, 0]\right]$$
$$3 \to \mathrm{M}[\{x \to \mathrm{LS}[\overline{v}, 0, 0], y \to \mathrm{LS}[\overline{u}, 0, 0], z \to \mathrm{LS}[\overline{w}, 0, 0]\}, \mathrm{CWS}[0, 0, 0]]$$
$$4 \to \mathrm{M}\left[\left\{x \to \mathrm{LS}\left[\overline{u} + \overline{v}, -\frac{\overline{uv}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{\overline{uv}v}\right], z \to \mathrm{LS}[\overline{v}, 0, 0]\right\}, \mathrm{CWS}[0, 0, 0]\right]$$
$$5 \to \mathrm{True}$$

## 6.3 Demo Run 1 — the Knot $8_{17}$

We are ready for a more substantial computation—the invariant of the knot $8_{17}$. We draw $8_{17}$ in the plane, with all but the neighbourhoods of the crossings dashed-out. We thus get a tangle $T_1$ which is the disjoint union of eight individual crossings (four positive and four negative). We number the 16 strands that appear in these eight crossings in the order of their eventual appearance within $8_{17}$, as seen below.




The 8-crossing tangle $T_1$ we just got has a rather boring $\zeta$ invariant, a disjoint merge of 8 $\rho^{\pm}$'s. We store it in `μ1`. Note that we used numerals as labels, and hence, in the expression below, top-bracketed numerals should be interpreted as symbols and not as integers. Note also that the program automatically converts two-digit numerical labels into alphabetical symbols, when these appear within Lie elements. Hence, in the output below, "`a`" is "10", "`c`" is "12", "`e`" is "14", and "`g`" is "16":

```
μ1 = R⁻[12, 1] R⁻[2, 7] R⁻[8, 3] R⁻[4, 11] R⁺[16, 5] R⁺[6, 13] R⁺[14, 9] R⁺[10, 15]
```

$$\mathrm{M}\big[\{1\to\mathrm{LS}[-\overline{c},0,0],\ 2\to\mathrm{LS}[0,0,0],\ 3\to\mathrm{LS}[-\overline{8},0,0],\ 4\to\mathrm{LS}[0,0,0],\ 5\to\mathrm{LS}[\overline{g},0,0],\ 6\to\mathrm{LS}[0,0,0],\ 7\to\mathrm{LS}[-\overline{2},0,0],\ 8\to\mathrm{LS}[0,0,0],\ 9\to\mathrm{LS}[\overline{e},0,0],\ 10\to\mathrm{LS}[0,0,0],\ 11\to\mathrm{LS}[-\overline{4},0,0],\ 12\to\mathrm{LS}[0,0,0],\ 13\to\mathrm{LS}[\overline{6},0,0],\ 14\to\mathrm{LS}[0,0,0],\ 15\to\mathrm{LS}[\overline{a},0,0],\ 16\to\mathrm{LS}[0,0,0]\},\ \mathrm{CWS}[0,0,0]\big]$$

Next is the key part of the computation. We "sew" together the strands of $T_1$ in order by first sewing 1 and 2 and naming the result 1, then sewing 1 and 3 and naming the result 1 once more, and so on until everything is sewn together to a single strand named 1. This is done by applying $dm_1^{1k}$ repeatedly to $\mu 1$, for $k = 2, \ldots, 16$, each time storing the result back again in $\mu 1$. Finally, we only wish to print the wheels part of the output, and this we do to degree 6:

```
Do[μ1 = μ1 // dm[1, k, 1], {k, 2, 16}];
Last[μ1]@{6}
```

$$\mathrm{CWS}\left[0,\ -\overline{11},\ 0,\ -\frac{31\ \overline{1111}}{12},\ 0,\ -\frac{1351\ \overline{111111}}{360}\right]$$

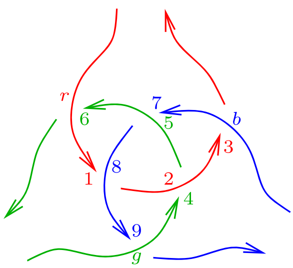
Let $A(X)$ be the Alexander polynomial of $8_{17}$. Namely, $A(X) = -X^{-3} + 4X^{-2} - 8X^{-1} + 11 - 8X + 4X^2 - X^3$. For comparison with the above computation, we print the series expansion of $\log A(e^x)$, also to degree 6:

```
Series[Log[-1/x^3 + 4/x^2 - 8/x + 11 - 8 x + 4 x^2 - x^3 /. x → e^x], {x, 0, 6}]
```

$$-x^2 - \frac{31\,x^4}{12} - \frac{1351\,x^6}{360} + O[x]^7$$

## 6.4 Demo Run 2—the Borromean Tangle

In a similar manner, we compute the invariant of the *rgb*-coloured Borromean tangle, shown below.



We label the edges near the crossings as shown, using the labels $\{r, 1, 2, 3\}$ for the $r$ component, $\{g, 4, 5, 6\}$ for the $g$ component, and $\{b, 7, 8, 9\}$ for the $b$ component. We let $\mu 2$ store the invariant of the disjoint union of six independent crossings labelled as in the Borromean tangle, we concatenate the numerically labelled strands into their corresponding letter-labelled strands, and we then print $\mu 2$, which now contains the invariant we seek:

```
μ2 = R⁻[r, 6] R⁺[2, 4] R⁻[g, 9] R⁺[5, 7] R⁻[b, 3] R⁺[8, 1];
(Do[μ2 = μ2 // dm[r, k, r], {k, 1, 3}]; Do[μ2 = μ2 // dm[g, k, g], {k, 4, 6}];
 Do[μ2 = μ2 // dm[b, k, b], {k, 7, 9}]; μ2)
```

$$\mathrm{M}\Big[\Big\{b \to \mathrm{LS}\Big[0,\ \overline{gr},\ \tfrac{1}{2}\overline{g\overline{gr}} + \overline{\overline{br}g} + \tfrac{1}{2}\overline{\overline{gr}r}\Big],$$
$$g \to \mathrm{LS}\Big[0,\ -\overline{br},\ \tfrac{1}{2}\overline{b\overline{br}} - \overline{b\overline{gr}} - \overline{\overline{br}g} + \tfrac{1}{2}\overline{\overline{br}r}\Big],$$
$$r \to \mathrm{LS}\Big[0,\ \overline{bg},\ \tfrac{1}{2}\overline{b\overline{bg}} + \overline{b\overline{gr}} + \tfrac{1}{2}\overline{\overline{bg}g}\Big]\Big\},\ \mathrm{CWS}\big[0,\ 0,\ 2\,\widehat{bgr}\big]\Big]$$

We then print the $r$-head part of the tree part of the invariant to degree 5 (the $g$-head and $b$-head parts can be computed in a similar way, or deduced from the cyclic symmetry of $r$, $g$, and $b$), and the wheels part to the same degree:

```
(r /. First[μ2])@{5}
```

$$\mathrm{LS}\Big[0,\ \overline{bg},\ \tfrac{1}{2}\overline{b\overline{bg}} + \overline{b\overline{gr}} + \tfrac{1}{2}\overline{\overline{bg}g},$$
$$\tfrac{1}{6}\overline{b\overline{b\overline{bg}}} + \tfrac{1}{2}\overline{b\overline{b\overline{gr}}} + \tfrac{1}{2}\overline{b\overline{g\overline{gr}}} + \tfrac{1}{4}\overline{b\overline{\overline{bg}g}} + \tfrac{1}{2}\overline{b\overline{\overline{gr}r}} + \tfrac{1}{6}\overline{\overline{\overline{bg}g}g},$$
$$\tfrac{1}{24}\overline{b\overline{b\overline{b\overline{bg}}}} + \tfrac{1}{6}\overline{b\overline{b\overline{b\overline{gr}}}} + \tfrac{1}{4}\overline{b\overline{b\overline{g\overline{gr}}}} + \tfrac{1}{12}\overline{b\overline{b\overline{\overline{bg}g}}} + \tfrac{1}{4}\overline{b\overline{b\overline{\overline{gr}r}}} +$$
$$\tfrac{1}{6}\overline{b\overline{g\overline{g\overline{gr}}}} + \tfrac{1}{4}\overline{b\overline{g\overline{\overline{gr}r}}} - \overline{b\overline{\overline{b\overline{gr}}g}} + \tfrac{1}{12}\overline{b\overline{\overline{\overline{bg}g}g}} - 2\overline{b\overline{\overline{\overline{br}g}g}} + \tfrac{1}{6}\overline{b\overline{\overline{\overline{gr}r}r}} +$$
$$\tfrac{1}{2}\overline{\overline{bg}\,\overline{b\overline{gr}}} - \overline{\overline{bg}\,\overline{\overline{br}g}} - \tfrac{1}{12}\overline{\overline{b\overline{bg}}\,\overline{bg}} - \tfrac{1}{2}\overline{\overline{b\overline{gr}}\,\overline{gr}} + \tfrac{1}{24}\overline{\overline{\overline{\overline{bg}g}g}g}\Big]$$

```
Last[μ2]@{5}
```

$$\mathrm{CWS}\Big[0,\ 0,\ 2\,\widehat{bgr},\ \widehat{bbgr} - \widehat{bgbr} + \widehat{bggr} - \widehat{bgrg} + \widehat{bgrr} - \widehat{brgr},$$
$$\frac{\widehat{bbbgr}}{3} - \frac{\widehat{bbgbr}}{2} + \frac{\widehat{bbggr}}{2} + \frac{\widehat{bbgrg}}{2} + \frac{\widehat{bbgrr}}{2} + \frac{\widehat{bbrbg}}{2} - \frac{3\,\widehat{bbrgr}}{2} + \frac{\widehat{bgbrr}}{2} - \frac{3\,\widehat{bggbr}}{2} +$$
$$\frac{\widehat{bgggr}}{3} - \frac{\widehat{bggrg}}{2} + \frac{\widehat{bggrr}}{2} + \frac{\widehat{bgrgg}}{2} - \frac{3\,\widehat{bgrrg}}{2} + \frac{\widehat{bgrrr}}{3} + \frac{\widehat{brggr}}{2} - \frac{\widehat{brgrr}}{2} + \frac{\widehat{brrgr}}{2}\Big]$$

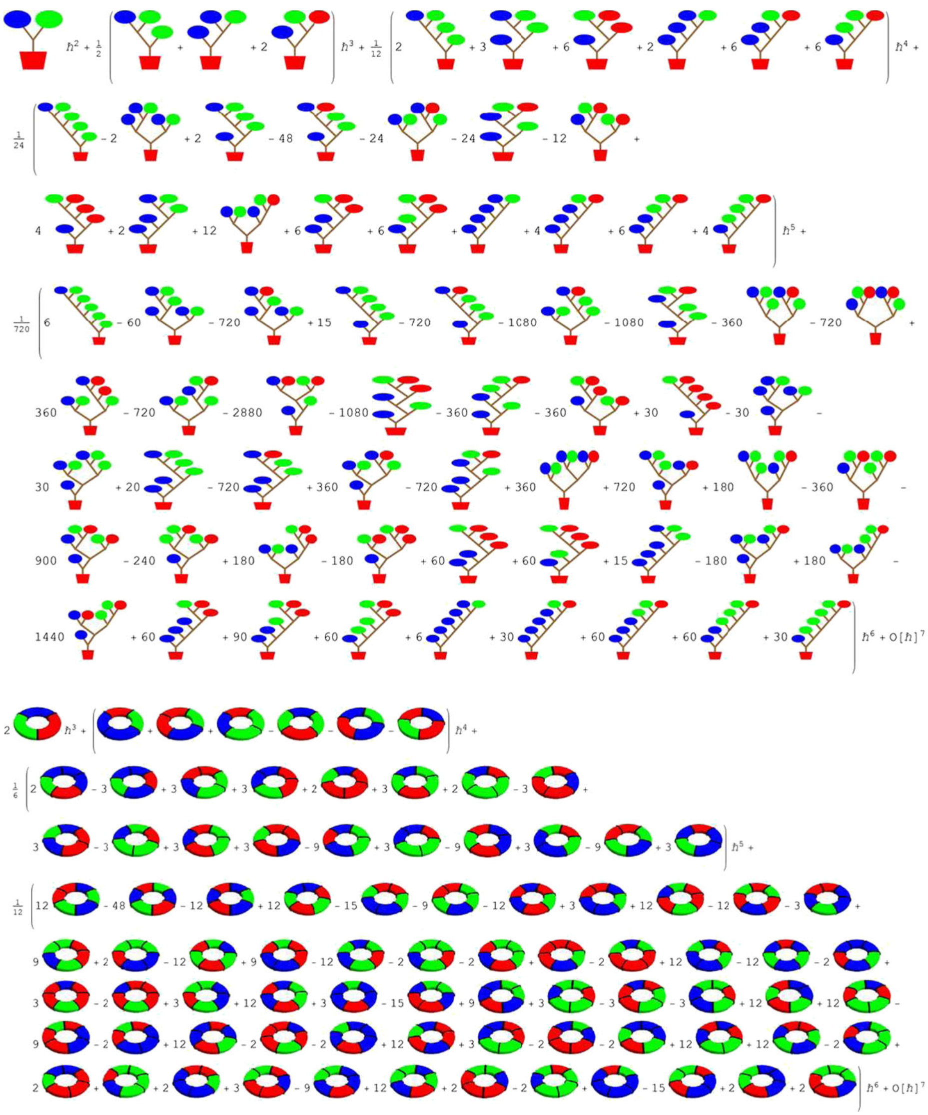


**Fig. 5** The redhead part of the tree part and the wheels part of the invariant of the Borromean tangle, to degree 6

A more graphically pleasing presentation of the same values, with the degree raised to 6, appears in Fig. 5. 759 760

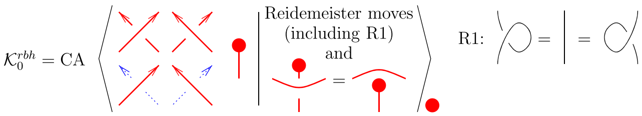
## 7 Sketch of the Relation with Finite Type Invariants 761

One way to view the invariant $\zeta$ of Section 5 is as a mysterious extension of the reasonably natural invariant $\zeta_0$ of Section 4. Another is as a solution to a universal problem—as we shall see in this section, $\zeta$ is a universal finite type invariant of objects in $\mathcal{K}_0^{\mathrm{rbh}}$. Given that $\mathcal{K}_0^{\mathrm{rbh}}$ 762 763 764 is closely related to $w\mathcal{T}$ (w-tangles), and given that much was already said on finite-type invariants of w-tangles in [5], this section will be merely a sketch, difficult to understand without reading much of [4] and sections 1–3 of [5], as well as the parts of section 4 that concern with caps.

Over all, defining $\zeta$ using the language of Sections 4 and 5 is about as difficult as using finite-type invariants. Yet computing it using the language of Sections 4 and 5 is much easier while proving invariance is significantly harder.

### 7.1 A circuit Algebra Description of $\mathcal{K}_0^{\mathrm{rbh}}$

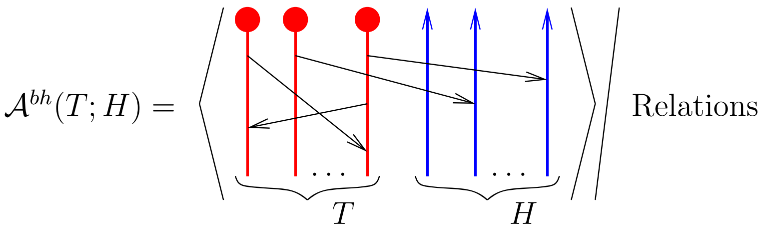
A w-tangle represents a collection of ribbon-knotted tubes in $\mathbb{R}^4$. It follows from Theorem 2.9 that every rKBH can be obtained from a w-tangle by capping some of its tubes and puncturing the rest, where puncturing a tube means "replacing it with its spine, a strand that runs along it". Using thick red lines to denote tubes, red bullets to denote caps, and dotted blue lines to denote punctured tubes, we find that




Note that punctured tubes (meanings strands or hoops) can only go under capped tubes (balloons), and that while it is allowed to slide tubes over caps, it is not allowed to slide them under caps. Further explanations and the meaning of "CA" are in [5]. The "red bullet" subscript on the right hand side indicates that we restrict our attention to the subspace in which all red strands are eventually capped. We leave it to the reader to interpret the operations *hm*, *tha*, and *tm* is this language (tm is non-obvious!).

### 7.2 Arrow Diagrams for $\mathcal{K}_0^{\mathrm{rbh}}$

As in [4, 5], one we finite-type invariants of elements on $\mathcal{K}_0^{\mathrm{rbh}}$ bi considering iterated differences of crossings and non-crossings (virtual crossings), and then again as in [4, 5], we find that the arrow-diagram space $\mathcal{A}^{\mathrm{bh}}(T; H)$ corresponding to these invariants may be described schematically as follows:



In the above, arrow tails may land only on the red "tail" strands, but arrow heads may land on either kind of strand. The "relations" are the TC and $\overrightarrow{4T}$ relations of [4, Section 2.3], the CP relation of [5, Section 4.2], and the relation $D_L = D_R = 0$, which corresponds to the R1 relation ($D_L$ and $D_R$ are defined in [4, Section 3]).

The operation *hm* acts on $\mathcal{A}^{\mathrm{bh}}$ by concatenating two head stands. The operation *tha* acts by duplicating a head strand (with the usual summation over all possible ways of reconnecting arrow-heads as in [4, Section 2.5.1.6]), changing the colour of one of the duplicates to red, and then concatenating it to the beginning of some tail strand.

We note that modulo the relations, one may eliminate all arrow-heads from all tail strands. For diagrams in which there are no arrow-heads on tail strands, the operation *tm* is defined by merging together two tail strands. The TC relation implies that arrow-tails on the resulting tail-strand can be ordered in any desired way.

As in [4, Section 3.5], $\mathcal{A}^{\mathrm{bh}}$ has an alternative model in which internal "2-in 1-out" trivalent vertices are allowed, and in which we also impose the $\overrightarrow{AS}$, $\overrightarrow{STU}$, and $\overrightarrow{IHX}$ relations (ibid.).

### 7.3 The Algebra Structure on $\mathcal{A}^{\mathrm{bh}}$ and its Primitives

For any fixed finite sets $T$ and $H$, the space $\mathcal{A}^{\mathrm{bh}}(T;H)$ is a co-commutative bi-algebra. Its product defined using the disjoint union followed by the *tm* operation on all tail strands and the *hm* operation on all head strands, and its co-product is the "sum of all splittings" as in [4, Section 3.2]. Thus by Milnor-Moore, $\mathcal{A}^{\mathrm{bh}}(T;H)$ is the universal enveloping algebra of its set of primitives $\mathcal{P}^{\mathrm{bh}}$. The latter is the set of connected diagrams in $\mathcal{A}^{\mathrm{bh}}$ (modulo relations), and those, as in [5, Section 3.2], are the trees and the degree $>1$ wheels. (Though note that even if $T = H = \{1,\dots,n\}$, the algebra structure on $\mathcal{A}^{\mathrm{bh}}(T;H)$ is different from the algebra structure on the space $\mathcal{A}^{w}(\uparrow_n)$ of ibid.). Identifying trees with $\mathrm{FL}(T)$ and wheels with $\mathrm{CW}^r(T)$, we find that

$$\mathcal{P}^{\mathrm{bh}}(T;H) \cong \mathrm{FL}(T)^H \times \mathrm{CW}^r(T) = M(T;H).$$

**Theorem 7.1** *By taking logarithms (using formal power series and the algebra structure of $\mathcal{A}^{bh}$), $\mathcal{P}^{bh}(T;H)$ inherits the structure of an MMA from the group-like elements of $\mathcal{A}^{bh}$. Furthermore, $\mathcal{P}^{bh}(T;H)$ and $M(T;H)$ are isomorphic as MMAs.*

*Sketch of the proof* Once it is established that $\mathcal{P}^{\mathrm{bh}}(T;H)$ is an MMA, that *tm* and *hm* act in the same way as on $M$ and that the tree part of the action of *tha* is given using the *RC* operation, it follows that the wheels part of the action of *tha* is given by some functional $J'$ which necessarily satisfies (19). But according to Remark 5.2, (19) and a few auxiliary conditions determine $J$ uniquely. These conditions are easily verified for $J'$, and hence $J' = J$. This concludes the proof.

Note that the above theorem and the fact that $\mathcal{P}^{\mathrm{bh}}(T;H)$ is an MMA provided an alternative proof of Proposition 5.1 which bypasses the hard computations of Section 10.4. In fact, personally, I first knew that $J$ exists and satisfies Proposition 5.1 using the reasoning of this section, and only then did I observe using the reasoning of Remark 5.2 that $J$ must be given by the formula in (18).

### 7.4 The Homomorphic Expansion $Z^{\mathrm{bh}}$

As in [4, Section 3.4] and [5, Section 3.1], there is a homomorphic expansion (a universal finite type invariant with good composition properties) $Z^{\mathrm{bh}}\colon \mathcal{K}_0^{\mathrm{rbh}} \to \mathcal{A}^{\mathrm{bh}}$ defined by mapping crossings to exponentials of arrows. It is easily verified that $Z^{\text{bh}}$ is a morphism of MMAs, and therefore it is determined by its values on the generators $\rho^{\pm}$ of $\mathcal{K}_0^{\text{rbh}}$, which are single crossings in the language of Section 7.1. Taking logarithms we find that $\log Z^{\text{bh}} = \zeta$ on the generators and hence always, and hence $\zeta$ is the logarithm of a universal finite type invariant of elements of $\mathcal{K}_0^{\text{rbh}}$.

## 8 The Relation with the BF Topological Quantum Field Theory

### 8.1 Tensorial Interpretation

Given a Lie algebra $\mathfrak{g}$, any element of $\text{FL}(T)$ can be interpreted as a function taking $|T|$ inputs in $\mathfrak{g}$ and producing a single output in $\mathfrak{g}$. Hence, putting aside issues of completion and convergence, there is a map $\tau_1 \colon \text{FL}(T) \to \text{Fun}(\mathfrak{g}^T \to \mathfrak{g})$, where in general, $\text{Fun}(X \to Y)$ denotes the space of functions from $X$ to $Y$. To deal with completions more precisely, we pick a formal parameter $\hbar$, multiply the degree $k$ part of $\tau_1$ by $\hbar^k$, and get a perfectly good $\tau = \tau_{\mathfrak{g}} : \text{FL}(T) \to \text{Fun}(\mathfrak{g}^T \to \mathfrak{g}[\![\hbar]\!])$, where in general, $V[\![\hbar]\!] := \mathbb{Q}[\![\hbar]\!] \otimes V$ for any vector space $V$. The map $\tau$ obviously extends to $\tau : \text{FL}(T)^H \to \text{Fun}(\mathfrak{g}^T \to \mathfrak{g}^H[\![\hbar]\!])$.

Similarly, if also $\mathfrak{g}$ is finite dimensional, then by taking traces in the adjoint representation we get a map $\tau = \tau_{\mathfrak{g}} : \text{CW}(T) \to \text{Fun}(\mathfrak{g}^T \to \mathbb{Q}[\![\hbar]\!])$. Multiplying this $\tau$ with the $\tau$ from the previous paragraph, we get $\tau = \tau_{\mathfrak{g}} : M(T; H) \to \text{Fun}(\mathfrak{g}^T \to \mathfrak{g}^H[\![\hbar]\!])$. Exponentiating, we get

$$e^{\tau} \colon M(T; H) \to \text{Fun}(\mathfrak{g}^T \to \mathcal{U}(\mathfrak{g})^{\otimes H}[\![\hbar]\!]).$$

### 8.2 $\zeta$ and BF Theory

Fix a finite dimensional Lie algebra $\mathfrak{g}$. In [7] (see especially section 4), Cattaneo and Rossi discuss the BF quantum field theory with fields $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ and $B \in \Omega^2(\mathbb{R}^4, \mathfrak{g}^*)$ and construct an observable "$U(A, B, \Xi)$" for each "long" $\mathbb{R}^2$ in $\mathbb{R}^4$; meaning, for each 2-sphere in $S^4$ with a prescribed behaviour at $\infty$. We interpret these as observables defined on our "balloons". The Cattaneo-Rossi observables are functions of a variable $\Xi \in \mathfrak{g}$, and they can be interpreted as power series in a formal parameter $\hbar$. Further, given the connection-field $A$, one may always consider its formal holonomy along a closed path (a "hoop") and interpret it as an element in $\mathcal{U}(\mathfrak{g})[\![\hbar]\!]$. Multiplying these hoop observables and also the Cattaneo-Rossi balloon observables, we get an observable $\mathcal{O}_\gamma$ for any KBH $\gamma$, taking values in $\text{Fun}(\mathfrak{g}^T \to \mathcal{U}(\mathfrak{g})^{\otimes H}[\![\hbar]\!])$.

**Conjecture 8.1** *If $\gamma$ is an rKBH, then $\langle \mathcal{O}_\gamma \rangle_{BF} = e^{\tau}(\zeta(\gamma))$.*

Of course, some interpretation work is required before Conjecture 8.1 even becomes a well-posed mathematical statement.

We note that the Cattaneo-Rossi observable does not depend on the ribbon property of the KBH $\gamma$. I hesitate to speculate whether this is an indication that the work presented in this paper can be extended to non-ribbon knots or an indication that somewhere within the rigorous mathematical analysis of BF theory an obstruction will arise that will force one to restrict to ribbon knots (yet I speculate that one of these possibilities holds true).

Most likely the work of Watanabe [28] is a proof of Conjecture 8.1 for the case of a single balloon and no hoops, and very likely, it contains all key ideas necessary for a complete proof of Conjecture 8.1.

## 9 The Simplest Non-Commutative Reduction and an Ultimate Alexander Invariant

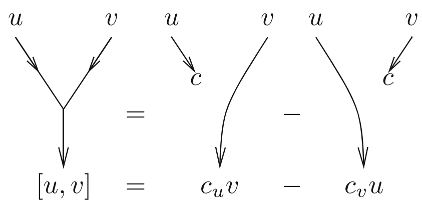
### 9.1 Informal

Let us start with some informal words. All the fundamental operations within the definition of $M$, namely $[.,.]$, $C_u^\gamma$, $RC_u^\gamma$ and $\operatorname{div}_u$, act by modifying trees and wheels near their extremities—their tails and their heads (for wheels, all extremities are tails). Thus, all operations will remain well-defined and will continue to satisfy the MMA properties if we extend or reduce trees and wheels by objects or relations that are confined to their "inner" parts.

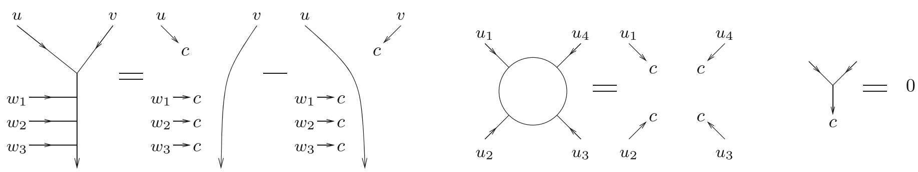
In this section, we discuss the "$\beta$-quotient of $M$", an extension/reduction of $M$ as discussed above, which is even better-computable than $M$. As we have seen in Section 6, objects in $M$, and in particular the invariant $\zeta$, are machine-computable. Yet the dimensions of FL and of CW grow exponentially in the degree, and so does the complexity of computations in $M$. Objects in the $\beta$-quotient are described in terms of commutative power series, their dimensions grow polynomially in the degree, and computations in the $\beta$-quotient are polynomial time. In fact, the power series appearing with the $\beta$-quotient can be "summed", and *non-perturbative* formulae can be given to everything in sight.

Yet $\zeta^\beta$, meaning $\zeta$ reduced to the $\beta$-quotient, remains strong enough to contain the (multi-variable) Alexander polynomial. I argue that in fact, the formulae obtained for the Alexander polynomial within this $\beta$-calculus are "better" than many standard formulae for the Alexander polynomial.

More on the relationship between the $\beta$-calculus and the Alexander polynomial (though nothing about its relationship with $M$ and $\zeta$), is in [6].




Still on the informal level, the $\beta$-quotient arises by allowing a new type of a "sink" vertex $c$ and imposing the $\beta$-relation, shown above, on both trees and wheels. One easily sees that under this relation, trees can be shaved to single arcs union "$c$-stubs", wheels become unions of $c$-stubs, and $c$-stubs "commute with everything":




Hence, $c$-stubs can be taken as generators for a commutative power series ring $R$ (with one generator $c_u$ for each possible tail label $u$), $\mathrm{CW}(T)$ becomes a copy of the ring $R$, elements of $\mathrm{FL}(T)$ becomes column vectors whose entries are in $R$ and whose entries correspond to the tail label in the remaining arc of a shaved tree, and elements of $\mathrm{FL}(T)^H$ can be regarded as $T \times H$ matrices with entries in $R$. Hence, in the $\beta$-quotient, the MMA $M$ reduces to an MMA $\{\beta_0(T; H)\}$ whose elements are $T \times H$ matrices of power series, with yet an additional power series to encode the wheels part. We will introduce $\beta_0$ more formally below, and then note that it can be simplified even further (with no further loss of information) to an MMA $\beta$ whose entries and operations involve rational functions, rather than power series.

*Remark 9.1* The $\beta$-relation arose from studying the (unique non-commutative) 2D Lie algebra $\mathfrak{g}_2 := \mathrm{FL}(\xi_1, \xi_2)/([\xi_1, \xi_2] = \xi_2)$, as in Section 8.1. Loosely, within $\mathfrak{g}_2$ the $\beta$-relation is a "polynomial identity" in a sense similar to the "polynomial identities" of the theory of PI-rings [25]. For a more direct relationship between this Lie algebra and the Alexander polynomial, see [web/chic1].

## 9.2 Less Informal

For a finite set $T$ let $R = R(T) := \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ denote the ring of power series with commuting generators $c_u$ corresponding to the elements $u$ of $T$, and let $L = L(T) := R \otimes \mathbb{Q}T$ be the the free $R$-module with generators $T$. Turn $L$ into a Lie algebra over $R$ by declaring that $[u, v] = c_u v - c_v u$ for any $u, v \in T$. Let $c\colon L \to R$ be the $R$-linear extension of $u \mapsto c_u$; namely,

$$\gamma = \sum_u \gamma_u u \in L \mapsto c_\gamma := \sum_u \gamma_u c_u \in R, \tag{23}$$

where the $\gamma_u$'s are coefficients in $R$. Note that with this definition, we have $[\alpha, \beta] = c_\alpha \beta - c_\beta \alpha$ for any $\alpha, \beta \in L$. There are obvious surjections $\pi : FL \to L$ and $\pi : CW \to R$ (strictly speaking, the first of those maps has a small cokernel yet becomes a surjection once the ground ring of its domain space is extended to $R$).

The following Lemma-Definition may appear scary, yet its proof is nothing more than high school level algebra, and the messy formulae within it mostly get renormalized away by the end of this section. Hang on!

**Lemma-Definition 9.2** The operations $C_u$, $RC_u$, bch, $\mathrm{div}_u$, and $J_u$ descend from FL/CWto $L/R$, and, for $\alpha, \beta, \gamma \in L$ (with $\gamma = \sum_v \gamma_v v$) they are given by

$$v /\!/ C_u^{-\gamma} = v /\!/ RC_u^{\gamma} = v \qquad \text{for } u \neq v \in T, \tag{24}$$

$$\rho /\!/ C_u^{-\gamma} = \rho /\!/ RC_u^{\gamma} = \rho \qquad \text{for } \rho \in R, \tag{25}$$

$$u /\!/ C_u^{-\gamma} = e^{-c_\gamma}\left(u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma}\gamma\right) \tag{26}$$

$$= e^{-c_\gamma}\left(\left(1 + c_u\gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma}\right) u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v\right), \tag{27}$$

$$u /\!/ RC_u^{\gamma} = \left(1 + c_u\gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma}\right)^{-1} \left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v\right), \tag{28}$$

$$\mathrm{bch}(\alpha, \beta) = \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1}\left(\frac{e^{c_\alpha} - 1}{c_\alpha}\alpha + e^{c_\alpha}\frac{e^{c_\beta} - 1}{c_\beta}\beta\right), \tag{29}$$

$$\mathrm{div}_u \gamma = c_u \gamma_u, \tag{30}$$

$$J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u\right). \tag{31}$$

*Proof (Sketch)* Equation (24) is obvious—$C_u$ or $RC_u$ conjugate or repeatedly conjugate $u$, but not $v$. Equation (25) is the statement that $C_u$ and $RC_u$ are $R$-linear, namely that they act on scalars as the identity. Informally this is the fact that 1-wheels commute with everything, and formally it follows from the fact that $\pi : FL \to L$ is a well-defined morphism of Lie algebras.

To prove (26), we need to compute $e^{-\mathrm{ad}\gamma}(u)$, and it is enough to carry this computation out within the 2D subspace of $L$ spanned by $u$ and by $\gamma$. Hence, the computation is an exercise in diagonalization—one needs to diagonalize the $2 \times 2$ matrix $\mathrm{ad}(-\gamma)$ in order to exponentiate it. Here, are some details: set $\delta = [-\gamma, u] = c_u\gamma - c_\gamma u$. Then, clearly $\mathrm{ad}(-\gamma)(\delta) = -c_\gamma\delta$, and hence $e^{-\mathrm{ad}\gamma}(\delta) = e^{-c_\gamma}\delta$. Also note that $\mathrm{ad}(-\gamma)(\gamma) = 0$, and hence $e^{-\mathrm{ad}\gamma}(\gamma) = \gamma$. Thus

$$u \mathbin{/\!\!/} C_u^{-\gamma} = e^{-\mathrm{ad}\gamma}(u) = e^{-\mathrm{ad}\gamma}\left(-\frac{\delta}{c_\gamma} + \frac{c_u\gamma}{c_\gamma}\right) = -\frac{e^{-c_\gamma}\delta}{c_\gamma} + \frac{c_u\gamma}{c_\gamma} = e^{-c_\gamma}\left(u + c_u\frac{e^{c_\gamma}-1}{c_\gamma}\gamma\right).$$

Equation (27) is simply (26) rewritten using $\gamma = \sum_v \gamma_v v$. To prove (28), take its right hand side and use (27) and (24) to get $u$ back again, and hence our formula for $RC_u^\gamma$ indeed inverts the formula already established for $C_u^{-\gamma}$.

Equation (29) amounts to writing the group law of a 2D Lie group in terms of its 2D Lie algebra, $L_0 := \mathrm{span}(\alpha, \beta)$, and this is again an exercise in $2 \times 2$ matrix algebra, though a slightly harder one. We work in the adjoint representation of $L_0$ and aim to compare the exponential of the left hand side of (29) with the exponential of its right hand side. If $a$ and $b$ are scalars, let $e(a, b)$ be the matrix representing $e^{\mathrm{ad}(a\alpha+b\beta)}$ on $L_0$ relative to the basis $(\alpha, \beta)$. Then using $[\alpha, \beta] = c_\alpha\beta - c_\beta\alpha$ we find that $e(a, b) = \exp\begin{pmatrix} bc_\beta & -ac_\beta \\ -bc_\alpha & ac_\alpha \end{pmatrix}$, and we need to show that $e(1, 0) \cdot e(0, 1) = e\left(\frac{c_\alpha + c_\beta}{e^{c_\alpha+c_\beta}-1}\frac{e^{c_\alpha}-1}{c_\alpha}, \frac{c_\alpha + c_\beta}{e^{c_\alpha+c_\beta}-1}e^{c_\alpha}\frac{e^{c_\beta}-1}{c_\beta}\right)$. Lazy bums do it as follows:

```
e[a_, b_] := MatrixExp[( b cβ   -a cβ
                         -b cα   a cα )];
e[1, 0].e[0, 1] == e[ (cα+cβ)/(e^(cα+cβ)-1) (e^cα-1)/cα , (cα+cβ)/(e^(cα+cβ)-1) e^cα (e^cβ-1)/cβ ] // Simplify

True
```

Equation 30 is the fact that $\mathrm{div}_u u = c_u$, along with the $R$-linearity of $\mathrm{div}_u$.

For (31), note that using (28), the coefficient of $u$ in $\gamma \mathbin{/\!\!/} RC_u^{s\gamma}$ is $\gamma_u e^{sc_\gamma}\left(1 + c_u\gamma_u\frac{e^{sc_\gamma}-1}{c_\gamma}\right)^{-1}$. Thus using (30) and the fact that $C_u$ acts trivially on $R$,

$$J_u(\gamma) = \int_0^1 ds\, \mathrm{div}_u(\gamma \mathbin{/\!\!/} RC_u^{s\gamma}) \mathbin{/\!\!/} C_u^{-s\gamma} = \int_0^1 ds\, \left(1 + c_u\gamma_u\frac{e^{sc_\gamma}-1}{c_\gamma}\right)^{-1} c_u\gamma_u e^{sc_\gamma}$$

$$= \log\left(1 + \frac{e^{sc_\gamma}-1}{c_\gamma}c_u\gamma_u\right)\Bigg|_0^1 = \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right).$$

□

### 9.3 The Reduced Invariant $\zeta^{\beta_0}$.

We now let $\beta_0(T; H)$ be the $\beta$-reduced version of $M(T; H)$. Namely, in parallel with Section 5.2 we define

$$\beta_0(T; H) := L(T)^H \times R^r(T) = R(T)^{T\times H} \times R^r(T).$$

In other words, elements of $\beta_0(T; H)$ are $T \times H$ matrices $A = (A_{ux})$ of power series in the variables $\{c_u\}_{u\in T}$, along with a single additional power series $\omega \in R^r$ ($R^r$ is $R$ modded out by its degree 1 piece) corresponding to the last factor above, which we write at the top left of $A$:

$$\beta_0(u, v, \ldots; x, y, \ldots) = \left\{ \begin{pmatrix} \omega & x & y & \cdots \\ u & A_{ux} & A_{uy} & \cdot \\ v & A_{vx} & A_{vy} & \cdot \\ \vdots & \cdot & \cdot & \ddots \end{pmatrix} : \omega \in R^r(T),\ A_{\cdot\cdot} \in R(T) \right\}$$

Continuing in parallel with Section 5.2 and using the formulae from Lemma-Definition 9.2, we turn $\{\beta_0(T; H)\}$ into an MMA with operations defined as follows (on a typical element of $\beta_0$, which is a decorated matrix $(A, \omega)$ as above):

- $t\sigma^u_v$ acts by renaming row $u$ to $v$ and sending the variable $c_u$ to $c_v$ everywhere. $t\eta^u$ acts by removing row $u$ and sending $c_u$ to 0. $tm^{uv}_w$ acts by adding row $u$ to row $v$ calling the result row $w$, and by sending $c_u$ and $c_v$ to $c_w$ everywhere.
- $h\sigma^x_y$ and $h\eta^x$ are clear. To define $hm^{xy}_z$, let $\alpha = (A_{ux})_{u\in T}$ and $\beta = (A_{uy})_{u\in T}$ denote the columns of $x$ and $y$ in $A$, let $c_\alpha := \sum_{u\in T} A_{ux}c_u$ and $c_\beta := \sum_{u\in T} A_{uy}c_u$ in parallel with (23), and let $hm^{xy}_z$ act by removing the $x$- and $y$-columns $\alpha$ and $\beta$ and introducing a new column, labelled $z$, and containing $\frac{c_\alpha+c_\beta}{e^{c_\alpha+c_\beta}-1}\left(\frac{e^{c_\alpha}-1}{c_\alpha}\alpha + e^{c_\alpha}\frac{e^{c_\beta}-1}{c_\beta}\beta\right)$, as in (29).
- We now describe the action of $tha^{ux}$ on an input $(A, \omega)$ as depicted below. Let $\gamma = \begin{pmatrix}\gamma_u \\ \gamma_{\text{rest}}\end{pmatrix}$ be the column of $x$, split into the "row $u$" part $\gamma_u$ and the rest, $\gamma_{\text{rest}}$. Let $c_\gamma$ be $\sum_{v\in T}\gamma_v c_v$ as in (23). Then $tha^{ux}$ acts as follows:

$$\begin{array}{c|cccc} \omega & x & \cdot & y & \cdot \\ \hline u & \gamma_u & \cdot & \alpha_u & \cdot \\ \vdots & \gamma_{\text{rest}} & \cdot & \alpha_{\text{rest}} & \cdot \end{array}$$

  – As dictated by (31), $\omega$ is replaced by $\omega + \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right)$.

  – As dictated by (24) and (28), every column $\alpha = \begin{pmatrix}\alpha_u \\ \alpha_{\text{rest}}\end{pmatrix}$ in $A$ (including the column $\gamma$ itself) is replaced by

$$\left(1 + c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\begin{pmatrix} e^{c_\gamma}\alpha_u \\ \alpha_{\text{rest}} - c_u\frac{e^{c_\gamma}-1}{c_\gamma}(c\gamma)_{\text{rest}}\end{pmatrix},$$

  where $(c\gamma)_{\text{rest}}$ is the column whose row $v$ entry is $c_v\gamma_v$, for any $v \neq u$.
- The "merge" operation $*$ is $\begin{array}{c|c}\omega_1 & H_1 \\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2 \\ \hline T_2 & A_2\end{array} := \begin{array}{c|cc}\omega_1+\omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2\end{array}$.
- • $t\epsilon_u = \begin{array}{c|c}0 & \emptyset \\ \hline u & \emptyset\end{array}$ and $h\epsilon_x = \begin{array}{c|c}0 & x \\ \hline \emptyset & \emptyset\end{array}$ (these values correspond to a matrix with an empty set of columns and a matrix with an empty set of rows, respectively).

We have concocted the definition of the MMA $\beta_0$ so that the projection $\pi : M \to \beta_0$ would be a morphism of MMAs. Hence, to completely compute $\zeta^{\beta_0} := \pi \circ \zeta$ on any rKBH (to all orders!), it is enough to note its values on the generators. These are determined by the values in Theorem 5.3: $\zeta^{\beta_0}(\rho^{\pm}_{ux}) = \begin{array}{c|c} 0 & x \\ \hline u & \pm 1 \end{array}$.

## 9.4 The Ultimate Alexander Invariant $\zeta^{\beta}$.

Some repackaging is in order. Noting the ubiquity of factors of the form $\frac{e^c-1}{c}$ in the previous section, it makes sense to multiply any column $\alpha$ of the matrix $A$ by $\frac{e^{c_\alpha}-1}{c_\alpha}$. Noting that row-$u$ entries (things like $\gamma_u$) often appear multiplied by $c_u$, we multiply every row by its corresponding variable $c_u$. Doing this and rewriting the formulae of the previous section in the new variables, we find that the variables $c_u$ only appear within exponentials of the form $e^{c_u}$. So, we set $t_u := e^{c_u}$ and rewrite everything in terms of the $t_u$'s. Finally, the only formula that touches $\omega$ is additive and has a log term. So, we replace $\omega$ with $e^{\omega}$. The result is "$\beta$-calculus", which was described in detail in [6]. A summary version follows. In these formulae, $\alpha$, $\beta$, $\gamma$, and $\delta$ denote entries, rows, columns, or submatrices as appropriate, and whenever $\alpha$ is a column, $\langle\alpha\rangle$ is the sum of is entries:

$$\beta(T;H) = \left\{ \begin{array}{c|ccc|l} \omega & x & y & \cdots & \omega \text{ and the } \alpha_{ux}\text{'s are rational functions in} \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot & \text{variables } t_u \text{, one for each } u \in T. \text{ When all} \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & t_u\text{'s are set to 1, } \omega \text{ is 1 and every } \alpha_{ux} \text{ is} \\ \vdots & \cdot & \cdot & \cdot & 0. \end{array} \right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \gamma \end{array} \mapsto \left( \begin{array}{c|c} \omega & H \\ \hline w & \alpha+\beta \\ T & \gamma \end{array} \right) /\!/ (t_u, t_v \to t_w),$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & H \\ \hline T & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & H \\ \hline T & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & H \\ \hline u & \alpha & \beta \\ T & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega(1+\alpha) & x & H \\ \hline u & \alpha(1+\langle\gamma\rangle/(1+\alpha)) & \beta(1+\langle\gamma\rangle/(1+\alpha)) \\ T & \gamma/(1+\alpha) & \delta-\gamma\beta/(1+\alpha) \end{array},$$

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} := \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2 \end{array},$$

$$\zeta^{\beta}(t\epsilon_u) = \begin{array}{c|c} 1 & \emptyset \\ \hline u & \emptyset \end{array}, \qquad \zeta^{\beta}(h\epsilon_x) = \begin{array}{c|c} 1 & x \\ \hline \emptyset & \emptyset \end{array}, \qquad \text{and} \qquad \zeta^{\beta}(\rho^{\pm}_{ux}) = \begin{array}{c|c} 1 & x \\ \hline u & t_u^{\pm 1}-1 \end{array}.$$

**Theorem 9.3** *If $K$ is a u-knot regarded as a 1-component pure tangle by cutting it open, then the $\omega$ part of $\zeta^{\beta}(\delta(K))$ is the Alexander polynomial of $K$.*

I know of three winding paths that constitute a proof of the above theorem:

- Use the results of Section 7 here, of [4, Section 3.7], and of [21].
- Use the results of Section 7 here, of [4, Section 3.9], and the known relation of the Alexander polynomial with the wheels part of the Kontsevich integral (e.g. [19]).

- Use the results of [18], where formulae very similar to ours appear.

Yet to me, the strongest evidence that Theorem 9.3 is true is that it was verified explicitly on very many knots—see the single example in Section 6.3 here and many more in [6].

In several senses, $\zeta^\beta$ is an "ultimate" Alexander invariant:

- The formulae in this section may appear complicated, yet note that if an rKBH consists of about $n$ balloons and hoops, its invariant is described in terms of only $O(n^2)$ polynomials and each of the operations *tm*, *hm*, and *tha* involves only $O(n^2)$ operations on polynomials.
- It is defined for tangles and has a prescribed behaviour under tangle compositions (in fact, it is defined in terms of that prescribed behaviour). This means that when $\zeta^\beta$ is computed on some large knot with (say) $n$ crossings, the computation can be broken up into $n$ steps of complexity $O(n^2)$ at the end of each the quantity computed is the invariant of some topological object (a tangle), or even into $3n$ steps at the end of each the quantity computed is the invariant of some rKBH[10].
- $\zeta^\beta$ contains also the multivariable Alexander polynomial and the Burau representation (overwhelmingly verified by experiment, not written-up yet).
- $\zeta^\beta$ has an easily prescribed behaviour under hoop- and balloon-doubling, and $\zeta^\beta \circ \delta$ has an easily prescribed behaviour under strand-doubling (not shown here).

## 10 Odds and Ends

### 10.1 Linking Numbers and Signs

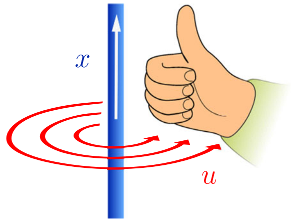
If $x$ is an oriented $S^1$ and $u$ is an oriented $S^2$ in an oriented $S^4$ (or $\mathbb{R}^4$) and the two are disjoint, their linking number $l_{ux}$ is defined as follows. Pick a ball $B$ whose oriented boundary is $u$ (using the "outward pointing normal" convention for orienting boundaries), and which intersects $x$ in finitely many transversal intersection points $p_i$. At any of these intersection points $p_i$, the concatenation of the orientation of $B$ at $p_i$ (thought of a basis to the tangent space of $B$ at $p_i$) with the tangent to $x$ at $p_i$ is a basis of the tangent space of $S^4$ at $p_i$, and as such it may either be positively oriented or negatively oriented. Define $\sigma(p_i) = +1$ in the former case and $\sigma(p_i) = -1$ in the latter case. Finally, let $l_{ux} := \sum_i \sigma(p_i)$. It is a standard fact that $l_{ux}$ is an isotopy invariant of $(u, x)$.

*Exercise 10.1* Verify that $l_{ux}(\rho^{\pm}_{ux}) = \pm 1$, where $\rho^+_{ux}$ and $\rho^-_{ux}$ are the positive and negative Hopf links as in Example 2.2. For the purpose of this exercise, the plane in which Fig. 1 is drawn is oriented counterclockwise, the 3D space it represents has its third coordinate oriented up from the plane of the paper, and $\mathbb{R}^4_{txyz}$ is oriented so that the $t$ coordinate is "first".

An efficient thumb rule for deciding the linking number signs for a balloon $u$ and a hoop $x$ presented using our standard notation as in Section 2.1 is the "right-hand rule" of the

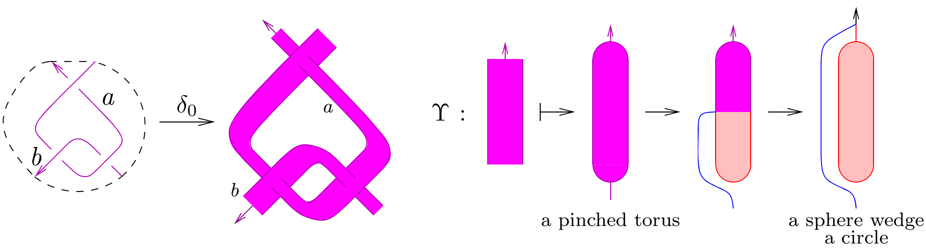
[10] A similar statement can be made for Alexander formulae based on the Burau representation. Yet note that such formulae still end with a computation of a determinant which may take $O(n^3)$ steps. Note also that the presentation of knots as braid closures is typically inefficient—typically a braid with $O(n^2)$ crossings is necessary in order to present a knot with just $n$ crossings.

figure below, shown here without further explanation. The lovely figure is adopted from [Wikipedia: Right-hand_rule].




## 10.2 A Topological Construction of $\delta$

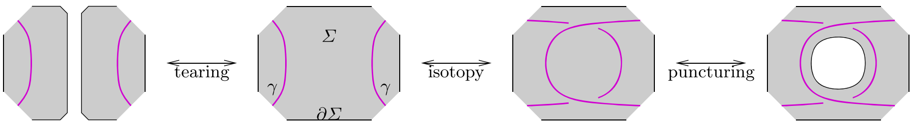
The map $\delta$ is a composition $\delta_0 \mathbin{/\!/} \Upsilon$ ("$\delta_0$ followed by $\Upsilon$", aka $\Upsilon \circ \delta_0$. See Section 10.5.). Here, $\delta_0$ is the standard "tubing" map $\delta_0$ (called $t'$ in Satoh's [26]), though with the tubes decorated by an additional arrowhead to retain orientation information. The map $\Upsilon$ caps and strings both ends of all tubes to $\infty$ and then uses, at the level of embeddings, the fact that a pinched torus is homotopy equivalent to a sphere wedge a circle:




It is worthwhile to give a completely "topological" definition of the tubing map $\delta_0$, thus giving $\delta = \delta_0 \mathbin{/\!/} \Upsilon$ a topological interpretation. We must start with a topological interpretation of v-tangles, and even before, with v-knots, also known as virtual knots.

The standard topological interpretation of v-knots (e.g. [20]) is that they are oriented knots drawn[11] on an oriented surface $\Sigma$, modulo "stabilization", which is the addition and/or removal of empty handles (handles that do not intersect with the knot). We prefer an equivalent, yet even more bare-bones approach. For us, a virtual knot is an oriented knot $\gamma$ drawn on a "virtual surface $\Sigma$ for $\gamma$". More precisely, $\Sigma$ is an oriented surface that may have a boundary, $\gamma$ is drawn on $\Sigma$, and the pair $(\Sigma, \gamma)$ is taken modulo the following relations:

- Isotopies of $\gamma$ on $\Sigma$ (meaning, in $\Sigma \times [-\epsilon, \epsilon]$).
- Tearing and puncturing parts of $\Sigma$ away from $\gamma$:




(We call $\Sigma$ a "virtual surface" because tearing and puncturing imply that we only care about it in the immediate vicinity of $\gamma$).

We can now define[12] a map $\delta_0$, defined on v-knots and taking values in ribbon tori in $\mathbb{R}^4$: given $(\Sigma, \gamma)$, embed $\Sigma$ arbitrarily in $\mathbb{R}^3_{xyz} \subset \mathbb{R}^4$. Note that the unit normal bundle of $\Sigma$ in $\mathbb{R}^4$ is a trivial circle bundle and it has a distinguished trivialization, constructed using its positive $t$-direction section and the orientation that gives each fibre a linking number $+1$ with the base $\Sigma$. We say that a normal vector to $\Sigma$ in $\mathbb{R}^4$ is "near unit" if its norm is between $1-\epsilon$ and $1+\epsilon$. The near-unit normal bundle of $\Sigma$ has as fibre an annulus that can be identified with $[-\epsilon, \epsilon] \times S^1$ (identifying the radial direction $[1-\epsilon, 1+\epsilon]$ with $[-\epsilon, \epsilon]$ in an orientation-preserving manner), and hence the near-unit normal bundle of $\Sigma$ defines an embedding of $\Sigma \times [-\epsilon, \epsilon] \times S^1$ into $\mathbb{R}^4$. On the other hand, $\gamma$ is embedded in $\Sigma \times [-\epsilon, \epsilon]$ so $\gamma \times S^1$ is embedded in $\Sigma \times [-\epsilon, \epsilon] \times S^1$, and we can let $\delta_0(\Sigma, \gamma)$ be the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$

which is a torus in $\mathbb{R}^4$, oriented using the given orientation of $\gamma$ and the standard orientation of $S^1$.

We leave it to the reader to verify that $\delta_0(\Sigma, \gamma)$ is ribbon, that it is independent of the choices made within its construction, that it is invariant under isotopies of $\gamma$ and under tearing and puncturing, that it is also invariant under the "overcrossing commute" relation of Fig. 3, and that it is equivalent to Satoh's tubing map.

The map $\delta_0$ has straightforward generalizations to v-links, v-tangles, framed-v-links, v-knotted-graphs, etc.

### 10.3 Monoids, Meta-Monoids, Monoid-Actions, and Meta-Monoid-Actions

How do we think about meta-monoid-actions? Why that name? Let us start with ordinary monoids.

[11] Here and below, "drawn on $\Sigma$" means "embedded in $\Sigma \times [-\epsilon, \epsilon]$".

[12] Following a private discussion with Dylan Thurston.

### *10.3.1 Monoids*

A monoid[13] $G$ gives rise to a slew of spaces and maps between them: the spaces would be the spaces of sequences $G^n = \{(g_1, \ldots, g_n) \colon g_i \in G\}$, and the maps will be the maps "that can be written using the monoid structure"—they will include, for example, the map $m^{ij}_i \colon G^n \to G^{n-1}$ defined as "store the product $g_i g_j$ as entry number $i$ in $G^{n-1}$ while erasing the original entries number $i$ and $j$ and re-numbering all other entries as appropriate". In addition, there is also an obvious binary "concatenation" map $* \colon G^n \times G^m \to G^{n+m}$ and a special element $\epsilon \in G^1$ (the monoid unit).

Equivalently but switching from "numbered registers" to "named registers", a monoid $G$ automatically gives rise to another slew of spaces and operations. The spaces are $G^X = \{f \colon X \to G\} = \{(x \to g_x)_{x \in X}\}$ of functions from a finite set $X$ to $G$, or as we prefer to say it, of $X$-indexed sequences of elements in $G$, or how computer scientists may say it, of associative arrays of elements of $G$ with keys in $X$. The maps between such spaces would now be the obvious "register multiplication maps" $m^{xy}_z \colon G^{X \cup \{x,y\}} \to G^{X \cup \{z\}}$ (defined whenever $x, y, z \notin X$ and $x \neq y$), and also the obvious "delete a register" map $\eta^x \colon G^X \to G^{X \backslash x}$, the obvious "rename a register" map $\sigma^x_y \colon G^{X \cup \{x\}} \to G^{X \cup \{y\}}$, and an obvious $* \colon G^X \times G^Y \to G^{X \cup Y}$, defined whenever $X$ and $Y$ are disjoint. Also, there are special elements, "units", $\epsilon_x \in G^{\{x\}}$.

This collection of spaces and maps between them (and the units) satisfies some properties. Let us highlight and briefly discuss two of those:

(1.) The "associativity property": For any $\Omega \in G^X$,

$$\Omega /\!\!/ m^{xy}_x /\!\!/ m^{xz}_x = \Omega /\!\!/ m^{yz}_y /\!\!/ m^{xy}_x. \tag{32}$$

This property is an immediate consequence of the associativity axiom of monoid theory. Note that it is a "linear property'—its subject, $\Omega$, appears just once on each side of the equality. Similar linear properties include $\Omega /\!\!/ \sigma^x_y /\!\!/ \sigma^y_z = \Omega /\!\!/ \sigma^x_z$, $\Omega /\!\!/ m^{xy}_z /\!\!/ \sigma^z_u = \Omega /\!\!/ m^{xy}_u$, etc., and there are also "multi-linear" properties like $(\Omega_1 * \Omega_2) * \Omega_3 = \Omega_1 * (\Omega_2 * \Omega_3)$, which are "linear" in each of their inputs.

(2.) If $\Omega \in G^{\{x,y\}}$, then

$$\Omega = (\Omega /\!\!/ \eta^y) * (\Omega /\!\!/ \eta^x) \tag{33}$$

(indeed, if $\Omega = (x \to g_x,\ y \to g_y)$, then $\Omega /\!\!/ \eta^y = (x \to g_x)$ and $\Omega /\!\!/ \eta^x = (y \to g_y)$ and so the right hand side is $(x \to g_x) * (y \to g_y)$, which is $\Omega$ back again), so an element of $G^{\{x,y\}}$ can be factored as an element of $G^{\{x\}}$ times an element of $G^{\{y\}}$. Note that $\Omega$ appears twice in the right hand side of this property, so this property is "quadratic". In order to write this property one must be able to "make two copies of $\Omega$".

### *10.3.2 Meta-Monoids*

**Definition 10.2** A meta-monoid is a collection $(G_X, m^{xy}_z, \sigma^x_z, \eta^x, *)$ of sets $G_X$, one for each finite set $X$ "of labels", and maps between them $m^{xy}_z, \sigma^x_z, \eta^x, *$ with the same domains and ranges as above, and special elements $\epsilon_x \in G_{\{x\}}$, and with the same **linear and multi-linear** properties as above.

[13] A monoid is a group sans inverses. You lose nothing if you think "group" whenever the discussion below states "monoid".

Very crucially, we do not insist on the non-linear property (33) of the above, and so we may not have the factorization $G_{\{x,y\}} = G_{\{x\}} \times G_{\{y\}}$, and in general, it need not be the case that $G_X = G^X$ for some monoid $G$. (Though of course, the case $G_X = G^X$ is an example of a meta-monoid, which perhaps may be called a "classical meta-monoid").

Thus a meta-monoid is like a monoid in that it has sets $G_X$ of "multi-elements" on which almost-ordinary monoid theoretic operations are defined. Yet, the multi-elements in $G_X$ need not simply be lists of elements as in $G^X$, and instead, they may be somehow "entangled". A relatively simple example of a meta-monoid which isn't a monoid is $H^{\otimes X}$ where $H$ is a Hopf algebra[14]. This simple example is similar to "quantum entanglement". But a meta-monoid is not limited to the kind of entanglement that appears in tensor powers. Indeed many of the examples within the main text of this paper aren't tensor powers and their "entanglement" is closer to that of the theory of tangles. This especially applied to the meta-monoid $w\mathcal{T}$ of Section 3.2.

#### *10.3.3 Monoid-Actions*

A monoid-action[15] of a monoid $G_1$ on another monoid $G_2$ is a single algebraic structure MA consisting of two sets $G_1$ (heads) and $G_2$ (tails), a binary operation defined on $G_1$, a binary operation defined on $G_2$, and a mixed operation $G_1 \times G_2 \to G_2$ (denoted $(x, u) \mapsto u^x$) which satisfy some well-known axioms, of which the most interesting are the associativities of the first two binary operations and the two action axioms $(uv)^x = u^x v^x$ and $u^{(xy)} = (u^x)^y$.

As in the case of individual monoids, a monoid-action MA gives rise to a slew of spaces and maps between them. The spaces are $\mathrm{MA}(T; H) := G_2^T \times G_1^H$, defined whenever $T$ and $H$ are finite sets of tail labels and head labels. The main operations[16] are $tm^{uv}_w\colon \mathrm{MA}(T \cup \{u, v\}; H) \to \mathrm{MA}(T \cup \{w\}; H)$ defined using the multiplication in $G_2$ (assuming $u, v, w \notin T$ and $u \neq v$), $hm^{xy}_z\colon \mathrm{MA}(T; H \cup \{x, y\}) \to \mathrm{MA}(T; H \cup \{z\})$ (assuming $x, y \notin H$ and $x \neq y$) defined using the multiplication in $G_1$, and $tha^{ux}\colon \mathrm{MA}(T; H) \to \mathrm{MA}(T; H)$ (assuming $x \in H$ and $u \in T$) defined using the action of $G_1$ on $G_2$. These operations have the following properties, corresponding to the associativity of $G_1$ and $G_2$ and to the two action axioms of the previous paragraph:

$$\begin{array}{ll} hm^{xy}_x \,/\!\!/\, hm^{xz}_x = hm^{yz}_y \,/\!\!/\, hm^{xy}_x, & tm^{uv}_u \,/\!\!/\, tm^{uw}_u = tm^{vw}_v \,/\!\!/\, tm^{uv}_u, \\ tm^{uv}_w \,/\!\!/\, tha^{wx} = tha^{ux} \,/\!\!/\, tha^{vx} \,/\!\!/\, tm^{uv}_w, & hm^{xy}_z \,/\!\!/\, tha^{uz} = tha^{ux} \,/\!\!/\, tha^{uy} \,/\!\!/\, hm^{xy}_z. \end{array} \tag{34}$$

There are also routine properties involving also $*$, $\eta$'s and $\sigma$'s as before.

#### *10.3.4 Meta-Monoid-Actions*

Finally, a meta-monoid-action is to a monoid-action like a meta-monoid is to a monoid. Thus it is a collection

$$(M(T; H), tm^{uv}_w, hm^{xy}_z, tha^{ux}, t\sigma^u_w, h\sigma^x_y, t\eta^u, h\eta^x, *, t\epsilon_u, h\epsilon_x)$$

[14] Or merely an algebra.

[15] Think "group-action".

[16] There are also $*$, $t\eta^u$, $h\eta^x$, $t\sigma^u_v$ and $h\sigma^x_y$ and units $t\epsilon_u$ and $h\epsilon_x$ as before.

of sets $M(T;H)$, one for each pair of finite sets $(T;H)$ of tail labels and head labels, and maps between them $tm^{uv}_w$, $hm^{xy}_z$, $tha^{ux}$, $t\sigma^u_v$, $h\sigma^x_y$, $t\eta^u$, $h\eta^x$, $*$, and units $t\epsilon_u$ and $h\epsilon_x$, with the same domains and ranges as above and with the same **linear and multi-linear** properties as above; most importantly, the properties in (34).

Thus a meta-monoid-action is like a monoid-action in that it has sets $M(T;H)$ of multi-elements on which almost-ordinary monoid theoretic operations are defined. Yet the multi-elements in $M(T;H)$ need not simply be lists of elements as in $G_2^T \times G_1^H$, and instead they may be somehow entangled.

#### *10.3.5 Meta-Groups / Meta-Hopf-Algebras*

Clearly, the prefix meta can be added to many other types of algebraic structures, though sometimes a little care must be taken. To define a "meta-group", for example, one may add to the definition of a meta-monoid in Section 10.3.2 a further collection of operations $S^x$, one for each $x \in X$, representing "invert the (meta-)element in register $x$". Except that the axiom for an inverse, $g \cdot g^{-1} = \epsilon$, is quadratic in $g$—one must have two copies of $g$ in order to write the axiom, and hence it cannot be written using $S^x$ and the operations in Section 10.3.2. Thus, in order to define a meta-group, we need to also include "meta-co-product" operations $\Delta^x_{yz}\colon G_{X\cup\{x\}} \to G_{X\cup\{y,z\}}$. These operations should satisfy some further axioms, much like within the definition of a Hopf algebra. The major ones are: a meta-co-associativity, a meta-compatibility with the meta-multiplication, and a meta-inverse axiom $\Omega /\!\!/ \Delta^x_{yz} /\!\!/ S^y /\!\!/ m^{yz}_x = (\Omega /\!\!/ \eta^x) * \epsilon_x$.

A strict analogy with groups would suggest another axiom: a meta-co-commutativity of $\Delta$, namely $\Delta^x_{yz} = \Delta^x_{zy}$. Yet, experience shows that it is better to sometimes not insist on meta-co-commutativity. Perhaps the name meta-group should be used when meta-co-commutativity is assumed, and "meta-Hopf-algebra" when it isn't.

Similarly one may extend "meta-monoid-actions" to "meta-group-actions" and/or "meta-Hopf-actions", in which new operations $t\Delta$ and $h\Delta$ are introduced, with appropriate axioms.

Note that $v\mathcal{T}$ and $w\mathcal{T}$ have a meta-co-product, defined using "strand doubling". It is not meta-co-commutative.

Note also that $\mathcal{K}^{\mathrm{rbh}}$ and $\mathcal{K}^{\mathrm{rbh}}_0$ have operations $h\Delta$ and $t\Delta$, defined using "hoop doubling" and "balloon doubling". The former is meta-co-commutative while the latter is not.

Note also that $M$ and $M_0$ have have an operation $h\Delta^x_{yz}$ defined by cloning one Lie word, and an operation $t\Delta^u_{vw}$ defined using the substitution $u \to v + w$. Both of these operations are meta-co-commutative.

Thus $\zeta_0$ and $\zeta$ cannot be homomorphic with respect to $t\Delta$. The discussion of trivalent vertices in [5, Section 4] can be interpreted as an analysis of the failure of $\zeta$ to be homomorphic with respect to $t\Delta$, but this will not be attempted in this paper.

### 10.4 Some Differentials and the Proof of Proposition 5.1

We prove Proposition 5.1, namely (19) through (21), by verifying that each of these equations holds at one point, and then by differentiating each side of each equation and showing that the derivatives are equal. While routine, this argument appears complicated because the spaces involved are infinite dimensional and the operations involved are non-commutative. In fact, even the well-known derivative of the exponential function, which appears in the definition of $C_u$ which appears in the definitions of $RC_u$ and of $J_u$, may surprise readers who are used to the commutative case $de^x = e^x dx$.

Recall that *FA* denotes the graded completion of the free associative algebra on some alphabet $T$, and that the exponential map $\exp\colon FL \to FA$ defined by $\gamma \mapsto \exp(\gamma) = e^\gamma := \sum_{k=0}^{\infty} \frac{\gamma^k}{k!}$ makes sense in this completion.

**Lemma 10.3** *If $\delta\gamma$ denotes an infinitesimal variation of $\gamma$, then the infinitesimal variation $\delta e^\gamma$ of $e^\gamma$ is given as follows:*

$$\delta e^\gamma = e^\gamma \cdot \left(\delta\gamma \mathbin{/\!/} \frac{1-e^{-ad\gamma}}{ad\gamma}\right) = \left(\delta\gamma \mathbin{/\!/} \frac{e^{ad\gamma}-1}{ad\gamma}\right) \cdot e^\gamma. \tag{35}$$

Above expressions such as $\frac{e^{\mathrm{ad}\gamma}-1}{\mathrm{ad}\gamma}$ are interpreted via their power series expansions, $\frac{e^{\mathrm{ad}\gamma}-1}{\mathrm{ad}\gamma} = 1 + \frac{1}{2}\mathrm{ad}\gamma + \frac{1}{6}(\mathrm{ad}\gamma)^2 + \dots$, and hence $\delta\gamma \mathbin{/\!/} \frac{e^{\mathrm{ad}\gamma}-1}{\mathrm{ad}\gamma} = \delta\gamma + \frac{1}{2}[\gamma, \delta\gamma] + \frac{1}{6}[\gamma, [\gamma, \delta\gamma]] + \dots$. Also, the precise meaning of (35) is that for any $\delta\gamma \in FL$, the derivative $\delta e^\gamma := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(e^{\gamma+\epsilon\delta\gamma} - e^\gamma\right)$ is given by the right-hand-side of that equation. Equivalently, $\delta e^\gamma$ is the term proportional to $\delta\gamma$ in $e^{\gamma+\delta\gamma}$, where during calculations, we may assume that "$\delta\gamma$ is an infinitesimal", meaning that anything quadratic or higher in $\delta\gamma$ can be regarded as equal to 0.

Lemma 10.3 is rather standard (e.g. [8, Section 1.5], [22, Section 7]). Here's a tweet:

*Proof of Lemma 10.3* With an infinitesimal $\delta\gamma$, consider $F(s) := e^{-s\gamma}e^{s(\gamma+\delta\gamma)} - 1$. Then, $F(0) = 0$ and $\frac{d}{ds}F(s) = e^{-s\gamma}(-\gamma)e^{s(\gamma+\delta\gamma)} + e^{-s\gamma}(\gamma + \delta\gamma)e^{s(\gamma+\delta\gamma)} = e^{-s\gamma}\delta\gamma e^{s(\gamma+\delta\gamma)} = e^{-s\gamma}\delta\gamma e^{s\gamma} = \delta\gamma \mathbin{/\!/} e^{-s\mathrm{ad}\gamma}$. So $e^{-\gamma}\delta\gamma = F(1) = \int_0^1 ds \frac{d}{ds}F(s) = \delta\gamma \mathbin{/\!/} \int_0^1 ds\, e^{-s\mathrm{ad}\gamma} = \delta\gamma \mathbin{/\!/} \frac{1-e^{-\mathrm{ad}\gamma}}{\mathrm{ad}\gamma}$. The second part of (35) is proven in a similar manner, starting with $G(s) := e^{s(\gamma+\delta\gamma)}e^{-s\gamma} - 1$.

**Lemma 10.4** *If $\gamma = \mathrm{bch}(\alpha, \beta)$ and $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ are infinitesimals related by $\gamma + \delta\gamma = \mathrm{bch}(\alpha + \delta\alpha, \beta + \delta\beta)$, then*

$$\delta\gamma \mathbin{/\!/} \frac{1-e^{-ad\gamma}}{ad\gamma} = \left(\delta\alpha \mathbin{/\!/} \frac{1-e^{-ad\alpha}}{ad\alpha} \mathbin{/\!/} e^{-ad\beta}\right) + \left(\delta\beta \mathbin{/\!/} \frac{1-e^{-ad\beta}}{ad\beta}\right) \tag{36}$$

*Proof* Use Leibniz' law on $e^\gamma = e^\alpha e^\beta$ to get $\delta e^\gamma = (\delta e^\alpha)e^\beta + e^\alpha(\delta e^\beta)$. Now use Lemma 10.3 three times to get

$$e^\gamma\left(\gamma \mathbin{/\!/} \frac{1-e^{-\mathrm{ad}\gamma}}{\mathrm{ad}\gamma}\right) = e^\alpha\left(\delta\alpha \mathbin{/\!/} \frac{1-e^{-\mathrm{ad}\alpha}}{\mathrm{ad}\alpha}\right)e^\beta + e^\alpha e^\beta\left(\delta\beta \mathbin{/\!/} \frac{1-e^{-\mathrm{ad}\beta}}{\mathrm{ad}\beta}\right),$$

conjugate the $e^\beta$ in the first summand to the other side of the parenthesis, and cancel $e^\gamma = e^\alpha e^\beta$ from both sides of the resulting equation. □

Recall that $C_u^\gamma$ and $RC_u^\gamma$ are automorphisms of FL. We wish to study their variations $\delta C_u^\gamma$ and $\delta RC_u^\gamma$ with respect to $\gamma$ (these variations are "infinitesimal" automorphisms of FL). We need a definition and a property first.

**Definition 10.5** For $u \in T$ and $\gamma \in \mathrm{FL}(T)$ let $\mathrm{ad}_u\{\gamma\} = \mathrm{ad}_u^\gamma : \mathrm{FL}(T) \to \mathrm{FL}(T)$ denote the derivation of $\mathrm{FL}(T)$ defined by its action of the generators as follows:

$$v /\!\!/ \mathrm{ad}_u\{\gamma\} = v /\!\!/ \mathrm{ad}_u^\gamma := \begin{cases} [\gamma, u] & v = u \\ 0 & \text{otherwise.} \end{cases}$$

**Property 10.6** $\mathrm{ad}_u$ is the infinitesimal version of both $C_u$ and $RC_u$. Namely, if $\delta\gamma$ is an infinitesimal, then $C_u^{\delta\gamma} = RC_u^{\delta\gamma} = 1 + \mathrm{ad}_u\{\delta\gamma\}$.

We omit the easy proof of this property and move on to $\delta C_u^\gamma$ and $\delta RC_u^\gamma$:

**Lemma 10.7** *$\delta C_u^\gamma = ad_u \left\{ \delta\gamma /\!\!/ \frac{e^{ad\gamma} - 1}{ad\gamma} /\!\!/ RC_u^{-\gamma} \right\} /\!\!/ C_u^\gamma$ and $\delta RC_u^\gamma = RC_u^\gamma /\!\!/ ad_u \left\{ \delta\gamma /\!\!/ \frac{1 - e^{-ad\gamma}}{ad\gamma} /\!\!/ RC_u^\gamma \right\}$.*

*Proof* Substitute $\alpha$ and $\delta\beta$ into (16) and get $RC_u^{\mathrm{bch}(\alpha,\delta\beta)} = RC_u^\alpha /\!\!/ RC_u^{\delta\beta /\!\!/ RC_u^\alpha}$, and hence using Property 10.6 for the infinitesimal $\delta\beta /\!\!/ RC_u^\alpha$ and Lemma 10.4 with $\delta\alpha = \beta = 0$ on $\mathrm{bch}(\alpha, \delta\beta)$,

$$RC_u^{\alpha + (\delta\beta /\!\!/ \frac{\mathrm{ad}\alpha}{1 - e^{-\mathrm{ad}\alpha}})} = RC_u^\alpha + RC_u^\alpha /\!\!/ \mathrm{ad}_u\{\delta\beta /\!\!/ RC_u^\alpha\}.$$

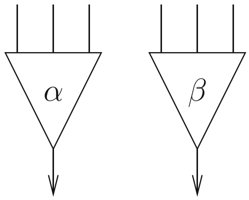
Now, replacing $\alpha \to \gamma$ and $\delta\beta \to \delta\gamma /\!\!/ \frac{1 - e^{-\mathrm{ad}\gamma}}{\mathrm{ad}\gamma}$, we get the equation for $\delta RC_u^\gamma$. The equation for $\delta C_U^\gamma$ now follows by taking the variation of $C_u^\gamma /\!\!/ RC_u^{-\gamma} = Id$. □

Our next task is to compute $\delta J_u(\gamma)$. Yet before we can do that, we need to know one of the two properties of $\mathrm{div}_u$ that matter for us (besides its linearity):

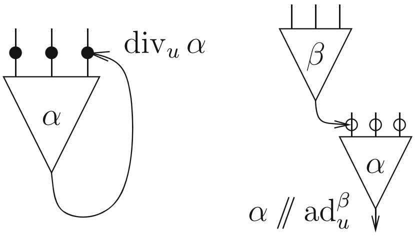
**Proposition 10.8** *For any $u, v \in T$ and any $\alpha, \beta \in FL$ and with $\delta_{uv}$ denoting the Kronecker delta function, the following "cocycle condition" holds: (compare with* [1, Proposition 3.20]*)*

$$\underbrace{(\mathrm{div}_u\alpha) /\!\!/ ad_v^\beta}_{A} - \underbrace{(\mathrm{div}_v\beta) /\!\!/ ad_u^\alpha}_{B} = \underbrace{\delta_{uv}\mathrm{div}_u[\alpha, \beta]}_{C} + \underbrace{\mathrm{div}_u(\alpha /\!\!/ ad_v^\beta)}_{D} - \underbrace{\mathrm{div}_v(\beta /\!\!/ ad_u^\alpha)}_{E}. \quad (37)$$

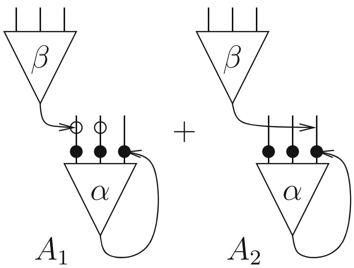
*Proof* Start with the case where $u = v$. We draw each contribution to each of the terms above and note that all of these contributions cancel, but we must first explain our drawing conventions. We draw $\alpha$ and $\beta$ as the "logic gates" appearing below. Each is really a linear combination, but (37) is bilinear so this doesn't matter. Each is really a tree, but the proof does not use this so we don't display this. Each may have many tail-legs labelled by other elements of $T$, but we care only about the legs labelled $u = v$ and so we display only those, and without real loss of generality, we draw it as if $\alpha$ and $\beta$ each have exactly three such tails.



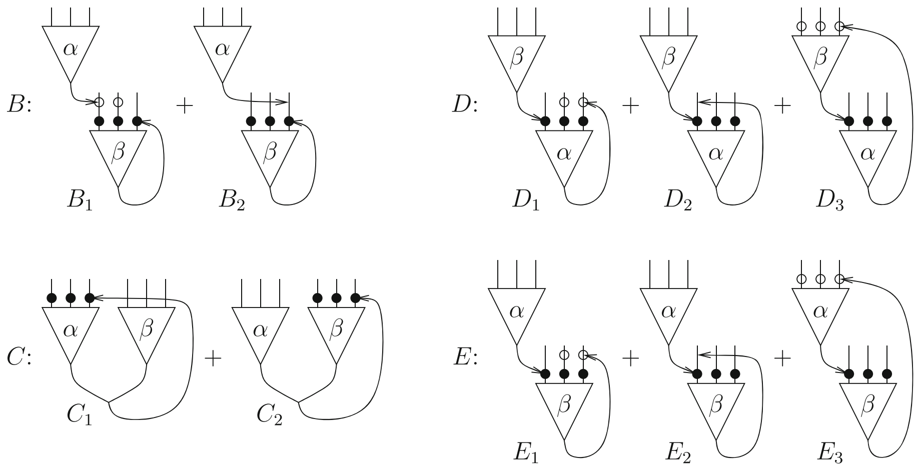

Objects such as $\mathrm{div}_u\alpha$ and $\alpha \mathbin{/\!/} \mathrm{ad}_u^\beta$ are obtained from $\alpha$ and $\beta$ by connecting the head of one near its own tails, or near the other's tails, in all possible ways. We draw just one summand from each sum, yet we indicate the other possible summands in each case by marking the other places where the relevant head could go with filled circles (•) or empty circles  (the filling of the circles has no algebraic meaning; it is there only to separate summations in cases where two summations appear in the same formula). I hope the pictures below explain this better than the words.




We illustrate our next convention with the pictorial representation of term $A$ of (37), $(\mathrm{div}_u\alpha) \mathbin{/\!/} \mathrm{ad}_u^\beta$, shown below. Namely, when the two relevant summations dictate that two heads may fall on the same arc, we split the sum into the generic part, $A_1$ below, in which the two heads do not fall on the same arc, and the exceptional part, $A_2$ below, in which the two heads do indeed fall on the same arc. The last convention is that • indicates the first summation, and ∘, the second. Hence in $A_1$, the $\alpha$ head may fall in three places, and after that, the $\beta$ head may only fall on one of the remaining relevant tails, whereas in $A_1$, the $\alpha$ is again free, but the $\beta$ head must fall on the same arc.

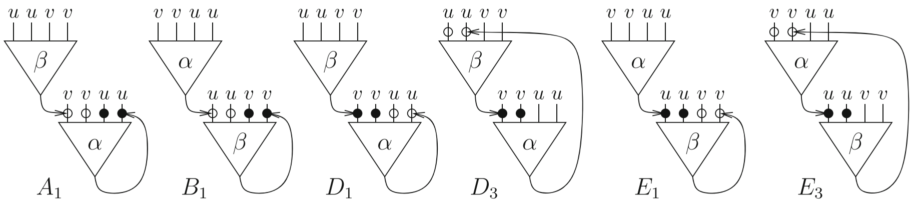


With all these conventions in place and with term $A$ as above, we depict terms $B$–$E$:




Clearly, $A_1 = D_1$, $B_1 = E_1$, and $D_3 = E_3$ (the last equality is the only place in this paper that we need the cyclic property of cyclic words). Also, by the Jacobi identity, $A_2 - D_2 = C_1$ and $E_2 - B_2 = C_2$. So altogether, $A - B = C + D - E$.

The case where $u \neq v$ is similar, except we have to separate between $u$ and $v$ tails, the terms analogous to $A_2$, $B_2$, $D_2$ and $E_2$ cannot occur, and $C = 0$:




Clearly, $A - B = D - E$. □

For completeness and for use within the proof of (21), here's the remaining property of div we need to know, presented without its easy proof:

**Proposition 10.9** *For any* $\gamma \in FL$, $\quad \gamma \mathbin{/\!/} t_w^{uv} \mathbin{/\!/} div_w = \gamma \mathbin{/\!/} div_u \mathbin{/\!/} t_w^{uv} + \gamma \mathbin{/\!/} div_v \mathbin{/\!/} t_w^{uv}$.

**Proposition 10.10** $\delta J_u(\gamma) = \delta\gamma \mathbin{/\!/} \dfrac{1 - e^{-ad\gamma}}{ad\gamma} \mathbin{/\!/} RC_u^{\gamma} \mathbin{/\!/} div_u \mathbin{/\!/} C_u^{-\gamma}$.

*Proof* Let $I_s := \gamma /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}$ denote the integrand in the definition of $J_u$. Then under $\gamma \to \gamma + \delta\gamma$, using Leibniz, the linearity of $\mathrm{div}_u$, and both parts of Lemma 10.7, we have

$$\begin{aligned}\delta I_s &= \delta\gamma /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad+\gamma /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{ad}_u \left\{\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma}\right\} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad-\gamma /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ \mathrm{ad}_u \left\{\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma}\right\} /\!\!/ C_u^{-s\gamma}.\end{aligned}$$

Taking the last two terms above as $D$ and $A$ of (37), with $\alpha = \gamma /\!\!/ RC_u^{s\gamma}$ and $\beta = \delta\gamma /\!\!/ \frac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma}$, and using $[\alpha,\beta] = [\gamma, \delta\gamma /\!\!/ \frac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma}] /\!\!/ RC_u^{s\gamma} = \delta\gamma /\!\!/ (1-e^{-\mathrm{ad}s\gamma}) /\!\!/ RC_u^{s\gamma}$, we get

$$\begin{aligned}\delta I_s &= \delta\gamma /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad+\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{ad}_u\{\gamma /\!\!/ RC_u^{s\gamma}\} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad-\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ \mathrm{ad}_u\{\gamma /\!\!/ RC_u^{s\gamma}\} /\!\!/ C_u^{-s\gamma}\\ &\quad-\delta\gamma /\!\!/ (1-e^{-\mathrm{ad}s\gamma}) /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma},\end{aligned}$$

and so, by combining the first and the last terms above,

$$\begin{aligned}\delta I_s &= \delta\gamma /\!\!/ e^{-\mathrm{ad}s\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad+\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{ad}_u\{\gamma /\!\!/ RC_u^{s\gamma}\} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\\ &\quad-\delta\gamma /\!\!/ \tfrac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ \mathrm{ad}_u\{\gamma /\!\!/ RC_u^{s\gamma}\} /\!\!/ C_u^{-s\gamma},\end{aligned}$$

and hence, once again using Lemma 10.7 to differentiate $RC_u^{s\gamma}$ and $C_u^{-s\gamma}$ (except that things are now simpler because $s\gamma$ and $\delta(s\gamma) = \frac{d}{ds}(s\gamma) = \gamma$ commute), we get

$$\delta I_s = \frac{d}{ds}\left(\delta\gamma /\!\!/ \frac{1-e^{-\mathrm{ad}s\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{s\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-s\gamma}\right).$$

Integrating with respect to the variable $s$ and using the fundamental theorem of calculus, we are done. □

*Proof of Equation* (19). We fix $\alpha$ and show that (19) holds for every $\beta$. For this it is enough to show that (19) holds for $\beta = 0$ (it trivially does), and that the derivatives of both sides of (19) in the radial direction are equal, for any given $\beta$. Namely, it is enough to verify that the variations of the two sides of (19) under $\beta \to \beta + \delta\beta$ are equal, where $\delta\beta$ is proportional to $\beta$. Indeed, using the chain rule, Lemma 10.4, Proposition 10.10, the fact that $\beta$ commutes with $\delta\beta$, and with $\gamma := \mathrm{bch}(\alpha,\beta)$,

$$\begin{aligned}\delta LHS &= \left(\delta\beta /\!\!/ \tfrac{1-e^{-\mathrm{ad}\beta}}{\mathrm{ad}\beta} /\!\!/ \tfrac{\mathrm{ad}\gamma}{1-e^{-\mathrm{ad}\gamma}}\right) /\!\!/ \tfrac{1-e^{-\mathrm{ad}\gamma}}{\mathrm{ad}\gamma} /\!\!/ RC_u^{\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\gamma}\\ &= \delta\beta /\!\!/ RC_u^{\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\gamma}.\end{aligned}$$

Similarly, using Proposition 10.10 and the fact that $\beta /\!\!/ RC_u^{\alpha}$ commutes with $\delta\beta /\!\!/ RC_u^{\alpha}$,

$$\delta RHS = \delta\beta /\!\!/ RC_u^{\alpha} /\!\!/ RC_u^{\beta /\!\!/ RC_u^{\alpha}} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\beta /\!\!/ RC_u^{\alpha}} /\!\!/ C_u^{-\alpha} = \delta\beta /\!\!/ RC_u^{\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\gamma},$$

where in the last equality, we have used (16) to combine the $RC$s and its inverse to combine the $C$s. □

*Proof of Equation* (20). Equation (20) clearly holds when $\alpha = 0$, so as before, it is enough to prove it after taking the radial derivative with respect to $\alpha$. So we need (ouch!)

$$\begin{aligned}
&\alpha /\!\!/ RC_u^\alpha /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\alpha} - \alpha /\!\!/ RC_v^\beta /\!\!/ RC_u^{\alpha /\!\!/ RC_v^\beta} /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\alpha /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} \\
&= -\beta /\!\!/ RC_u^\alpha /\!\!/ \mathrm{ad}_u^{\alpha /\!\!/ RC_u^\alpha} /\!\!/ \frac{1 - e^{-\mathrm{ad}(\beta /\!\!/ RC_u^\alpha)}}{\mathrm{ad}(\beta /\!\!/ RC_u^\alpha)} /\!\!/ RC_v^{\beta /\!\!/ RC_u^\alpha} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta /\!\!/ RC_u^\alpha} /\!\!/ C_u^{-\alpha} \\
&\qquad - \beta /\!\!/ RC_u^\alpha /\!\!/ J_v /\!\!/ \mathrm{ad}_u^{-\alpha /\!\!/ RC_u^\alpha} /\!\!/ C_u^{-\alpha}.
\end{aligned}$$

This we simplify using (13) and (14), cancel the $C_u^{-\alpha}$ on the right, and get

$$\begin{aligned}
&\alpha /\!\!/ RC_u^\alpha /\!\!/ \mathrm{div}_u - \alpha /\!\!/ RC_u^\alpha /\!\!/ RC_v^{\beta /\!\!/ RC_u^\alpha} /\!\!/ \mathrm{div}_u /\!\!/ C_v^{-\beta /\!\!/ RC_u^\alpha} \\
&\stackrel{?}{=} -\beta /\!\!/ RC_u^\alpha /\!\!/ \mathrm{ad}_u^{\alpha /\!\!/ RC_u^\alpha} /\!\!/ \frac{1 - e^{-\mathrm{ad}(\beta /\!\!/ RC_u^\alpha)}}{\mathrm{ad}(\beta /\!\!/ RC_u^\alpha)} /\!\!/ RC_v^{\beta /\!\!/ RC_u^\alpha} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta /\!\!/ RC_u^\alpha} \\
&\qquad - \beta /\!\!/ RC_u^\alpha /\!\!/ J_v /\!\!/ \mathrm{ad}_u^{-\alpha /\!\!/ RC_u^\alpha}.
\end{aligned}$$

We note that above $\alpha$ and $\beta$ only appear within the combinations $\alpha /\!\!/ RC_u^\alpha$ and $\beta /\!\!/ RC_u^\alpha$, so we rename $\alpha /\!\!/ RC_u^\alpha \to \alpha$ and $\beta /\!\!/ RC_u^\alpha \to \beta$:

$$\begin{aligned}
&\alpha /\!\!/ \mathrm{div}_u - \alpha /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_u /\!\!/ C_v^{-\beta} \\
&\stackrel{?}{=} -\beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \frac{1 - e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta} - \beta /\!\!/ J_v /\!\!/ \mathrm{ad}_u^{-\alpha}.
\end{aligned} \tag{38}$$

Equation (38) still contains a $J_v$ in it, so in order to prove it, we have to differentiate once again. So note that it holds at $\beta = 0$, multiply by $-1$, and take the radial variation with respect to $\beta$ (note that $\frac{d}{ds} \left. \frac{1-e^{-\mathrm{ad}(s\beta)}}{\mathrm{ad}(s\beta)} \right|_{s=1} = \frac{e^{-\mathrm{ad}(\beta)}(1 + \mathrm{ad}(\beta) - e^{\mathrm{ad}(\beta)})}{\mathrm{ad}(\beta)}$):

$$\begin{aligned}
&\alpha /\!\!/ RC_v^\beta /\!\!/ \mathrm{ad}_v^{\beta /\!\!/ RC_v^\beta} /\!\!/ \mathrm{div}_u /\!\!/ C_v^{-\beta} - \alpha /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_u /\!\!/ \mathrm{ad}_v^{\beta /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} \\
&\stackrel{?}{=} \beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \tfrac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta} \\
&\quad + \beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \tfrac{e^{-\mathrm{ad}(\beta)}(1 + \mathrm{ad}(\beta) - e^{\mathrm{ad}(\beta)})}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta} \\
&\quad + \beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \tfrac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta /\!\!/ \mathrm{ad}_v^{\beta /\!\!/ RC_v^\beta} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\beta} \\
&\quad + \beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \tfrac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_v /\!\!/ \mathrm{ad}_v^{-\beta /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} \\
&\quad + \beta /\!\!/ RC_v^\beta /\!\!/ \mathrm{div}_v /\!\!/ CC_v^{-\beta} /\!\!/ \mathrm{ad}_u^{-\alpha}
\end{aligned} \tag{39}$$

We massage three independent parts of the above desired equality at the same time:

- The div and the ad on the left hand side make terms $D$ and $A$ of (37), with $\alpha /\!\!/ RC_v^\beta \to \alpha$ and $\beta /\!\!/ RC_v^\beta \to \beta$. We replace them by terms $A$ and $E$.
- We combine the first two terms of the right hand side using $\frac{1-e^{-a}}{a} + \frac{e^{-a}(1+a-e^a)}{a} = e^{-a}$.
- In (14), $C_u^{-\alpha /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} = C_v^{-\beta /\!\!/ RC_u^\alpha} /\!\!/ C_u^{-\alpha}$, take an infinitesimal $\alpha$ and use Property 10.6 and Lemma 10.7 to get

$$\mathrm{ad}_u^{-\alpha /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} = \mathrm{ad}_v^{-\beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} + C_v^{-\beta} /\!\!/ \mathrm{ad}_u^{-\alpha}. \tag{40}$$

  The last of that matches the last of (39), so we can replace the last of (39) with the start of (40).

All of this done, (39) becomes the lowest point of this paper:

$$\begin{aligned}
\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} C_v^{-\beta} - \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} C_v^{-\beta} \\
\overset{?}{=} \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} e^{-\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} C_v^{-\beta} \\
+ \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \tfrac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} C_v^{-\beta} \\
+ \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \tfrac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{-\beta \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} C_v^{-\beta} \\
+ \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_u^{-\alpha \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} C_v^{-\beta} \\
- \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{-\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} C_v^{-\beta}
\end{aligned}$$

Next, we cancel the $C_v^{-\beta}$ at the right of every term, and a pair of repeating terms to get

$$\begin{aligned}
\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v \overset{?}{=} \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} e^{-\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \\
+ \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v \\
- \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} RC_v^\beta} \\
- \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{-\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta}
\end{aligned}$$

The two middle terms above differ only in the order of $\mathrm{ad}_v$ and $\mathrm{div}_v$. So we apply (37) again and get

$$\begin{aligned}
&\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v \overset{?}{=} \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} e^{-\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \\
&+\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta} \mathbin{/\!\!/} \mathrm{div}_v - \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta} \\
&+ \left[ \beta \mathbin{/\!\!/} RC_v^\beta, \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \right] \mathbin{/\!\!/} \mathrm{div}_v - \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{div}_v \mathbin{/\!\!/} \mathrm{ad}_v^{-\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta}
\end{aligned}$$

In the above, the two terms that do not end in $\mathrm{div}_v$ cancel each other. We then remove the $\mathrm{div}_v$ at the end of all remaining terms, thus making our quest only harder. Finally, we note that $RC_v^\beta$ is a Lie algebra morphism, so we can pull it out of the bracket in the penultimate term, getting

$$\begin{aligned}
\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \overset{?}{=} \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} e^{-\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta \\
+\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta} + \left[ \beta, \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \right] \mathbin{/\!\!/} RC_v^\beta
\end{aligned}$$

The bracketing with $\beta$ in the last term above cancels the $\mathrm{ad}(\beta)$ denominator there, and then that term combines with the first term of the right hand side to yield

$$\beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_u^{\alpha \mathbin{/\!\!/} RC_v^\beta} \overset{?}{=} \beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} RC_v^\beta + \beta \mathbin{/\!\!/} RC_v^\beta \mathbin{/\!\!/} \mathrm{ad}_v^{\beta \mathbin{/\!\!/} \mathrm{ad}_u^\alpha \mathbin{/\!\!/} \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} \mathbin{/\!\!/} RC_v^\beta}$$

We make our task harder again,

$$RC_v^\beta /\!\!/ \mathrm{ad}_u^{\alpha /\!\!/ RC_v^\beta} \overset{?}{=} \mathrm{ad}_u^\alpha /\!\!/ RC_v^\beta + RC_v^\beta /\!\!/ \mathrm{ad}_v^{\beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta}$$

and then we both pre-compose and post-compose with the isomorphism $C_v^{-\beta}$, getting

$$\mathrm{ad}_u^{\alpha /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta} \overset{?}{=} C_v^{-\beta} /\!\!/ \mathrm{ad}_u^\alpha + \mathrm{ad}_v^{\beta /\!\!/ \mathrm{ad}_u^\alpha /\!\!/ \frac{1-e^{-\mathrm{ad}(\beta)}}{\mathrm{ad}(\beta)} /\!\!/ RC_v^\beta} /\!\!/ C_v^{-\beta}$$

The above is (40), with $\alpha$ replaced by $-\alpha$, and hence it holds true. □

*Proof of Equation* (21). As before, the equation clearly holds at $\gamma = 0$, so we take its radial derivative. That of the left hand side is

$$\gamma /\!\!/ tm_w^{uv} /\!\!/ RC_w^{\gamma /\!\!/ tm_w^{uv}} /\!\!/ \mathrm{div}_w /\!\!/ C_w^{-\gamma /\!\!/ tm_w^{uv}}$$

Using (15) and then Proposition 10.9, this becomes

$$\gamma /\!\!/ RC_u^\gamma /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ (\mathrm{div}_u + \mathrm{div}_v) /\!\!/ tm_w^{uv} /\!\!/ C_w^{-\gamma /\!\!/ tm_w^{uv}}.$$

Now using the reverse of (15), proven by reading the horizontal arrows within its proof backwards, this becomes

$$\gamma /\!\!/ RC_u^\gamma /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ (\mathrm{div}_u + \mathrm{div}_v) /\!\!/ C_v^{-\gamma /\!\!/ RC_u^\gamma} /\!\!/ C_u^{-\gamma} /\!\!/ tm_w^{uv}.$$

On the other hand, the radial variation of the right hand side of (21) is

$$\begin{aligned}\gamma /\!\!/ RC_u^\gamma /\!\!/ \mathrm{div}_u /\!\!/ C_u^{-\gamma} /\!\!/ tm_w^{uv} + \gamma /\!\!/ RC_u^\gamma /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\gamma /\!\!/ RC_u^\gamma} /\!\!/ C_u^{-\gamma} /\!\!/ tm_w^{uv} \\ + \gamma /\!\!/ RC_u^\gamma /\!\!/ \mathrm{ad}_u^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \frac{1-e^{-\mathrm{ad}(\gamma /\!\!/ RC_u^\gamma)}}{\mathrm{ad}(\gamma /\!\!/ RC_u^\gamma)} /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\gamma /\!\!/ RC_u^\gamma} /\!\!/ C_u^{-\gamma} /\!\!/ t_w^{uv} \\ + \gamma /\!\!/ RC_u^\gamma /\!\!/ J_v /\!\!/ \mathrm{ad}_u^{-\gamma /\!\!/ RC_u^\gamma} /\!\!/ C_u^{-\gamma} /\!\!/ t_w^{uv}\end{aligned}$$

Equating the last two formulae while eliminating the common term (the second term in each) and removing all trailing $C_u^{-\gamma} /\!\!/ t_w^{uv}$'s (thus making the quest harder), we need to show that

$$\begin{aligned}\gamma /\!\!/ RC_u^\gamma /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \mathrm{div}_u /\!\!/ C_v^{-\gamma /\!\!/ RC_u^\gamma} = \gamma /\!\!/ RC_u^\gamma /\!\!/ \mathrm{div}_u \\ + \gamma /\!\!/ RC_u^\gamma /\!\!/ \mathrm{ad}_u^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \frac{1-e^{-\mathrm{ad}(\gamma /\!\!/ RC_u^\gamma)}}{\mathrm{ad}(\gamma /\!\!/ RC_u^\gamma)} /\!\!/ RC_v^{\gamma /\!\!/ RC_u^\gamma} /\!\!/ \mathrm{div}_v /\!\!/ C_v^{-\gamma /\!\!/ RC_u^\gamma} \\ + \gamma /\!\!/ RC_u^\gamma /\!\!/ J_v /\!\!/ \mathrm{ad}_u^{-\gamma /\!\!/ RC_u^\gamma}.\end{aligned}$$

Nicely enough, the above is (38) with $\alpha = \beta = \gamma /\!\!/ RC_u^\gamma$. □

## 10.5 Notational Conventions and Glossary

For $n \in \mathbb{N}$ let $\underline{n}$ denote some fixed set with $n$ elements, say $\{1, 2, \ldots, n\}$.

Often, within this paper, we use postfix notation for operator evaluations, so $f(x)$ may also be denoted $x /\!\!/ f$. Even better, we use $f /\!\!/ g$ for "composition done right", meaning $f /\!\!/ g = g \circ f$, meaning that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $X \xrightarrow{f /\!\!/ g} Z$ rather than the uglier (though equally correct) $X \xrightarrow{g \circ f} Z$. We hope that this notation will be adopted by others, to be used alongside and eventually instead of $g \circ f$, much as we hope that $\tau$ will be used alongside and eventually instead of the presently popular $\pi := \tau/2$. In LaTeX, $/\!\!/ =$ `\sslash` $\in$ `stmaryrd.sty`.

In the few paragraphs that follow, $X$ is an arbitrary set. Though within this paper such $X$'s will usually be finite, and their elements will thought of as labels. Hence, if $f \in G^X$ is a function $f: X \to G$ where $G$ is some other set, we think of $f$ as a collection of elements of $G$ labelled by the elements of $X$. We often write $f_x$ to denote $f(x)$.

If $f \in G^X$ and $x \in X$, we let $f\backslash x$ denote the restricted function $f|_{X\backslash x}$ in which $x$ is removed from the domain of $f$. In other words, $f\backslash x$ is "the collection $f$, with the element labelled $x$ removed". We often neglect to state the condition $x \in X$. Thus, when writing $f\backslash x$ we implicitly assume that $x \in X$.

Likewise, we write $f\backslash\{x, y\}$ for "$f$ with $x$ and $y$ removed from its domain" and as before this includes the implicit assumption that $\{x, y\} \subset X$.

If $f_1: X_1 \to G$ and $f_2: X_2 \to G$ and $X_1$ and $X_2$ are disjoint, we denote by $f \cup g$ the obvious "union function" with domain $X_1 \cup X_2$ and range $G$. In fact, whenever we write $f \cup g$, we make the implicit assumption that the domains of $f_1$ and $f_2$ are disjoint.

In the spirit of "associative arrays" as they appear in various computer languages, we use the notation $(x \to a, y \to b, \ldots)$ for "inline function definition". Thus, () is the empty function, and if $f = (x \to a, y \to b)$, then the domain of $f$ is $\{x, y\}$ and $f_x = a$ and $f_y = b$.

We denote by $\sigma^x_y$ the operation that renames the key $x$ in an associative array to $y$. Namely, if $f \in G^X$, $x \notin X$, and $y \notin X\backslash x$, then

$$\sigma^x_y f = (f\backslash x) \cup (y \to f_x).$$

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## Glossary of Notations (Greek letters, then Latin, then symbols)

| | | |
|---|---|---|
| $\alpha, \beta, \gamma$ | Free Lie series | Sec. 4 |
| $\alpha, \beta, \gamma, \delta$ | Matrix parts | Sec. 9.4 |
| $\beta$ | A repackaging of $\beta$ | Sec. 9.4 |
| $\beta_0$ | A reduction of $M$ | Sec. 9.3 |
| $\delta$ | A map $u\mathcal{T}/v\mathcal{T}/w\mathcal{T} \to \mathcal{K}^{\text{rbh}}$ | Sec. 2.2 |
| $\delta\alpha, \delta\beta, \delta\gamma$ | Infinitesimal free Lie series | Sec. 10.4 |
| $\epsilon_a$ | Units | Sec. 3.2 |
| $\Pi$ | The MMA "of groups" | Sec. 3.4 |
| $\pi$ | The fundamental invariant | Sec. 2.3 |
| $\pi$ | The projection $\mathcal{K}^{\text{rbh}}_0 \to \mathcal{K}^{\text{rbh}}$ | Prop. 3.6 |
| $\rho^{\pm}_{ux}$ | $\pm$-Hopf links in 4D | Ex. 2.2 |
| $\sigma^x_y$ | Re-labelling | Sec. 10.5 |
| $\tau$ | Tensorial interpretation map | Sec. 8.1 |
| $\omega$ | The wheels part of $M/\zeta$ | Sec.5 |
| $\omega$ | The scalar part in $\beta/\beta_0$ | Sec. 9.3 |
| $\Upsilon$ | Capping and sliding | Sec.10.2 |
| $\zeta$ | The main invariant | Sec. 5 |

| | | |
|---|---|---|
| $\zeta_0$ | The tree-level invariant | Sec. 4 |
| $\zeta^\beta$ | A $\beta$-valued invariant | Sec. 9.4 |
| $\zeta^{\beta_0}$ | A $\beta_0$-valued invariant | Sec. 9.3 |
| $A$ | The matrix part in $\beta/\beta_0$ | Sec. 9.3 |
| $a, b, c$ | Strand labels | Sec. 2.2 |
| $\mathrm{ad}_u^\gamma$, $\mathrm{ad}_u\{\gamma\}$ | Derivations of FL | Def. 105 |
| $\mathcal{A}^{\mathrm{bh}}$ | Space of arrow diagrams | Sec. 7.2 |
| bch | Baker-Campbell-Hausdorff | Sec. 4.2 |
| $C_u^\gamma$ | Conjugating a generator | Sec. 4.2 |
| CA | Circuit algebra | Sec. 7.1 |
| CW | Cyclic words | Sec. 5.1 |
| $\mathrm{CW}^r$ | CW mod degree 1 | Sec. 5.1 |
| $c$ | A “sink” vertex | Sec. 9.1 |
| $c_u$ | A “$c$-stub” | Sec. 9.1 |
| $\mathrm{div}_u$ | The “divergence” $FL \to CW$ | Sec. 5.1 |
| $dm_c^{ab}$ | Double/diagonal multiplication | Sec. 3.2 |
| FA | Free associative algebra | Sec. 5.1 |
| FL | Free Lie algebra | Sec. 4.2 |
| $\mathrm{Fun}(X \to Y)$ | Functions $X \to Y$ | Sec. 8.1 |
| $H$ | Set of head/hoop labels | Sec. 2 |
| $h\epsilon_x$ | Units | Ex. 2.2, Sec. 4.2,5.2 |
| $h\eta$ | Head delete | Sec. 3,4.2,5.2 |
| $hm_z^{xy}$ | Head multiply | Sec. 3,4.2,5.2 |
| $h\sigma_y^x$ | Head re-label | Sec. 3,4.2,5.2 |
| $J_u$ | The “spice” FL $\to$ CW | Sec. 5.1 |
| $\mathcal{K}^{\mathrm{rbh}}$ | All rKBHs | Def. 2.1 |
| $\mathcal{K}_0^{\mathrm{rbh}}$ | Conjectured version of $\mathcal{K}^{\mathrm{rbh}}$ | Sec. 3.3 |
| $l_{ux}$ | 4D linking numbers | Sec. 10.1 |
| $l_x$ | Longitudes | Sec. 2.3 |
| $M$ | The “main” MMA | Sec. 5.2 |
| $M_0$ | The MMA of trees | Sec. 4.2 |
| MMA | Meta-monoid-action | Def. 3.2, Sec. 10.3.4 |
| $m_u$ | Meridians | Sec. 2.3 |
| $m_c^{ab}$ | Strand concatenation | Sec 3.2 |
| OC | Overcrossings commute | Fig. 3 |
| $\mathcal{P}^{\mathrm{bh}}$ | Primitives of $\mathcal{A}^{\mathrm{bh}}$ | Sec. 7.3 |
| $R$ | Ring of $c$-stubs | Sec. 9.2 |
| $R^r$ | $R$ mod degree 1 | Sec. 9.3 |
| R1,R1',R2,R3 | Reidemeister moves | Sec. 2.2, 7.1 |
| $RC_u^\gamma$ | Repeated $C_u^\gamma$ / reverse $C_u^{-\gamma}$ | Sec. 4.2 |
| rKBH | Ribbon knotted balloons&hoops | Def. 2.1 |
| $S$ | Set of strand labels | Sec. 2.2 |
| $T$ | Set of tail / balloon labels | Sec. 2 |
| $t\epsilon^u$ | Units | Ex. 2.2, Sec. 4.2,5.2 |
| $tha^{ux}$ | Tail by head action | Sec. 3,4.2,5.2 |
| $t\eta^u$ | Tail delete | Sec. 3,4.2,5.2 |
| $tm_w^{uv}$ | Tail multiply | Sec. 3,4.2,5.2 |
| $t\sigma_y^x$ | Tail re-label | Sec. 3,4.2,5.2 |

| | | |
|---|---|---|
| $t, x, y, z$ | Coordinates | Sec. 2 |
| UC | Undercrossings commute | Fig. 3 |
| u-tangle | A usual tangle | Sec. 2.2 |
| $u\mathcal{T}$ | All u-tangles | Sec. 2.2 |
| $u, v, w$ | Tail / balloon labels | Sec. 2 |
| v-tangle | A virtual tangle | Sec. 2.4 |
| $v\mathcal{T}$ | All v-tangles | Sec. 2.4 |
| w-tangle | A virtual tangle mod OC | Sec. 2.4 |
| $w\mathcal{T}$ | All w-tangles | Sec. 2.4 |
| $x, y, z$ | Head / hoop labels | Sec. 2 |
| $Z^{\mathrm{bh}}$ | An $\mathcal{A}^{\mathrm{bh}}$-valued expansion | Sec. 7.4 |
| $*$ | Merge operation | Sec. 3,4.2,5.2 |
| $/\!\!/$ | Composition done right | Sec. 10.5 |
| $x /\!\!/ f$ | Postfix evaluation | Sec. 10.5 |
| $f \backslash x$ | Entry removal | Sec. 10.5 |
| $x \to a$ | Inline function definition | Sec. 10.5 |
| $\overline{uv}$ | "Top bracket form" | Sec. 6 |
| $\overbrace{uv}$ | A cyclic word | Sec. 6 |

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