A Missing Link Between 2D and 4D — Proposed Research at the University of Sydney, 2019

Executive Summary. In [WKO] Zsuzsanna Dancso of the University of Sydney and I found a surprising relation between the topology of certain knotted objects in $\mathbb{R}^4$ and the "Kashiwara-Vergne" (KV) equations, which first arose in the context of harmonic analysis on Lie groups [KV]. In [AKKN], building on the same basic principle of "homomorphic expansions" yet applying that principle to completely different objects, Alekseev, Kawazumi, Kuno, and Naef found that the same KV equations are also related to the "Goldman-Turaev Lie bialgebra", a decidedly 2D construct. There has to be a direct way, not via the complicated Kashiwara-Vergne equations, to see that the Goldman-Turaev Lie bialgebra is related to knotted objects in $\mathbb{R}^4$, but as of yet, we don't know what that direct way is. I hope that a long visit by myself to Sydney will allow Dr. Dancso and myself to uncover that missing link.

In More Words. A recurring theme in mathematics is that finding a filtration on an object makes it easier to study that object. If K is a vector space, for example, a descending filtration on K is a sequence of vector spaces $K = K_0 \supset K_1 \supset K_2 \supset K_3 \cdots$. We think of K_3 as "that part of K whose study we are willing to postpone by 3 days", and of K_{14} as "that part of K whose study we are willing to delay by a whole fortnight". With this in mind we end up spreading the study of K over our whole sabbatical and perhaps more, and in day n we only need to study $A_n := K_n/K_{n+1}$. In good cases, even if K is a space of intricate things, each A_n is a space of simpler things, and our process of systematic procrastination pays off. One famous example is the case of homology — in that case K is additionally equipped with a differential d that is appropriately compatible with the filtration, and careful procrastination – venturing also towards K_n/K_{n+r} – leads one to the study of "spectral sequences".

The K's we care about are a bit different — they are spaces of linear combinations of topological things, like braids or knots or 2-knots or curves on surfaces. They carry some algebraic structures that arises from operations defined on topological things: concatenations, doubling operations, intersections, etc. They carry filtrations that resemble the filtration of a group ring by powers of its augmentation ideal. And the following question always arises:

> To what extent does the study of K resemble the sum total of all of our daily chores, namely the study of the "associated graded" space $A := \prod A_n = \prod K_n/K_{n+1}$?

A more precise formulation is "is there a homomorphic expansion, an appropriately non-degenerate map $Z\colon K \to A$ that preserves all available algebraic structure"? (Fully precise formulations are available too; for example, in [WKO]). Let us highlight two specific examples:

Knotted Objects in 4D. Here K is a certain space of knotted surfaces in $\mathbb{R}^4$ (some conditions apply, some singularities allowed). The corresponding associated graded space is a space whose basic ingredients are the free Lie algebra $FL(S)$ and the vector space of cyclic words $CW(S)$ on some alphabet S, and the construction of a homomorphic expansion turns out to be equivalent to a solution of the Kashiwara-Vergne problem [KV]: Find $f, g \in FL(x, y)$ so that

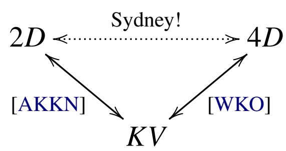
$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})f + (e^{\operatorname{ad} y} - 1)g, \quad (1)$$

$$\frac{1}{2}\operatorname{tr}_u\left(\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x} - 1} + \frac{\operatorname{ad} x}{e^{\operatorname{ad} x} - 1} - \frac{\operatorname{ad}\operatorname{BCH}(x,y)}{e^{\operatorname{ad}\operatorname{BCH}(x,y)} - 1}\right)(u)\right) = \operatorname{div}_x f + \operatorname{div}_y g. \quad (2)$$

The details are in [WKO]. (The KV statement is about convolutions on Lie groups and algebras, even if this is not visible above).

The "2D" Goldman Turaev Lie Bialgebra. Here K is the space of formal linear combinations of homotopy classes of curves on a surface Σ, or alternatively, of conjugacy classes in $\pi_1(\Sigma)$. This space carries the "Goldman bracket", defined on pairs of curves by mapping such a pair to the signed sum – over all of their intersection points – of the "smoothing" corresponding to such an intersection: ⤫ $\mapsto \pm$)(. A similar definition using self-intersections defines the "Turaev co-bracket" on K (some intricacies suppressed). The corresponding associated graded space is now related to the free associative algebra $FA(S)$ and the space of cyclic words $CW(S)$, and by what may appear to be a miracle, the problem of finding a homomorphic expansion is equivalent to the same Kashiwara-Vergne problem, (1) and (2). The details are in [AKKN].

What We Plan. We don't believe in miracles, and so we believe that there must be a direct explanation for the appearance of the Kashiwara-Vergne equations in both 2D and 4D topology. There must be some direct link completing the triangle on the right. Zsuzsanna Dancso and I hope to find it during my visit to Sydney in September-October 2019.



[AKKN] A. Alekseev, N. Kawazumi, Y. Kuno, and F. Naef, The Goldman-Turaev Lie bialgebra in Genus Zero and the Kashiwara-Vergne Problem, Adv. in Math. **326** (2018) 1–53, arXiv:1703.05813.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[WKO] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956.

D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem,* Math. Ann. **367** (2017) 1517–1586, arXiv:1405.1955.

D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects III: Double Tree Construction,* in preparation, ωεβ/wko3.

D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects IV: Some Computations,* in preparation, ωεβ/wko4, arXiv:1511.05624.