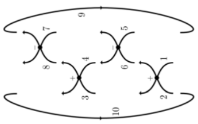
(ב) | Dror Bar-Natan:

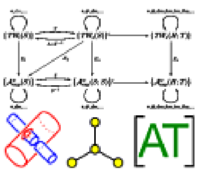
Publications

(in approximate reverse chronological order)


arXiv:1708.04853

A Polynomial Time Knot Polynomial (joint with Roland van der Veen, 21 pages, posted August 2017, to appear in the Proceedings of the American Mathematical Society).

We present the strongest known knot invariant that can be computed effectively (in polynomial time).


WKO4.pdf
pensieve
`FreeLie.m`
`AwCalculus.m`

Finite Type Invariants of w-Knotted Objects IV: Some Computations (49 pages, posted November 2015, arXiv:1511.05624)

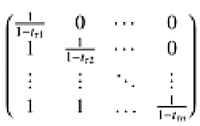
In the previous three papers in this series, [WKO1]-[WKO3], Z. Dancso and I studied a certain theory of "homomorphic expansions" of "w-knotted objects", a certain class of knotted objects in 4-dimensional space. When all layers of interpretation are stripped off, what remains is a study of a certain number of equations written in a family of spaces $\mathcal{A}^w$, closely related to degree-completed free Lie algebras and to degree-completed spaces of cyclic words.

The purpose of this paper is to introduce mathematical and computational tools that enable explicit computations (up to a certain degree) in these $\mathcal{A}^w$ spaces and to use these tools to solve the said equations and verify some properties of their solutions, and as a consequence, to carry out the computation (up to a certain degree) of certain knot-theoretic invariants discussed in [WKO1]-[WKO3] and in my related paper [KBH].


OnOnInvariants.pdf
pensieve

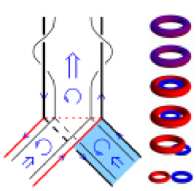
On Raoul Bott's "On Invariants of Manifold" (2 pages, posted August 2015, to appear in Bott's collected works, vol. 5)

I'm not sure how to introduce a review paper. So rather than commenting on the paper as whole, I will concentrate on my subjective view of just one paragraph - a paragraph which I think I influenced and which ended up influencing me very deeply.


UofG.pdf
pensieve

A Note on the Unitarity Property of the Gassner Invariant (3 pages, posted June 2014, updated August 2014, Bulletin of Chelyabinsk State University (Mathematics, Mechanics, Informatics) **3-358-17** (2015) 22-25, arXiv:1406.7632)

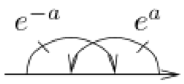
We give a 3-page description of the Gassner invariant (or representation) of braids (or pure braids), along with a description and a proof of its unitarity property.


WKO2.pdf
pensieve

Finite Type Invariants of w-Knotted Objects II: Tangles, Foams and the Kashiwara-Vergne Problem (joint with Zsuzsanna Dancso, 57 pages, posted May 2014, updated October 2015, Mathematische Annalen **367** (2017) 1517-1586, partially replaces WKO, arXiv:1405.1955)

This is the second in a series of papers dedicated to studying w-knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.). These are classes of knotted objects that are wider but weaker than their "usual" counterparts. To get (say) w-knots from usual knots (or u-knots), one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation beyond the ordinary collection of Reidemeister moves, called the "overcrossings commute" relation, making w-knotted objects a bit weaker once again. Satoh studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbf{R}^4$.

In this article we study finite type invariants of w-tangles and w-trivalent graphs (also referred to as w-tangled foams). Much as the spaces A of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, the spaces A^w of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-foams is essentially the same as a solution of the Kashiwara-Vergne conjecture and much of the Alekseev-Torossian work on Drinfel'd associators and Kashiwara-Vergne can be re-interpreted as a study of w-foams.



WKO1.pdf
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Finite Type Invariants of w-Knotted Objects I: w-Knots and the Alexander Polynomial (joint with Zsuzsanna Dancso, 52 pages, posted May 2014, updated April 2016, Algebraic and Geometric Topology **16-2** (2016) 1063-1133, partially replaces WKO, arXiv:1405.1956)

This is the first in a series of papers studying w-knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.). These are classes of knotted objects which are wider but weaker than their "usual" counterparts. To get (say) w-knots from usual knots (or u-knots), one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation beyond the ordinary collection of Reidemeister moves, called the "overcrossings commute" relation, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke and was shown to be isomorphic to the McCool group of "basis-conjugating" automorphisms of a free group F_n - the smallest subgroup of $Aut(F_n)$ that contains both braids and permutations. Brendle and Hatcher, in work that traces back to Goldsmith, have shown this group to be a group of movies of flying rings in $\mathbf{R}^3$. Satoh studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbf{R}^4$. So w-knotted objects are algebraically and topologically interesting.

In this article we study finite type invariants of w-braids and w-knots. Following Berceanu and Papadima, we construct homomorphic universal finite type invariants of w-braids. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces A of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces A^w of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras, and in later papers of this series we re-interpret Alekseev-Torossian's work on Drinfel'd associators and the Kashiwara-Vergne problem as a study of w-knotted trivalent graphs.

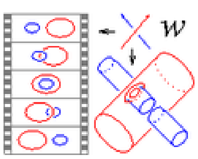
The true value of w-knots, though, is likely to emerge later, for we expect them to serve as a warmup example for what we expect will be even more interesting - the study of virtual knots, or v-knots. We expect v-knotted objects to provide the global context whose associated graded structure will be the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras.

$$w_{sl_2,2}(\pi_{2m}(D))=2R_m(D)c_2^m+\dots$$
$$\Updownarrow$$
$$JJ(h)(K)\cdot\bar{C}(h)(K)=1$$

arXiv:1401.0754

Proof of a Conjecture of Kulakova et al. Related to the sl_2 Weight System (joint with Huan Vo, European Journal of Combinatorics **45** (2015) 65-70, arXiv:1401.0754).

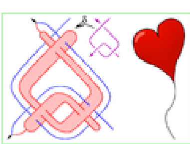
In this article, we show that a conjecture raised in [KLMR] (arXiv:1307.4933), which regards the coefficient of the highest term when we evaluate the sl_2 weight system on the projection of a diagram to primitive elements, is equivalent to the Melvin-Morton-Rozansky conjecture, proved in [BNG] (MMR).



paper's home
WKO.pdf

Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne. (joint with Zsuzsanna Dancso, 100 pages, posted September 2013, updated November 2013, arXiv:1309.7155)

This paper was split in two and became the first two parts of a four-part series (WKO1, WKO2, WKO3, WKO4). The remaining relevance of this paper's home is due to the series of videotaped lectures (wClips) that are linked there.



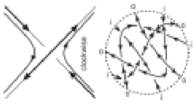
paper's home
KBH.pdf

Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant (56 pages, posted August 2013, updated November 2017, Acta Mathematica Vietnamica **40-2** (2015) 271-329, arXiv:1308.1721)

Balloons are two-dimensional spheres. Hoops are one dimensional loops. Knotted Balloons and Hoops (KBH) in 4-space behave much like the first and second fundamental groups of a topological space - hoops can be composed as in π_1, balloons as in π_2, and hoops "act" on balloons as π_1 acts on π_2. We observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops.

We give an ansatz for a tree and wheel (that is, free-Lie and cyclic word) -valued invariant ζ of (ribbon) KBHs in terms of the said compositions and action and we explain its relationship with finite type invariants. We speculate that ζ is a complete evaluation of the BF topological quantum field theory in 4D, though we are not sure what that means. We show that a certain "reduction and repackaging" of ζ is an "ultimate Alexander invariant" that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that should be a wonderful playground.

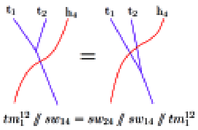
Khovanov Homology for Alternating Tangles (joint with Hernando Burgos-Soto, Journal of Knot Theory and



arXiv:1305.1695
Pensieve

its Ramifications **23-2** (2014), 18 pages, posted May 2013, updated March 2014, arXiv:1305.1695).

We describe a "concentration on the diagonal" condition on the Khovanov complex of tangles, show that this condition is satisfied by the Khovanov complex of the single crossing tangles, and prove that it is preserved by alternating planar algebra compositions. Hence, this condition is satisfied by the Khovanov complex of all alternating tangles. Finally, in the case of links, our condition is equivalent to a well known result which states that the Khovanov homology of a non-split alternating link is supported in two lines. Thus our condition is a generalization of Lee's Theorem to the case of tangles.



MetaMonoids.pdf
Pensieve

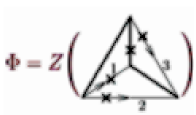
Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial (joint with Sam Selmani, 15 pages, posted February 2013, updated February 2014, Journal of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689).

We introduce a new invariant of tangles along with an algebraic framework in which to understand it. We claim that the invariant contains the classical Alexander polynomial of knots and its multivariable extension to links. We argue that of the computationally efficient members of the family of Alexander invariants, it is the most meaningful.

CDMReview.pdf
Pensieve

Review of a Book by Chmutov, Duzhin, and Mostovoy (Bull. Amer. Math. Soc. **50** (2013) 685-690, posted February 2013).

Merely 30 years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible, Lie algebras are rigid. Knots are irregular, Lie algebras are symmetric. The list of knots is a lengthy mess, the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen, Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar, Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.

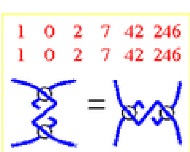


paper's home
ktgs.pdf

Homomorphic Expansions for Knotted Trivalent Graphs (joint with Zsuzsanna Dancso, 23 pages, posted March 2011, updated August 2012, Journal of Knot Theory and Its Ramifications **22-1** (2013), arXiv:1103.1896).
It had been known since old times that there exists a universal finite type invariant ("an expansion") Z^{old} for Knotted Trivalent Graphs (KTGs), and that it can be chosen to intertwine between some of the standard operations on KTGs and their chord-diagrammatic counterparts (so that relative to those operations, it is "homomorphic"). Yet perhaps the most important operation on KTGs is the "edge unzip" operation, and while the behavior of Z^{old} under edge unzip is well understood, it is not plainly homomorphic as some "correction factors" appear.
In this paper we present two (equivalent) ways of modifying Z^{old} into a new expansion Z, defined on "dotted Knotted Trivalent Graphs" (dKTGs), which is homomorphic with respect to a large set of operations. The first is to replace "edge unzips" by "tree connect sums", and the second involves somewhat restricting the circumstances under which edge unzips are allowed. As we shall explain, the newly defined class dKTG of knotted trivalent graphs retains all the good qualities that KTGs have - it remains firmly connected with the Drinfel'd theory of associators and it is sufficiently rich to serve as a foundation for an "Algebraic Knot Theory". As a further application, we present a simple proof of the good behavior of the LMO invariant under the Kirby II (band-slide) move.

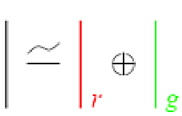
arXiv:1010.0754

Pentagon and Hexagon Equations Following Furusho (joint with Zsuzsanna Dancso, 7 pages, posted October 2010, Proceedings of the American Mathematical Society **140-4** (2012) 1243-1250).
In [arXiv:math/0702128] H. Furusho proves the beautiful result that of the three defining equations for associators, the pentagon implies the two hexagons (see also [Willwacher's arXiv:1009.1654]). In this note we present a simpler proof for this theorem (although our paper is less dense, and hence only slightly shorter). In particular, we package the use of algebraic geometry and Groethendieck-Teichmuller groups into a useful and previously known principle, and, less significantly, we eliminate the use of spherical braids.



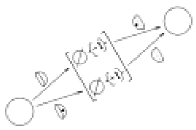
paper's home

Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots (joint with Iva Halacheva, Louis Leung, and Fionntan Roukema, 8 pages, posted September 2009, updated January 2011, Experimental Mathematics **20-3** (2011) 282-287, arXiv:0909.5169).
We compute many dimensions of spaces of finite type invariants of virtual knots (of several kinds) and the dimensions of the corresponding spaces of "weight systems", finding everything to be in agreement with the conjecture that "every weight system integrates".



paper's home

The Karoubi Envelope and Lee's Degeneration of Khovanov Homology (joint with Scott Morrison, 8 pages, posted June 2006, Algebraic & Geometric Topology **6** (2006) 1459-1469, arXiv:math.GT/0606542).
We give a simple proof of Lee's result from arXiv:math.GT/0210213, that the dimension of the Lee variant of the Khovanov homology of an c-component link is 2^c, regardless of the number of crossings. Our method of proof is entirely local and hence we can state a Lee-type theorem for tangles as well as for knots and links. Our main tool is the "Karoubi envelope of the cobordism category", a certain enlargement of the cobordism category which is mild enough so that no information is lost yet strong enough to allow for some simplifications that are otherwise unavailable.


paper's home

Fast Khovanov Homology Computations (13 pages, posted June 2006, updated May 2007, arXiv:math.GT/0606318, Journal of Knot Theory and Its Ramifications, **16-3** (2007) 243-255).
We introduce a *local* algorithm for Khovanov Homology computations - that is, we explain how it is possible to "cancel" terms in the Khovanov complex associated with a ("local") tangle, hence canceling the many associated "global" terms in one swoosh early on. This leads to a dramatic improvement in computational efficiency. Thus our program can rapidly compute certain Khovanov homology groups that otherwise would have taken centuries to evaluate.

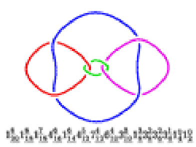

paper's home
Cobordism.pdf

Khovanov's Homology for Tangles and Cobordisms (39 pages, posted October 2004, updated April 2006, Geometry and Topology **9** (2005) 1443-1499, arXiv:math.GT/0410495).
We give a fresh introduction to the Khovanov Homology theory for knots and links, with special emphasis on its extension to tangles, cobordisms and 2-knots. By staying within a world of topological pictures a little longer than in other articles on the subject, the required extension becomes essentially tautological. And then a simple application of an appropriate functor (a "TQFT") to our pictures takes them to the familiar realm of complexes of (graded) vector spaces and ordinary homological invariants.

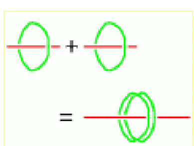
ENCYCLOPEDIA OF MATHEMATICAL PHYSICS
article's home

Finite Type Invariants (9 pages, posted August 2004, arXiv:math.GT/0408182).
This is an overview article on finite type invariants, written for the *Encyclopedia of Mathematical Physics*.

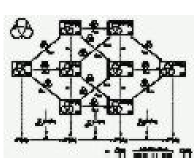
A Correction to "Groups of Ribbon Knots" by Ka Yi Ng (joint with Ofer Ron, 2 pages, posted September 2003). No longer available.


KHTables.pdf
KHTables.ps.gz

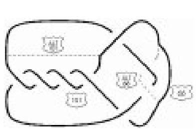
Khovanov Homology for Knots and Links with up to 11 Crossings (74 pages, posted May 2003, updated August 2004).
We provide tables of the ranks of the Khovanov homology of all prime knots and links with up to 11 crossings.


paper's home
TwoApplications.pdf
TwoApplications.ps
arXiv:math.QA/0204311

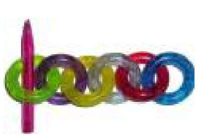
Two Applications of Elementary Knot Theory to Lie Algebras and Vassiliev Invariants (joint with Thang T. Q. Lê and Dylan P. Thurston, Geometry and Topology **7-1** (2003) 1-31, posted April 2002, arXiv:math.QA/0204311).
Using elementary equalities between various cables of the unknot and the Hopf link, we prove the Wheels and Wheeling conjectures of [BGRT:WheelsWheeling] and [Deligne:Letter], which give the exact Kontsevich integral of the unknot and a map intertwining two natural products on a space of diagrams. It turns out that the Wheeling map is given by the Kontsevich integral of a cut Hopf link (a bead on a wire), and its intertwining property is analogous to the computation of *1+1=2* on an abacus. The Wheels conjecture is proved from the fact that the *k*-fold connected cover of the unknot is the unknot for all *k*.
Along the way, we find a formula for the invariant of the general *(k,l)* cable of a knot. Our results can also be interpreted as a new proof of the multiplicativity of the Duflo-Kirillov map *S(g) --> U(g)* for metrized Lie (super-)algebras *g*.


paper's home

On Khovanov's Categorification of the Jones Polynomial (posted September 2001, Algebraic and Geometric Topology **2-16** (2002) 337-370, arXiv:math.QA/0201043, updated August 2004).
The working mathematician fears complicated words but loves pictures and diagrams. We thus give a no-fancy-anything picture-rich glimpse into Khovanov's novel construction of "the categorification of the Jones polynomial". For the same low cost we also provide some computations, including some that show that Khovanov's invariant is strictly stronger than the Jones polynomial and including a table of the values of Khovanov's invariant for all prime knots with up to 11 crossings.

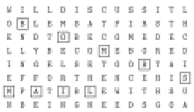

paper's home

Bracelets and the Goussarov Filtration of the Space of Knots (posted November 26, 2001, *Invariants of knots and 3-manifolds (Kyoto 2001)*, Topology and Geometry Monographs **4**, 1-12, arXiv:math.GT/0111267).
Following Goussarov's paper "Interdependent Modifications of Links and Invariants of Finite Degree" we describe an alternative finite type theory of knots. While (as shown by Goussarov) the alternative theory turns out to be equivalent to the standard one, it nevertheless has its own share of intrinsic beauty.


paper's home

A Rational Surgery Formula for the LMO Invariant (joint with Ruth Lawrence, posted May 15, 2000, Israel Journal of Mathematics **140** (2004) 29-60, arXiv:math.GT/0007045).
We write a formula for the LMO invariant of a rational homology sphere presented as a rational surgery on a link in S^3. Our main tool is a careful use of the Århus integral and the (now proven) "Wheels" and "Wheeling" conjectures of B-N, Garoufalidis, Rozansky and Thurston. As steps, side benefits and asides we give explicit formulas for the

values of the Kontsevich integral on the Hopf link and on Hopf chains, and for the LMO invariant of lens spaces and Seifert fibered spaces. We find that the LMO invariant does not separate lens spaces, is far from separating general Seifert fibered spaces, but does separate Seifert fibered spaces which are integral homology spheres.


paper's home
StatSci.pdf
StatSci.ps

Solving the Bible Code Puzzle (joint with Brendan McKay, Gil Kalai and Maya Bar-Hillel; *Statistical Science* **14-2** (1999) 150-173)
A paper of Witztum, Rips and Rosenberg in the journal *Statistical Science* in 1994 made the extraordinary claim that the Hebrew text of the Book of Genesis encodes events which did not occur until millennia after the text was written. In reply, we argue that Witztum, Rips and Rosenberg's case is fatally defective, indeed that their result merely reflects on the choices made in designing their experiment and collecting the data for it. We present extensive evidence in support of that conclusion. We also report on many new experiments of our own, all of which failed to detect the alleged phenomenon.

$$(-1)^{m|X|}\int^{(m)} G\,dX = (\det\Lambda)^m \int^{FG} G\,dX$$
AarhusIII.pdf
AarhusIII.ps
AarhusIII.tar.gz

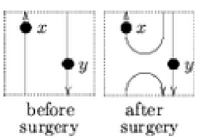
The Århus integral of rational homology 3-spheres III: The Relation with the Le-Murakami-Ohtsuki Invariant (joint with Stavros Garoufalidis, Lev Rozansky and Dylan P. Thurston, Selecta Mathematica, New Series **10** (2004) 305-324, arXiv:math.QA/9808013).
Continuing the work started in Part I and Part II of this series, we prove the relationship between the Århus integral and the invariant *LMO* defined by T.Q.T. Le, J. Murakami and T. Ohtsuki in q-alg/9512002. The basic reason for the relationship is that both constructions afford an interpretation as "integrated holonomies". In the case of the Århus integral, this interpretation was the basis for everything we did in Part I and Part II. The main tool we used there was "formal Gaussian integration". For the case of the *LMO* invariant, we develop an interpretation of a key ingredient, the map j_m, as "formal negative-dimensional integration". The relation between the two constructions is then an immediate corollary of the relationship between the two integration theories.

A story of "Cooked Lists" and "Permutation Races."
Chance.pdf

The Torah Codes: Puzzle and Solution (joint with Maya Bar-Hillel and Brendan McKay , Chance **11-2** (1998) 13-19)
A plain-English account of some of our investigations into "Bible codes".


paper's home

On the Witztum-Rips-Rosenberg Sample of Nations (joint with Brendan McKay and Shlomo Sternberg, draft, April 1998; first edition: March 1998).
We study the Witztum-Rips-Rosenberg (WRR) sample of nations and find clear evidence that their results were obtained by selective data manipulation and are therefore invalid. Our tool is the study of variations - we vary the sample of nations in many ways, and find that the variations are almost always "worse" than the original. We argue that the only way this can be possible is if the original was "tuned" in one way or another. Finally, we show that "tuning" is a sufficiently strong process that can by itself produce results similar to WRR's.



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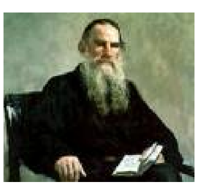
AarhusII.ps
AarhusII.tar.gz

The Århus integral of rational homology 3-spheres II: Invariance and Universality (joint with Stavros Garoufalidis, Lev Rozansky and Dylan P. Thurston, Selecta Mathematica, New Series **8** (2002) 341-371, arXiv:math.QA/9801049).
We continue the work started in Part I, and prove the invariance and universality in the class of finite type invariants of the object defined and motivated there, namely the Århus integral of rational homology 3-spheres. Our main tool in proving invariance is a translation scheme that translates statements in multi-variable calculus (Gaussian integration, integration by parts, etc.) to statements about diagrams. Using this scheme the straight-forward "philosophical" calculus-level proofs of Part I become straight-forward honest diagram-level proofs here. The universality proof is standard and utilizes a simple "locality" property of the Kontsevich integral.


paper's home

Equidistant Letter Sequences in Tolstoy's "War and Peace" (joint with Brendan McKay, draft, December 1997; first edition: September 1997).
In [WRR1], Witztum, Rips and Rosenberg found a surprising correlation between famous rabbis and their dates of birth and death, as they appear as equidistant letter sequences in the Book of Genesis. We make a smaller or equal number of mistakes, and find the same phenomenon in Tolstoy's eternal creation "War and Peace".

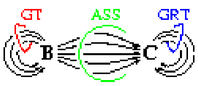
Integrate the Holonomies
AarhusI.pdf
AarhusI.ps
AarhusI.tar.gz

The Århus integral of rational homology 3-spheres I: A highly non trivial flat connection on S^3 (joint with Stavros Garoufalidis, Lev Rozansky and Dylan P. Thurston, Selecta Mathematica, New Series **8** (2002) 315-339, arXiv:q-alg/9706004).
Path integrals don't really exist, but it is very useful to dream that they do and figure out the consequences. Apart from describing much of the physical world as we now know it, these dreams also lead to some highly non-trivial mathematical theorems and theories. We argue that even though non-trivial flat connections on S^3 don't really exist, it is beneficial to dream that one exists (and, in fact, that it comes from the non-existent Chern-Simons path integral). Dreaming the right way, we are led to a rigorous construction of a universal finite-type invariant of rational homology spheres. We show that this invariant recovers the Rozansky and Ohtsuki invariants and that it is

essentially equal to the LMO (Le-Murakami-Ohtsuki) invariant.
This is part I of a 4-part series, containing the introductions and answers to some frequently asked questions. Theorems are stated but not proved in this part, and it can be viewed as a "research announcement". Part II of this series is titled "Invariance and Universality", Part III is titled "The Relation with the Le-Murakami-Ohtsuki Invariant", and part IV will be titled "The Relation with the Rozansky and Ohtsuki Invariants".

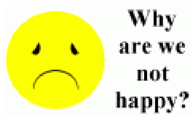

Wheels.pdf
Wheels.uu

Wheels, Wheeling, and the Kontsevich Integral of the Unknot (joint with Stavros Garoufalidis, Lev Rozansky and Dylan P. Thurston, posted March 1997, Israel Journal of Mathematics **119** (2000) 217-237, arXiv:q-alg/9703025).
We conjecture an exact formula for the Kontsevich integral of the unknot, and also conjecture a formula (also conjectured independently by Deligne) for the relation between the two natural products on the space of uni-trivalent diagrams. The two formulas use the related notions of "Wheels" and "Wheeling". We prove these formulas "on the level of Lie algebras" using standard techniques from the theory of Vassiliev invariants and the theory of Lie algebras.



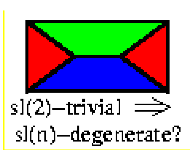
GT1.pdf
pensieve

On Associators and the Grothendieck-Teichmuller Group I (Selecta Mathematica, New Series **4** (1998) 183-212, June 1996, updated October 1998, arXiv:q-alg/9606021).
We present a formalism within which the relationship (discovered by Drinfel'd) between associators (for quasi-triangular quasi-Hopf algebras) and (a variant of) the Grothendieck-Teichmuller group becomes simple and natural, leading to a great simplification of Drinfel'd's original work. In particular, we re-prove that rational associators exist and can be constructed iteratively.



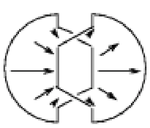
Fundamental.pdf
Fundamental.ps
Fundamental.uu

The Fundamental Theorem of Vassiliev Invariants (joint with Alexander Stoimenow, Geometry and Physics, (J.E. Andersen, J. Dupont, H. Pedersen, and A. Swann, eds.), lecture notes in pure and applied mathematics 184, Marcel Dekker, New-York 1997, pp. 101-134, arXiv:q-alg/9702009).
An exposition of four approaches to the proof of "The Fundamental Theorem of Vassiliev Invariants", saying that every weight system can be integrated to an invariant. We argue that each of these approaches (topological-combinatorial, geometric, physical, and algebraic) is, in some sense, wrong. The first and most natural approach simply fails, but while the other three succeed, they still appear unnatural. We express our hopes that these difficulties are an indication that there's something hiding around there, waiting to be discovered. The only new mathematics in this paper is a repackaging of Hutchings' topological-combinatorial argument in terms of the snake lemma.

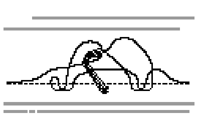


4CT.pdf
4CT.ps 4CT.uu

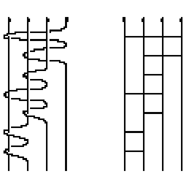
Lie Algebras and the Four Color Theorem (Combinatorica **17-1** (1997) 43-52, last updated October 1999, arXiv:q-alg/9606016).
Contains an appealing statement about Lie algebras that is equivalent to the Four Color Theorem. The notions appearing in the statement also appear in the theory of finite-type invariants of knots (Vassiliev invariants) and 3-manifolds.


tube.pdf
tube.ps

An Elementary Proof That All Spanning Surfaces of a Link Are Tube-Equivalent (joint with Jason Fulman and Louis H. Kauffman, June 1995, updated March 1998, Journal of Knot Theory and its Ramifications **7-7** (1998) 873-879).
The standard proof that the potential function provides a model for the Alexander-Conway polynomial depends on the fact that all Seifert surfaces of a link are tube-equivalent. Proofs that all Seifert surfaces of a link are tube-equivalent use machinery such as the Thom-Pontrjagin construction. Here we present an elementary geometric argument in the case of links in the three dimensional sphere which allows one to visualize the additions and removals of tubes.


poly.pdf
poly.dvi

Polynomial Invariants are Polynomial (Mathematical Research Letters **2** (1995) 239-246, arXiv:q-alg/9606025).
Contains a proof that (as conjectured by Lin and Wang [arXiv:dg-ga/9411015] when a Vassiliev invariant of type m is evaluated on a knot projection having n crossings, the result is bounded by a constant times n^m. Thus the well known analogy between Vassiliev invariants and polynomials justifies (well, at least *explains*) the odd title of this note.

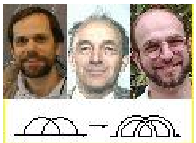

glN.pdf
glN.dvi

Vassiliev and Quantum Invariants of Braids (Proceedings of Symposia in Applied Mathematics **51** (1996) 129-144, Amer. Math. Soc., arXiv:q-alg/9607001).
Contains a proof of the fact that all Vassiliev invariants of braids come (in the natural sense) from *gl(N)* and its representations, and thus, in the light the fact that Vassiliev invariants separate braids (see my paper *Vassiliev Homotopy String Link Invariants*), *the gl(N)* invariants separate braids. A nice corollary of that is that every Vassiliev invariant of braids extends to a Vassiliev invariant (of the same degree) of string links.

$$\mathcal{G}_4\mathcal{E}_5 = R_{311}$$
$$\mathcal{G}_5\mathcal{E}_5 = 2R_{311}$$
$$\mathcal{G}_6\mathcal{E}_5 = 3R_{311} + R_{2111}$$
$$\mathcal{G}_7\mathcal{E}_5 = 5R_{311} + R_{2111}$$

table.pdf
table.dvi

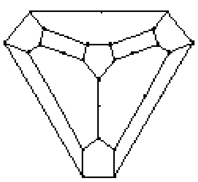
Some Computations Related to Vassiliev Invariants (18 pp, last updated May 5, 1996, available online, not meant for publication).
Contains tables of dimensions of over 500 spaces of Chinese Characters, as well as some tables of dimensions of spaces of Vassiliev invariants of knots, braids, and string links. Also contains the decompositions into irreducibles of the representations of the symmetric groups naturally associated with Chinese Characters. The data summarized in these tables is available in a mathematica readable format (also viewable as plain text) in a rather large data file (>800Kb compressed, >9Mb in full!), table.m. (If you're using Netscape, you may want to click SHIFT-LEFT on table.m when downloading it, to prevent automatic decompression).



mmr.pdf
mmr.ps

On the Melvin-Morton-Rozansky Conjecture (joint with Stavros Garoufalidis, July 1994, last updated January 1996, Inventiones Mathematicae **125** (1996) 103-133).
We prove a conjecture made by Melvin and Morton, saying that a certain specialization of the colored Jones polynomial is equal to the inverse of the Conway polynomial (in particular, the Conway polynomial is computable from the Jones polynomial and its cablings, something that was not known before). Later, Rozansky gave (here) a non-rigorous Chern-Simons path integral "proof" of that conjecture, which suggests a generalization (which we also prove) to arbitrary Lie algebras. Rozansky's techniques do not appear to be related to ours. Other reasons to read our paper:

- We prove the theorem on the level of weight systems and then use a general principle to show that in some cases (ours included), this is sufficient. We expect that there will be other applications to this method of proof.
- We use a nice (and new) relation between the weight system of the Conway polynomial and the intersection graph of a chord diagram.
- We discuss an amusing (and new) relation between immanants and the algebra generated by the coefficients of the Conway polynomial.
- We give a general formula for the weight system corresponding to the Lie algebra su(2) in an arbitrary representation.



nat.pdf
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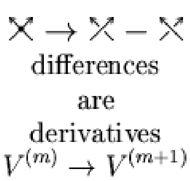
Non-Associative Tangles (in *Geometric topology*, proceedings of the Georgia international topology conference, W. H. Kazez, ed., 139-183, Amer. Math. Soc. and International Press, Providence, 1997).
We give a *first* completely combinatorial construction of a universal Vassiliev invariant, along lines suggested by Drinfel'd's work on quasi-Hopf algebras (previous papers on the subject did not give a combinatorial construction of an associator Phi). We describe a mathematica program implementing our algorithm, compute an associator up to degree 7, and compute our invariant in a few simple cases.

homotopy.pdf
homotopy.tar.gz

Vassiliev Homotopy String Link Invariants (February 1993, last updated January 1999, Journal of Knot Theory and its Ramifications **4-1** (1995) 13-32).
I show that the main conjectures of *On the Vassiliev Knot Invariants* become theorems when the attention is restricted to string links considered only up to *homotopy*. That is, the corresponding map into surfaces is injective (so all homotopy invariants come from surfaces), and Vassiliev homotopy invariants separate homotopy string links. The later result is proven by showing that the Milnor mu invariants are Vassiliev invariants. Along the way we also find that Vassiliev invariants of braids separate braids.



paper's home
OnVassiliev.pdf

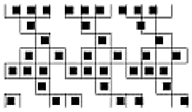
On the Vassiliev Knot Invariants (August 1992, last updated January 2007, Topology **34** (1995) 423-472).
An introduction to Vassiliev invariants. Contains the definition, proofs that the various knot polynomials are Vassiliev invariants (appropriately parametrized and expanded), the basic constructions (of weight systems from Vassiliev invariants and from Lie algebras), a discussion of the Hopf algebra of chord diagrams, The Kontsevich integral proving that every weight system comes from an invariant, the diagrammatic PBW theorem and Chinese characters, the map into marked surfaces, an analysis of the space of weight systems coming from that map (exactly all classical algebras), and some more.
If you're a newcomer to the field and you're asking me, that's the paper to read!

thesis.pdf

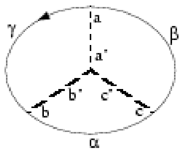
Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory (Ph.D. thesis, 109 pp, Princeton Univeristy June 1991). Contents:

- Introduction, The Feynman rules for Chern-Simons theory in flat space.
- My paper *Perturbative Chern-Simons Theory,* in a slightly older format.
- A general method for translating the BRST argument to the level of Feynman diagrams.
- The first "formal" proof (and the only one fully written up) that Chern-Simons perturbation theory gives knot invariants to all orders. The proof is on the formal algebraic level and does not deal (unfortunately) with divergencies. Modulo this point, it is also the first proof that every weight system can be integrated to a Vassiliev invariant.
- My paper *Weights of Feynman Diagrams and the Vassiliev Knot Invariants.*

- A somewhat different derivation of the results in my paper with Witten *Perturbative Expansion of Chern-Simons Theory with Non-Compact Gauge Group.*


the paper

Perceived Depth Images (appeared (in a shorter form) as *Random Dot Stereograms* in The Mathematica Journal **1-3** (1991) 69-75).
Describes a Mathematica package for creating perceived depth images - these things that look 3D when you look at them with your eyes crossed. For the mathematica package itself, click here. For a primitive but working 9 line version of that package, click here.

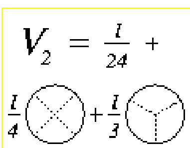


weights.pdf
weights.ps

Weights of Feynman Diagrams and the Vassiliev Knot Invariants (22 pp, February 1991, last updated June 1995).
My first paper on Vassiliev invariants, the first place where the relation between Vassiliev invariants and Lie algebras was noticed, and the first place where it was shown that there are more than finitely many Vassiliev invariants (by showing that the coefficients of the Conway polynomial are Vassiliev invariants). This paper is almost entirely a subset of my *On the Vassiliev Knot Invariants.* Perhaps the only thing which is still of interest in it is an algorithm for computing the weight systems associated with the symplectic groups.

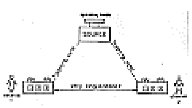

NonCompact.pdf
NonCompact.dvi

Perturbative Expansion of Chern-Simons Theory with Non-Compact Gauge Group (joint with Edward Witten, Communications in Mathematical Physics **141** (1991) 423-440).
A discussion of the semi-classical approximation for Chern-Simons theory with a non-compact gauge group. After finding the correct gauge fixing, we discuss the somewhat non-standard eta invariant that enters the computation of the phase of the path integral, and a certain anomaly related to it.

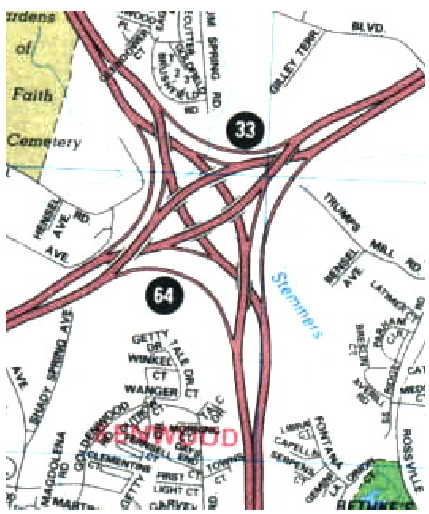


pcs.pdf
pcs.ps

Perturbative Chern-Simons Theory (43 pp, April 1990, last updated September 1995, Journal of Knot Theory and its Ramifications **4-4** (1995) 503-548).
Contains an introduction to perturbation theory in the context of Chern-Simons theory and knots, a discussion of the first order perturbation theory (linking, self-linking, and the torsion-related anomaly that forces the introduction of framings), a proof that the second order pertubation theory converges and yields a familiar knot invariant whose reduction mod 2 is the arf invariant, and a discussion of what is expected to happen at higher orders.


the paper

Two Examples in Non-Commutative Probability (Foundations of Physics **19** (1989) 97-104).
Mainly an exposition of the Bell inequality from the point of view of non-commutative (quantum) probability. Also contains a short discussion of the Heisenberg uncertainty principle from the same point of view.



An interchange near Baltimore