

```

found = False; While[ ! found,
  sol = FindRoot @@
    Join[{eqns}, ({#, Random[] + (2 Random[] - 1) I} & /@ vars), {MaxIterations -> 1000}];
  found = And[
    And @@ ((0 < Im[#]) & /@ (vars /. sol)),
    Abs[(Plus @@ (Vol /@ (vars /. sol))) - 2 Vol[omega]] < 0.0001
  ]
]; sol

seqns = DeleteCases[
  Simplify[
    eqns /. z12 -> 1/z14 /. z41 -> 1/z43 /. z13 -> 1/(z31 z34) /. z23 -> 1/(z24 z42) /.
      z21 -> 1/(z22 z44) /. z11 -> 1/(z32 z33) /. z14 -> z22 z32 z42 /.
      z31 -> z22 z32 z33 z43 z44 /. z22 -> 1/(z24 z32 z34 z42 z44) /.
      z44 -> (-z24 z34 z42 + z24 z32 z33 z34 z42 + z33 z43 - z32 z33^2 z43) /
      (-z24 z34 z42 + z24 z32 z33 z34 z42 + z33^2 z43 - z32 z33^2 z43 + z33 z34 z43 -
      z33^2 z34 z43 + z24 z32 z33 z34 z42 z43 - z24 z32 z33^2 z34 z42 z43 -
      z24 z32 z33 z34^2 z42 z43 + z24 z32 z33^2 z34^2 z42 z43) /. z43 ->  $\frac{-1 + z32 z33}{z32 (-z33 + z33^2)}$ 
  ],
  True
]

svars = Union[Cases[seqns, _Symbol, Infinity]]

```

"Mob[z, a, b, c]" stands for the Mobius function that maps a, b, c to 0, 1, Infinity, respectively.

```

Format[Mob[z_, a_, b_, c_]] := SuperscriptBox[z, {a, b, c}]

(sprods = FS @@ prods /. ((# -> Mob[#, 0, 1, Infinity]) & /@ vars) /. {
  Mob[z_, a_, b_, c_] ' := Mob[z, b, c, a],
  Mob[z_, a_, b_, c_] '' := Mob[z, c, a, b]
} /. Mob[z_, a_, b_, c_] ^ (-1) := Mob[z, c, b, a] // DisplayForm

FS[z12{0,1,∞} z14{0,1,∞}, z11{0,1,∞} z32{0,1,∞} z33{0,1,∞},
z13{0,1,∞} z31{0,1,∞} z34{0,1,∞}, z11{0,1,∞} z21{0,1,∞} z31{0,1,∞} z41{0,1,∞},
z23{0,1,∞} z24{0,1,∞} z42{0,1,∞}, z12{0,1,∞} z22{0,1,∞} z32{0,1,∞} z42{0,1,∞},
z11{∞,0,1} z12{1,∞,0} z14{∞,0,1} z21{1,∞,0} z22{∞,0,1} z24{1,∞,0} z32{1,∞,0} z42{∞,0,1},
z13{0,1,∞} z23{0,1,∞} z33{0,1,∞} z43{0,1,∞}, z41{0,1,∞} z43{0,1,∞},
z11{1,∞,0} z13{∞,0,1} z21{∞,0,1} z22{1,∞,0} z31{1,∞,0} z32{∞,0,1} z41{∞,0,1} z43{1,∞,0},
z11{∞,0,1} z13{1,∞,0} z23{∞,0,1} z24{1,∞,0} z33{1,∞,0} z34{∞,0,1} z41{1,∞,0} z43{∞,0,1},
z13{1,∞,0} z23{∞,0,1} z31{∞,0,1} z32{∞,0,1} z33{1,∞,0} z41{1,∞,0} z42{1,∞,0} z43{∞,0,1},
z21{0,1,∞} z22{0,1,∞} z44{0,1,∞}, z21{1,∞,0} z31{∞,0,1} z34{1,0,∞} z44{0,∞,1},
z12{1,∞,0} z14{∞,0,1} z22{∞,0,1} z23{1,∞,0} z32{1,∞,0} z33{∞,0,1} z42{∞,0,1} z44{1,∞,0},
z11{1,∞,0} z21{∞,0,1} z31{1,∞,0} z33{∞,0,1} z34{∞,0,1} z41{∞,0,1} z43{1,∞,0} z44{1,∞,0}]

```

```
(sprods = sprods /. Cases[
  List @@ sprods,
  Mob[z1_, a1_, b1_, c1_] Mob[z2_, a2_, b2_, c2_] => Sequence[
    Mob[z1, a1, b1, c1] Mob[z2, a2, b2, c2] -> 1,
    Mob[z1, c1, a1, b1] Mob[z2, b2, c2, a2] -> 1,
    Mob[z1, b1, c1, a1] Mob[z2, c2, b2, a2] -> 1
  ],
  {1}, 1
]
) // DisplayForm
FS[1, z11{0,1,∞} z32{0,1,∞} z33{0,1,∞},
  z13{0,1,∞} z31{0,1,∞} z34{0,1,∞}, z11{0,1,∞} z21{0,1,∞} z31{0,1,∞} z41{0,1,∞},
  z23{0,1,∞} z24{0,1,∞} z42{0,1,∞}, z12{0,1,∞} z22{0,1,∞} z32{0,1,∞} z42{0,1,∞},
  z11{∞,0,1} z12{1,∞,0} z14{∞,0,1} z21{1,∞,0} z22{∞,0,1} z24{1,∞,0} z32{1,∞,0} z42{∞,0,1},
  z13{0,1,∞} z23{0,1,∞} z33{0,1,∞} z43{0,1,∞}, z41{0,1,∞} z43{0,1,∞},
  z11{1,∞,0} z13{∞,0,1} z21{∞,0,1} z22{1,∞,0} z31{1,∞,0} z32{∞,0,1} z41{∞,0,1} z43{1,∞,0},
  z11{∞,0,1} z13{1,∞,0} z23{∞,0,1} z24{1,∞,0} z33{1,∞,0} z34{∞,0,1} z41{1,∞,0} z43{∞,0,1},
  z13{1,∞,0} z23{∞,0,1} z31{∞,0,1} z32{∞,0,1} z33{1,∞,0} z41{1,∞,0} z42{1,∞,0} z43{∞,0,1},
  z21{0,1,∞} z22{0,1,∞} z44{0,1,∞}, z21{1,∞,0} z31{∞,0,1} z34{1,0,∞} z44{0,∞,1},
  z12{1,∞,0} z14{∞,0,1} z22{∞,0,1} z23{1,∞,0} z32{1,∞,0} z33{∞,0,1} z42{∞,0,1} z44{1,∞,0},
  z11{1,∞,0} z21{∞,0,1} z31{1,∞,0} z33{∞,0,1} z34{∞,0,1} z41{∞,0,1} z43{1,∞,0} z44{1,∞,0}]
```

```
Cases[
  List @@ sprods,
  Mob[z1_, a1_, b1_, c1_] Mob[z2_, a2_, b2_, c2_] => Sequence[
    Mob[z1, a1, b1, c1] Mob[z2, a2, b2, c2] -> 1,
    Mob[z1, c1, a1, b1] Mob[z2, b2, c2, a2] -> 1,
    Mob[z1, b1, c1, a1] Mob[z2, c2, b2, a2] -> 1
  ]
] // DisplayForm
{z41{0,1,∞} z43{0,1,∞} -> 1, z41{∞,0,1} z43{1,∞,0} -> 1, z41{1,∞,0} z43{∞,1,0} -> 1}
```

```
sprods // InputForm
FS[1, Mob[z11, 0, 1, Infinity]*Mob[z32, 0, 1, Infinity]*Mob[z33, 0, 1, Infinity], Mob[z13,
  Mob[z11, 0, 1, Infinity]*Mob[z21, 0, 1, Infinity]*Mob[z31, 0, 1, Infinity]*Mob[z41, 0, 1,
  Mob[z12, 0, 1, Infinity]*Mob[z22, 0, 1, Infinity]*Mob[z32, 0, 1, Infinity]*Mob[z42, 0, 1,
  Mob[z21, 1, Infinity, 0]*Mob[z22, Infinity, 0, 1]*Mob[z24, 1, Infinity, 0]*Mob[z32, 1, I
  Mob[z13, 0, 1, Infinity]*Mob[z23, 0, 1, Infinity]*Mob[z33, 0, 1, Infinity]*Mob[z43, 0, 1,
  Mob[z11, 1, Infinity, 0]*Mob[z13, Infinity, 0, 1]*Mob[z21, Infinity, 0, 1]*Mob[z22, 1, In
  Mob[z43, 1, Infinity, 0], Mob[z11, Infinity, 0, 1]*Mob[z13, 1, Infinity, 0]*Mob[z23, Inf
  Mob[z41, 1, Infinity, 0]*Mob[z43, Infinity, 0, 1], Mob[z13, 1, Infinity, 0]*Mob[z23, Inf
  Mob[z41, 1, Infinity, 0]*Mob[z42, 1, Infinity, 0]*Mob[z43, Infinity, 0, 1], Mob[z21, 0, :
  Mob[z21, 1, Infinity, 0]*Mob[z31, Infinity, 0, 1]*Mob[z34, 1, 0, Infinity]*Mob[z44, 0, In
  Mob[z23, 1, Infinity, 0]*Mob[z32, 1, Infinity, 0]*Mob[z33, Infinity, 0, 1]*Mob[z42, Infi
  Mob[z11, 1, Infinity, 0]*Mob[z21, Infinity, 0, 1]*Mob[z31, 1, Infinity, 0]*Mob[z33, Infin
  Mob[z44, 1, Infinity, 0]]
```

**s1 = FS @@ prods**

$$\text{FS}\left[z_{12} z_{14}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, z_{11} z_{21} z_{31} z_{41}, z_{23} z_{24} z_{42},\right. \\ \left. z_{12} z_{22} z_{32} z_{42}, z_{13} z_{23} z_{33} z_{43}, z_{41} z_{43}, z_{21} z_{22} z_{44}, \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''},\right. \\ \left. z_{11}' z_{22}' z_{31}' z_{43}' z_{13}'' z_{21}'' z_{32}'' z_{41}'', z_{11}' z_{31}' z_{43}' z_{44}' z_{21}'' z_{33}'' z_{34}'' z_{41}'',\right. \\ \left. z_{12}' z_{21}' z_{24}' z_{32}' z_{11}'' z_{14}'' z_{22}'' z_{42}'', z_{12}' z_{23}' z_{32}' z_{44}' z_{14}'' z_{22}'' z_{33}'' z_{42}'',\right. \\ \left. z_{13}' z_{33}' z_{41}' z_{42}' z_{23}'' z_{31}'' z_{32}'' z_{43}'', z_{13}' z_{24}' z_{33}' z_{41}' z_{11}'' z_{23}'' z_{34}'' z_{43}''\right]$$

**s2 = s1 /. {z14 → 1/z12, z14' → 1/z12'}**

$$\text{FS}\left[1, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, z_{11} z_{21} z_{31} z_{41}, z_{23} z_{24} z_{42}, z_{12} z_{22} z_{32} z_{42},\right. \\ \left. z_{13} z_{23} z_{33} z_{43}, z_{41} z_{43}, z_{21} z_{22} z_{44}, \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{11}' z_{22}' z_{31}' z_{43}' z_{13}'' z_{21}'' z_{32}'' z_{41}'',\right. \\ \left. z_{11}' z_{31}' z_{43}' z_{44}' z_{21}'' z_{33}'' z_{34}'' z_{41}'', z_{21}' z_{24}' z_{32}' z_{11}'' z_{22}'' z_{42}'', z_{23}' z_{32}' z_{44}' z_{22}'' z_{33}'' z_{42}'',\right. \\ \left. z_{13}' z_{33}' z_{41}' z_{42}' z_{23}'' z_{31}'' z_{32}'' z_{43}'', z_{13}' z_{24}' z_{33}' z_{41}' z_{11}'' z_{23}'' z_{34}'' z_{43}''\right]$$

**s3 = s2 /. {z41' z43' → 1, z41' → 1/z43'}**

$$\text{FS}\left[1, 1, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, \frac{z_{13} z_{23} z_{33}}{z_{41}}, z_{11} z_{21} z_{31} z_{41},\right. \\ \left. z_{23} z_{24} z_{42}, z_{12} z_{22} z_{32} z_{42}, z_{21} z_{22} z_{44}, z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'',\right. \\ \left. z_{13}' z_{33}' z_{42}' z_{23}'' z_{31}'' z_{32}'', \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'',\right. \\ \left. z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}'', z_{21}' z_{24}' z_{32}' z_{11}'' z_{22}'' z_{42}'', z_{23}' z_{32}' z_{44}' z_{22}'' z_{33}'' z_{42}''\right]$$

**s4 = s3 /. FS[x\_/z41, y\_\*z41] → FS[x\*y]**

$$\text{FS}\left[1, 1, z_{11} z_{13} z_{21} z_{23} z_{31} z_{33}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34},\right. \\ \left. z_{23} z_{24} z_{42}, z_{12} z_{22} z_{32} z_{42}, z_{21} z_{22} z_{44}, z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'',\right. \\ \left. z_{13}' z_{33}' z_{42}' z_{23}'' z_{31}'' z_{32}'', \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'',\right. \\ \left. z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}'', z_{21}' z_{24}' z_{32}' z_{11}'' z_{22}'' z_{42}'', z_{23}' z_{32}' z_{44}' z_{22}'' z_{33}'' z_{42}''\right]$$

**s5 = s4 /. z42 → 1/(z23 z24)**

$$\text{FS}\left[1, 1, 1, \frac{z_{12} z_{22} z_{32}}{z_{23} z_{24}}, z_{11} z_{13} z_{21} z_{23} z_{31} z_{33}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34},\right. \\ \left. z_{21} z_{22} z_{44}, z_{21}' z_{24}' z_{32}' z_{11}'' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)'', z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'',\right. \\ \left. z_{13}' \left(\frac{1}{z_{23} z_{24}}\right)' z_{33}' z_{23}'' z_{31}'' z_{32}'', z_{23}' z_{32}' z_{44}' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)' z_{33}'',\right. \\ \left. \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'', z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}''\right]$$

$$\text{FS}\left[1, 1, 1, \frac{z_{12} z_{22} z_{32}}{z_{23} z_{24}}, z_{11} z_{13} z_{21} z_{23} z_{31} z_{33}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, z_{21} z_{22} z_{44}, z_{21}' z_{24}' z_{32}'' z_{11}'' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)'', z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'', z_{13}' \left(\frac{1}{z_{23} z_{24}}\right)' z_{33}' z_{23}'' z_{31}'' z_{32}'', z_{23}' z_{32}' z_{44}' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)'' z_{33}'', \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'', z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}''\right]$$

$$\text{FS}\left[1, 1, 1, \frac{z_{12} z_{22} z_{32}}{z_{23} z_{24}}, z_{11} z_{13} z_{21} z_{23} z_{31} z_{33}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, z_{21} z_{22} z_{44}, z_{21}' z_{24}' z_{32}'' z_{11}'' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)'', z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'', z_{13}' \left(\frac{1}{z_{23} z_{24}}\right)' z_{33}' z_{23}'' z_{31}'' z_{32}'', z_{23}' z_{32}' z_{44}' z_{22}'' \left(\frac{1}{z_{23} z_{24}}\right)'' z_{33}'', \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'', z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}''\right]$$

$$s_6 = s_5 /. z_{23}^{-1} z_{24}^{-1} \rightarrow w_1' w_2'$$

$$\text{FS}\left[1, 1, 1, z_{11} z_{13} z_{21} z_{23} z_{31} z_{33}, z_{11} z_{32} z_{33}, z_{13} z_{31} z_{34}, z_{21} z_{22} z_{44}, z_{12} z_{22} z_{32} w_1' w_2', z_{11}' z_{22}' z_{31}' z_{13}'' z_{21}'' z_{32}'', z_{13}' z_{33}' (w_1' w_2')' z_{23}'' z_{31}'' z_{32}'', \frac{z_{21}' z_{31}''}{z_{44}' z_{34}''}, z_{13}' z_{24}' z_{33}' z_{11}'' z_{23}'' z_{34}'', z_{11}' z_{31}' z_{44}' z_{21}'' z_{33}'' z_{34}'', z_{21}' z_{24}' z_{32}'' z_{11}'' z_{22}'' (w_1' w_2')'', z_{23}' z_{32}' z_{44}' z_{22}'' z_{33}'' (w_1' w_2')''\right]$$

$$t_1 = s_1 /. ((\# \rightarrow \#[0]) \& /@ vars) /. Derivative[p_][z_[0]] \Rightarrow z[p]$$

$$\text{FS}\left[z_{12}[0] z_{14}[0], z_{11}[0] z_{32}[0] z_{33}[0], z_{13}[0] z_{31}[0] z_{34}[0], z_{11}[0] z_{21}[0] z_{31}[0] z_{41}[0], z_{23}[0] z_{24}[0] z_{42}[0], z_{12}[0] z_{22}[0] z_{32}[0] z_{42}[0], z_{11}[2] z_{12}[1] z_{14}[2] z_{21}[1] z_{22}[2] z_{24}[1] z_{32}[1] z_{42}[2], z_{13}[0] z_{23}[0] z_{33}[0] z_{43}[0], z_{41}[0] z_{43}[0], z_{11}[1] z_{13}[2] z_{21}[2] z_{22}[1] z_{31}[1] z_{32}[2] z_{41}[2] z_{43}[1], z_{11}[2] z_{13}[1] z_{23}[2] z_{24}[1] z_{33}[1] z_{34}[2] z_{41}[1] z_{43}[2], z_{13}[1] z_{23}[2] z_{31}[2] z_{32}[2] z_{33}[1] z_{41}[1] z_{42}[1] z_{43}[2], z_{21}[0] z_{22}[0] z_{44}[0], \frac{z_{21}[1] z_{31}[2]}{z_{34}[2] z_{44}[1]}, z_{12}[1] z_{14}[2] z_{22}[2] z_{23}[1] z_{32}[1] z_{33}[2] z_{42}[2] z_{44}[1], z_{11}[1] z_{21}[2] z_{31}[1] z_{33}[2] z_{34}[2] z_{41}[2] z_{43}[1] z_{44}[1]\right]$$

```
t1 = FS[z12[0] z14[0], z11[0] z32[0] z33[0], z13[0] z31[0] z34[0],
      z11[0] z21[0] z31[0] z41[0], z23[0] z24[0] z42[0], z12[0] z22[0] z32[0] z42[0],
      z11[2] z12[1] z14[2] z21[1] z22[2] z24[1] z32[1] z42[2], z13[0] z23[0] z33[0] z43[0],
      z41[0] z43[0], z11[1] z13[2] z21[2] z22[1] z31[1] z32[2] z41[2] z43[1],
      z11[2] z13[1] z23[2] z24[1] z33[1] z34[2] z41[1] z43[2],
      z13[1] z23[2] z31[2] z32[2] z33[1] z41[1] z42[1] z43[2],
      z21[0] z22[0] z44[0], z12[1] z14[2] z22[2] z23[1] z32[1] z33[2] z42[2] z44[1],
      z11[1] z21[2] z31[1] z33[2] z34[2] z41[2] z43[1] z44[1]]
```

```
FS[z12[0] z14[0], z11[0] z32[0] z33[0], z13[0] z31[0] z34[0],
  z11[0] z21[0] z31[0] z41[0], z23[0] z24[0] z42[0], z12[0] z22[0] z32[0] z42[0],
  z11[2] z12[1] z14[2] z21[1] z22[2] z24[1] z32[1] z42[2], z13[0] z23[0] z33[0] z43[0],
  z41[0] z43[0], z11[1] z13[2] z21[2] z22[1] z31[1] z32[2] z41[2] z43[1],
  z11[2] z13[1] z23[2] z24[1] z33[1] z34[2] z41[1] z43[2],
  z13[1] z23[2] z31[2] z32[2] z33[1] z41[1] z42[1] z43[2],
  z21[0] z22[0] z44[0], z12[1] z14[2] z22[2] z23[1] z32[1] z33[2] z42[2] z44[1],
  z11[1] z21[2] z31[1] z33[2] z34[2] z41[2] z43[1] z44[1]]
```

```
Length[t1]
```

```
15
```

```
n = 4;
```

```
eqns = (# == 1) & /@ (prods = {
  z11 z21 z31 z41, z12 z22 z32 z42, z13 z23 z33 z43,
  z12 z14, z34 z13 z31, z24 z42 z23, z21 z22 z44, z41 z43, z11 z33 z32,
  z32 ' z11 ' z41 ' ' z43 ' z13 ' ' z31 ' z21 ' ' z22 ' ,
  z33 ' ' z32 ' z22 ' ' z44 ' z14 ' ' z12 ' z42 ' ' z23 ' ,
  z34 ' ' z13 ' z43 ' ' z41 ' z11 ' ' z33 ' z23 ' ' z24 ' ,
  z24 ' z42 ' ' z32 ' z11 ' ' z21 ' z22 ' ' z12 ' z14 ' ' ,
  z41 ' z43 ' ' z33 ' z32 ' ' z42 ' z23 ' ' z13 ' z31 ' ' ,
  z43 ' z41 ' ' z31 ' z34 ' ' z44 ' z21 ' ' z11 ' z33 ' ' ,
  z21 ' z31 ' ' (z34 ' ' z44 ' ) ^ (-1)
}) /. {(z_) ' => 1 - 1/z, (z_) ' ' => 1 / (1 - z)}
```

```
vars = Union[Cases[eqns, _Symbol, Infinity]]
```

```
{
  sol =
  FindRoot @@ Join[{eqns}, ({#, Exp[Random[] Pi I]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[ ((First /@ eqns) /. sol) - 1]],
  Plus @@ (Vol /@ (vars /. sol))
}
```

```
L = PD[Knot[5, 2]]
```

```
n = Length[L]
```

```

dt = EQs @@ Flatten[
  {
    Table[
      t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
      {v, n}, {j, 4}
    ],
    Table[e @@ Position[L, i], {i, 2 n}] /. e[x_, y_] → {e[x, y], e[y, x]} /. {
      e[{v_, a_}, {u_, b_}] ⇒
        t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
      e[{v_, a_}, {u_, b_}] ⇒ t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
        t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
    }
  ] /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] ⇒ {
    t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
  }
] /.
t[z[v_, j_], a_, b_] ⇒ t[z[v, j]] ~ Join ~ (t[a, b] /. {3 → 4, 4 → 3}) /; EvenQ[j];

rels = List @@ Union[
  dt /. p[t[z_, a_, b_], t[w_, c_, d_]] ⇒
    EQs[t[z, a, b] ~ to ~ t[w, d, c], t[w, c, d] ~ to ~ t[z, b, a]] /. (
    EQs[xx_. * (t[x_, ab_] ~ to ~ t[y_, cd_]), zz_. * (t[y_, cd_] ~ to ~ t[z_, ef_])] ×
  ⇒ EQs[xx * y[cd] * zz * (t[x, ab] ~ to ~ t[z, ef])]
) /. (t[z_, ab_] ~ to ~ t[z_, ab_]) ⇒ z[ab] /.
(w_z)[ab_] ⇒ Switch[Sort[{ab}],
  {1, 2} | {3, 4}, w[0],
  {1, 3} | {2, 4}, w[1],
  {1, 4} | {2, 3}, w[2]
]
] /. eq_Times ⇒ (EQ @@ eq);

merge = Position[Count[#, z[n, _][__]] & /@ rels, 4];
rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
rels = Append[rels, EQ @@ Product[
  z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
  If[L[[k, 4]] > L[[k, 2]] || L[[k, 2]] - L[[k, 4]] > 1,
    z[k, 1][1] * z[k, 4][2],
    z[k, 3][1] * z[k, 2][2]
  ],
  {k, n}
] /. {EQ[z_[0], z_[1], z_[2]] → EQ[-1], EQ[-1, -1] → EQ[]}
] /. z[i_, j_] ⇒ ToExpression["z" <> ToString[i] <> ToString[j]]

```

```

RandomPermutation[n_Integer] := Last /@ Sort[{Random[], #} & /@ Range[n]];
$Log = False;
Red[rels_List] := Module[
  {s = rels, ks, k, t, at1, at2},
  pr[k_] := (k + 1) ~Mod~ 3; prpr[k_] := (k + 2) ~Mod~ 3;
  Label[Start]; s = DeleteCases[s, EQ[]];
  ks = RandomPermutation[Length[s]]; While[ks != {},
    k = First[ks]; ks = Rest[ks];
    Replace[s[[k]], EQ[z1_[p1_], z2_[p2_]] => (
      $Log && Print[StringForm["Doing bigon: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
      t = Delete[s, k] //. {EQ[z1[p1 // pr], z2[p2 // prpr]] -> EQ[],
        EQ[z1[p1 // prpr], z2[p2 // pr]] -> EQ[]};
      If[Length[at1 = Position[t, z1]] != 1 || Length[at2 = Position[t, z2]] != 1,
        $Log && Print[StringForm["Bad bigon: at1=`; at2=`; t=`", at1, at2, t]],
        at1 = at1[[1, 1]]; at2 = at2[[1, 1]];
        s = If[at1 == at2,
          $Log && Print["Equal ats", t[[at1]]]; t /. EQ[z1[p1], z2[p2]] -> EQ[],
          Append[Delete[t, {{at1}, {at2}}],
            EQ[t[[at1]], t[[at2]]] /. EQ[z1[p1], z2[p2]] -> EQ[]
        ]];
      Goto[Start]
    ]];
  ]];
ks = RandomPermutation[Length[s]]; While[ks != {},
  k = First[ks]; ks = Rest[ks];
  Replace[s[[k]], EQ[z1_[p1_], z2_[p2_], z3_[p3_]] => (
    $Log && Print[StringForm["Doing triangle: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
    {w1, w2} = Unique[{w, w}];
    t = Delete[s, k] //. {
      z3[p3] -> EQ[w1[0], w2[0]],
      z2[p2] -> EQ[w1[2], w2[1]], z1[p1] -> EQ[w1[1], w2[2]],
      EQ[z3[p3 // pr], z3[p3 // prpr]] -> EQ[-1, w1[2], w2[1], w1[1], w2[2]],
      EQ[z2[p2 // pr], z2[p2 // prpr]] -> EQ[-1, w1[0], w2[0], w1[1], w2[2]],
      EQ[z1[p1 // pr], z1[p1 // prpr]] -> EQ[-1, w1[0], w2[0], w1[2], w2[1]],
      EQ[z1[p1 // prpr], z2[p2 // pr]] -> w1[0],
      EQ[z1[p1 // pr], z2[p2 // prpr]] -> w2[0],
      EQ[z2[p2 // prpr], z3[p3 // pr]] -> w1[1],
      EQ[z3[p3 // pr], z1[p1 // prpr]] -> w2[1],
      EQ[z3[p3 // prpr], z1[p1 // pr]] -> w1[2],
      EQ[z2[p2 // pr], z3[p3 // prpr]] -> w2[2],
      EQ[z_ [0], z_ [1], z_ [2]] -> EQ[-1], EQ[-1, -1] -> EQ[]
    };
  If[FreeQ[t, z1 | z2 | z3],
    s = t; Goto[Start],
    $Log && Print[StringForm["Failed triangle; t=`; problems at `",
      t, Select[t, !FreeQ[#, z1 | z2 | z3] &]]
  ]];
  ]];
s
]

```

```
Join[PachnerRules[z1, z2, z3, w1, w2], {  
  EQ[z3[1], z3[2]] → EQ[-1, w1[2], w2[1], w1[1], w2[2]],  
  EQ[z2[1], z2[2]] → EQ[-1, w1, w2, w1[1], w2[2]],  
  EQ[z1[1], z1[2]] → EQ[-1, w1, w2, w1[2], w2[1]],  
  EQ[m[z_, 0], m[z_, 1], m[z_, 2]] → EQ[-1], EQ[-1, -1] → EQ[]  
}]
```