

```
<< \ c : / drorbn / projects / KAtlas / KnotTheory.m
```

```
Loading KnotTheory`...
```

```
EasyKnots =
```

```
{Knot[4, 1], Knot[5, 2], Knot[6, 1], Knot[7, 2], Knot[8, 1], Knot[9, 2], Knot[9, 42]}
{Knot[4, 1], Knot[5, 2], Knot[6, 1], Knot[7, 2], Knot[8, 1], Knot[9, 2], Knot[9, 42]}
```

```
Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
```

```
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1 / (1 - z)})
```

```
2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
```

```
2.02988
```

```
PachnerTest[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +
  Vol[1 / (z * w)] - Vol[z (1 - w) / (1 - z * w)] - Vol[w (1 - z) / (1 - z * w)]}
```

```
PachnerTest[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],
  Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
```

```
{0.878326, 0.879622, -5.43821 × 10-12}
```

```
SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat, OneIdentity}]
```

```
ClearAll[Rels]
```

```
General::spell1: Possible spelling error: new symbol name "Rels" is similar to existing symbol "Reals".
```

```

Rels[K_] := Rels[K] = Rels[PD[K]];
Rels[pd_PD] := Module[
  {n, dt, z, e, u, v, j, im4, dm4, w, zz,
   xx, yy, ab, merge, rels, x, y, a, b, c, d, d1, d2, d3, eq},
  im4[a_] := a + 1 /. 5 -> 1; dm4[a_] := a - 1 /. 0 -> 4;
  n = Length[pd];
  dt = EQs @@ Flatten[
    {
      Table[
        t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
        {v, n}, {j, 4}
      ],
      Table[e @@ Position[pd, i], {i, 2 n}] /. e[x_, y_] -> {e[x, y], e[y, x]} /. {
        e[{v_, a_}, {u_, b_}] ->
          t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
        e[{v_, a_}, {u_, b_}] -> t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
          t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
      }
    ] /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] -> {
      t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
    }
  ] /.
  t[z[v_, j_], a_, b_] -> t[z[v, j]] ~ Join ~ (t[a, b] /. {3 -> 4, 4 -> 3}) /; EvenQ[j];
  rels = List @@ Union[
    dt /. p[t[z_, a_, b_], t[w_, c_, d_]] ->
      EQs[t[z, a, b] ~ to ~ t[w, d, c], t[w, c, d] ~ to ~ t[z, b, a]] /. (
        EQs[xx_. * (t[x_, ab_] ~ to ~ t[y_, cd_]), zz_. *
          (t[y_, cd_] ~ to ~ t[z_, ef_])] ->
          EQs[xx * y[cd] * zz * (t[x, ab] ~ to ~ t[z, ef])]
      ) /. (t[z_, ab_] ~ to ~ t[z_, ab_]) -> z[ab] /.
      (w_z)[ab_] -> Switch[Sort[{ab}],
        {1, 2} | {3, 4}, w[0],
        {1, 3} | {2, 4}, w[1],
        {1, 4} | {2, 3}, w[2]
      ]
    ] /. eq_Times -> (EQ @@ eq);
  merge = Position[Count[#, z[n, _][_] ] & /@ rels, 4];
  rels = {
    Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]],
    z1 ~ EQ ~ (EQ @@ Product[
      z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
      If[pd[[k, 4]] > pd[[k, 2]] || pd[[k, 2]] - pd[[k, 4]] > 1,
        z[k, 1][1] * z[k, 4][2],
        z[k, 3][1] * z[k, 2][2]
      ],
      {k, n}
    ]) /. {EQ[z_[0], z_[1], z_[2]] -> EQ[-1], EQ[-1, -1] -> EQ[]}
  ] /. z[i_, j_][k_] -> m[ToExpression["z" <> ToString[i] <> ToString[j]], k]
]

```

General::spell1: Possible spelling error: new symbol name "rels" is similar to existing symbol "Rels".

```
rels = Rels[Knot[4, 1]]
```

```
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
 EQ[m[z13, 0], m[z23, 0]], EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]],
 EQ[m[z14, 0], m[z22, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]],
 EQ[m[z12, 1], m[z13, 2], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z31, 2], m[z34, 1]],
 EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z23, 2], m[z24, 1], m[z33, 2], m[z34, 1]],
 EQ[m[z11, 0], m[z33, 0], m[z41, 0]], EQ[m[z32, 0], m[z42, 0]],
 EQ[m[z21, 2], m[z22, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]],
 EQ[m[z21, 0], m[z31, 0], m[z43, 0]], EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2],
 m[z33, 1], m[z34, 2], m[z42, 1], m[z43, 2]], EQ[m[z12, 0], m[z24, 0], m[z44, 0]],
 EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1], m[z23, 2], m[z41, 2], m[z44, 1]],
 EQ[m[z13, 2], m[z14, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z43, 2], m[z44, 1]],
 EQ[m[z11, 2], m[z12, 1], m[z21, 2], m[z24, 1], m[z31, 2], m[z32, 1],
 m[z32, 1], m[z33, 2], m[z41, 0], m[z41, 1], m[z41, 1], m[z42, 0], m[z42, 2],
 m[z42, 2], m[z43, 0], m[z43, 1], m[z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]]},
 EQ[z1, m[z12, 2], m[z14, 0], m[z14, 1], m[z22, 2], m[z24, 0], m[z24, 1],
 m[z31, 1], m[z33, 0], m[z33, 2], m[z41, 1], m[z43, 0], m[z43, 2]]}
```

```
$Log = 20;
```

```
RandomPermutation[l_List] := Last /@ Sort[Random[], #] & /@ l];
```

```
m[z_, k1_][k2_] := m[z, (k1 + k2) ~Mod~ 3];
```

```
PachnerRules[z1_, z2_, z3_, w1_, w2_] := {
 z3 → EQ[w1, w2], z2 → EQ[w1[2], w2[1]], z1 → EQ[w1[1], w2[2]],
 EQ[z1[2], z2[1]] → w1, EQ[z1[1], z2[2]] → w2, EQ[z2[2], z3[1]] → w1[1],
 EQ[z3[1], z1[2]] → w2[1], EQ[z3[2], z1[1]] → w1[2], EQ[z2[1], z3[2]] → w2[2]
};
```

```
RemoveBigon[s_, k_Integer] := (Replace[List@@s[[1, k]], {z1_m, z2_m} => (
 ($Log > 50) &&
 Print[StringForm["Doing bigon: k=``; s[[1,k]]=``; s=``", k, s[[1, k]], s]];
 t = s //. {EQ[z1, z2] → EQ[], EQ[z1[1], z2[2]] → EQ[], EQ[z1[2], z2[1]] → EQ[]};
 at = Position[t[[1]], # // First] & /@ {z1, z2};
 If[(Length /@ at) === {0, 0},
 ($Log > 25) && Print["Annular face removed..."]; Return[t]];
 If[(Length /@ at) != {1, 1},
 ($Log > 25) &&
 Print[StringForm["Bad bigon: at=``; t=``", at, t]]; Return[Failed],
 at1 = at[[1, 1, 1]]; at2 = at[[2, 1, 1]];
 Return[{
 Append[Delete[t[[1]], {{at1}, {at2}}],
 EQ[t[[1, at1]], t[[1, at2]]] /. EQ[z1, z2] → EQ[],
 t[[2]] /. {z1 → DeleteCases[t[[1, at2]], z2],
 z2 → DeleteCases[t[[1, at1]], z1]} //.
 {EQ[m[z_, 0], m[z_, 1], m[z_, 2]] → EQ[-1], EQ[-1, -1] → EQ[]}}
 ]}
);
Return[Failed]
);
```

```

RemoveTriangle[Failed, _] = Failed;
RemoveTriangle[s_, k_Integer] := (Replace[List@s[[1, k]], {z1_m, z2_m, z3_m} => (
  ($Log > 50) &&
  Print[StringForm["Doing triangle: k=`; s[[1,k]]=`; s=`; Lengths = `",
    k, s[[1, k]], s, Length /@ s[[1]]];
  {w1, w2} = m[#, 0] & /@ Unique[{w, w}];
  t = {
    Delete[s[[1]], k] //. (rules = PachnerRules[z1, z2, z3, w1, w2]),
    s[[2]] //. Join[rules, {
      EQ[z3[1], z3[2]] -> EQ[-1, w1[2], w2[1], w1[1], w2[2]],
      EQ[z2[1], z2[2]] -> EQ[-1, w1, w2, w1[1], w2[2]],
      EQ[z1[1], z1[2]] -> EQ[-1, w1, w2, w1[2], w2[1]],
      EQ[m[z_, 0], m[z_, 1], m[z_, 2]] -> EQ[-1], EQ[-1, -1] -> EQ[]
    }
  ]];
  If[FreeQ[t, z123 = First[z1] | First[z2] | First[z3]],
    Return[t],
    ($Log > 25) && Print[StringForm[
      "Failed triangle; t=`; problems at `", t, Select[t, !FreeQ[#, z123] &]]
  ]
)];
Return[Failed]
);
ContractDomino[s_, k1_Integer, k2_Integer] := (
  ($Log > 70) && Print[StringForm["{k1, k2} = {`, `}; s[[1]][[{k1,k2}]]={`, `}",
    k1, k2, s[[1]][[k1]], s[[1]][[k2]]];
  Replace[s[[1]][[k1, k2]],
    {EQ[_m, _m, w1_m, w2_m], EQ[wa_m, wb_m, _m, _m]} => (
    ($Log > 50) && Print[StringForm["Found domino at k1=` , k2=`", k1, k2]];
    {z1, z2, z3} = m[#, 0] & /@ Unique[{z, z, z}];
    t = Insert[
      s //. (Reverse /@ PachnerRules[z1, z2, z3, w1, w2]), EQ[z1, z2, z3], {1, -1}];
    t = RemoveTriangle[RemoveTriangle[t, k2], k1];
    ($Log > 50) && Print["t=", t];
    Return[t]
  ) /; (w1 == wa[1] && w2 == wb[2])
];
Return[Failed]
);
Red[rels_List] := Module[
  {s = rels, ks, k, t, at, at1, at2, k1, k2},
  Label[Start]; s[[1]] = RandomPermutation[DeleteCases[s[[1]], EQ[]]];
  If[(ev = Complement[Union[Cases[s[[2]], _Symbol, Infinity], Union[
    Cases[s[[1]], _Symbol, Infinity]])] != {z1}, Print["Extra vars: ", ev]];
  Do[If[Failed != (t = RemoveBigon[s, k]), s = t; Goto[Start]], {k, Length[s[[1]]}];
  Do[If[Failed != (t = RemoveTriangle[s, k]), s = t; Goto[Start]],
    {k, Length[s[[1]]}];
  ($Log > 50) && Print["Contracting dominos; Lengths=", Length /@ s[[1]], " s=", s];

```

```

Do[
  If[{4, 4} == (Length /@ s[[1]][[{k1, k2}]) &&
    Failed != (t = ContractDomino[s, k1, k2]), s = t; Goto[Start]],
    {k2, 2, Length[s[[1]]]}, {k1, k2 - 1}];
  s
]

{rels = Rels[Knot[6, 1]]; s = Red[rels], Length /@ s[[1]], Length /@ s}
{{EQ[m[w$94575, 0], m[w$94575, 1], m[w$94576, 1], m[w$94576, 2], m[w$94581, 0]],
  EQ[m[w$94575, 0], m[w$94575, 2], m[w$94576, 2], m[w$94581, 1], m[w$94581, 2],
  m[z$94560, 1], m[z$94560, 1]], EQ[m[w$94575, 2], m[w$94576, 0], m[w$94576, 1],
  m[w$94581, 0], m[w$94581, 1], m[z$94560, 0], m[z$94560, 2], m[z$94560, 2]],
  EQ[m[w$94575, 1], m[w$94576, 0], m[w$94581, 2], m[z$94560, 0]]},
EQ[z1, m[w$94575, 1], m[w$94575, 1], m[w$94575, 2], m[w$94575, 2],
m[w$94576, 1], m[w$94576, 1], m[w$94581, 1], m[w$94581, 1], m[w$94581, 2],
m[w$94581, 2], m[z$94560, 2], m[z$94560, 2]], {5, 7, 8, 4}, {4, 13}}

vars = Complement[Union[Cases[s, _Symbol, Infinity]], {z1}]
{w$94575, w$94576, w$94581, z$94560}

eqns = ReplacePart[
  (Plus @@ # == 2 Pi I) & /@ Rest[Flatten[s]] /.
  {z1 -> 0, m[w_, 0] -> Log[w], m[w_, 1] -> Log[1 - 1/w], m[w_, 2] -> Log[1/(1 - w)]},
  4 Pi I, {-1, 2}
]

{Log[1/(1 - w$94575)] + Log[w$94575] + Log[1/(1 - w$94576)] +
  Log[1 - 1/w$94581] + Log[1/(1 - w$94581)] + 2 Log[1 - 1/z$94560] == 2 i pi,
  Log[1/(1 - w$94575)] + Log[1 - 1/w$94576] + Log[w$94576] + Log[1 - 1/w$94581] +
  Log[w$94581] + 2 Log[1/(1 - z$94560)] + Log[z$94560] == 2 i pi,
  Log[1 - 1/w$94575] + Log[w$94576] + Log[1/(1 - w$94581)] + Log[z$94560] == 2 i pi,
  2 Log[1 - 1/w$94575] + 2 Log[1/(1 - w$94575)] + 2 Log[1 - 1/w$94576] +
  2 Log[1 - 1/w$94581] + 2 Log[1/(1 - w$94581)] + 2 Log[1/(1 - z$94560)] == 4 i pi}

```

```
{
  sol =
    FindRoot @@ Join[{eqns /. zl -> 1}, ({#, omega} & /@ vars), {MaxIterations -> 1000}],
    Max[Abs[[(First /@ (eqns /. zl -> 1)) /. sol] - 1]],
    Plus @@ (Vol /@ (vars /. sol))
}
{{w$94575 -> 0.322042 + 0.15778 i, w$94576 -> 0.851808 + 0.911292 i,
  w$94581 -> 0.895123 + 1.55249 i, z$94560 -> 0.0433154 + 0.6412 i}, 12.6061, 3.16396}
```

```
briefs = vars /. (sols = NSolve[eqns /. zl -> 1, vars])
```

```
$Aborted
```

```
Sort[{Plus @@ Vol /@ #, #} & /@ briefs]
```

```
{{2.82812, {0.122561 + 0.744862 i, 0.122561 + 0.744862 i, 0.78492 + 1.30714 i}}}
```

Jeff Weeks: Read about holonomies for CS/Vol computations within SnapPea in `gluingequations.c`; `hyperbolic structure.c`

```
leqns = Rest[Flatten[s]] /. eq_EQ => Plus @@ Log /@ eq /.
```

```
{m[w_, 0] => w, m[w_, 1] => 1 - 1/w, m[w_, 2] => 1/(1 - w), zl -> 1}
```

```
{Log[1 -  $\frac{1}{w51667}$ ] + Log[ $\frac{1}{1 - w51667}$ ] + 2 Log[ $\frac{1}{1 - w51671}$ ] +
```

```
Log[w51671] + Log[ $\frac{1}{1 - w51686}$ ] + Log[ $\frac{1}{1 - w51689}$ ] + Log[w51689],
```

```
Log[1 -  $\frac{1}{w51667}$ ] + Log[ $\frac{1}{1 - w51686}$ ] + 2 Log[w51686] + Log[ $\frac{1}{1 - w51689}$ ],
```

```
Log[w51667] + Log[w51671] + 2 Log[1 -  $\frac{1}{w51686}$ ] + Log[1 -  $\frac{1}{w51689}$ ], Log[w51667] +
```

```
Log[ $\frac{1}{1 - w51671}$ ] + 2 Log[ $\frac{1}{1 - w51686}$ ] + 2 Log[w51686] + 4 Log[ $\frac{1}{1 - w51689}$ ] + 3 Log[w51689]}
```

leqns /. sols

$$\begin{aligned}
& \{ \{-7.35767 \times 10^{-12} + 6.28319 \, i, -2.6632 \times 10^{-12} + 6.28319 \, i, \\
& \quad 7.50933 \times 10^{-12} + 2.86163 \times 10^{-12} \, i, -1.46922 \times 10^{-11} + 18.8496 \, i\}, \\
& \{-8.98615 \times 10^{-13} - 6.28319 \, i, 1.00364 \times 10^{-13} - 6.28319 \, i, -5.25968 \times 10^{-14} - 1.97873 \times 10^{-13} \, i, \\
& \quad -5.47118 \times 10^{-13} - 18.8496 \, i\}, \{2.43494 \times 10^{-12} + 6.28319 \, i, -1.33227 \times 10^{-14} + 6.28319 \, i, \\
& \quad 2.67064 \times 10^{-13} + 6.28319 \, i, 5.24025 \times 10^{-13} + 18.8496 \, i\}, \{1.39599 \times 10^{-12} - 6.28319 \, i, \\
& \quad 2.17326 \times 10^{-13} - 6.28319 \, i, 1.4736 \times 10^{-12} - 6.28319 \, i, -2.30038 \times 10^{-13} - 18.8496 \, i\}, \\
& \{4.34097 \times 10^{-14} + 6.28319 \, i, 2.22045 \times 10^{-14} - 1.79412 \times 10^{-13} \, i, \\
& \quad -3.52496 \times 10^{-15} - 6.28319 \, i, -5.01821 \times 10^{-14} + 6.28319 \, i\}, \\
& \{1.59872 \times 10^{-14} - 6.28319 \, i, 5.9952 \times 10^{-15} - 7.77156 \times 10^{-15} \, i, -1.90681 \times 10^{-14} + 6.28319 \, i, \\
& \quad 1.90958 \times 10^{-14} - 6.28319 \, i\}, \{4.8459 \times 10^{-12} + 6.28319 \, i, 7.40019 \times 10^{-12} - 5.78471 \times 10^{-12} \, i, \\
& \quad -1.57578 \times 10^{-11} + 6.28319 \, i, 1.54836 \times 10^{-11} + 12.5664 \, i\}, \{1.30562 \times 10^{-13} - 6.28319 \, i, \\
& \quad 2.13141 \times 10^{-12} - 4.04121 \times 10^{-14} \, i, -6.02002 \times 10^{-12} + 6.28319 \, i, 9.3916 \times 10^{-12} - 12.5664 \, i\}, \\
& \{-1.22367 \times 10^{-13} - 1.84297 \times 10^{-14} \, i, 3.71647 \times 10^{-14} + 6.28319 \, i, \\
& \quad -8.37663 \times 10^{-14} + 4.37428 \times 10^{-14} \, i, -4.32709 \times 10^{-14} + 12.5664 \, i\}, \\
& \{-2.32946 \times 10^{-13} + 3.08864 \times 10^{-13} \, i, 1.47382 \times 10^{-14} + 6.28319 \, i, \\
& \quad 1.69309 \times 10^{-14} + 1.05471 \times 10^{-13} \, i, 8.77354 \times 10^{-14} + 1.15463 \times 10^{-13} \, i\}, \\
& \{2.19689 \times 10^{-12} - 6.28319 \, i, -1.14058 \times 10^{-11} - 6.28319 \, i, 2.5442 \times 10^{-12} + 2.9946 \times 10^{-12} \, i, \\
& \quad 4.09438 \times 10^{-12} - 6.43485 \times 10^{-13} \, i\}, \{-1.67979 \times 10^{-12} + 6.28319 \, i, -2.49561 \times 10^{-12} + 6.28319 \, i, \\
& \quad 1.18196 \times 10^{-12} - 4.56302 \times 10^{-14} \, i, -1.36904 \times 10^{-13} + 1.25011 \times 10^{-12} \, i\}, \\
& \{5.14089 \times 10^{-13} + 6.28319 \, i, 1.31561 \times 10^{-13} + 6.28319 \, i, -2.49578 \times 10^{-13} + 6.28319 \, i, \\
& \quad -1.15297 \times 10^{-13} + 12.5664 \, i\}, \{-9.24871 \times 10^{-13} - 6.28319 \, i, \\
& \quad 9.98202 \times 10^{-13} - 6.28319 \, i, -7.7427 \times 10^{-13} - 6.28319 \, i, 9.42524 \times 10^{-13} - 12.5664 \, i\}, \\
& \{-2.05003 \times 10^{-13} + 6.28319 \, i, -1.27343 \times 10^{-13} - 4.38982 \times 10^{-13} \, i, \\
& \quad 3.31457 \times 10^{-13} + 4.77618 \times 10^{-13} \, i, -1.81188 \times 10^{-13} + 6.28319 \, i\}, \\
& \{-2.40918 \times 10^{-14} - 6.28319 \, i, 2.51243 \times 10^{-13} - 1.60982 \times 10^{-14} \, i, \\
& \quad -2.7206 \times 10^{-13} - 6.32827 \times 10^{-14} \, i, 3.72147 \times 10^{-13} - 6.28319 \, i\}, \\
& \{8.27782 \times 10^{-13} + 12.5664 \, i, 1.38223 \times 10^{-13}, -7.41851 \times 10^{-13} + 6.28319 \, i, \\
& \quad 5.57776 \times 10^{-13} + 12.5664 \, i\}, \{3.36842 \times 10^{-13} + 8.24896 \times 10^{-14} \, i, 1.40998 \times 10^{-14} + 6.28319 \, i, \\
& \quad -8.49321 \times 10^{-15} + 1.58984 \times 10^{-13} \, i, -2.79776 \times 10^{-14} + 6.28319 \, i\}, \\
& \{-7.57561 \times 10^{-13} + 1.4877 \times 10^{-14} \, i, 4.12337 \times 10^{-13} + 6.28319 \, i, \\
& \quad -6.56863 \times 10^{-13} + 1.12355 \times 10^{-13} \, i, 3.89466 \times 10^{-13} + 6.28319 \, i\}, \\
& \{-2.98461 \times 10^{-12} + 6.28319 \, i, 6.06659 \times 10^{-12} + 6.28319 \, i, -1.59536 \times 10^{-12} + 6.28319 \, i, \\
& \quad 1.03983 \times 10^{-12} + 6.28319 \, i\}, \{-1.39944 \times 10^{-12} - 6.28319 \, i, \\
& \quad -1.53899 \times 10^{-12} - 6.28319 \, i, 6.84813 \times 10^{-13} - 6.28319 \, i, -2.39675 \times 10^{-12} - 6.28319 \, i\}, \\
& \{-2.36144 \times 10^{-13} + 2.19769 \times 10^{-13} \, i, -8.26172 \times 10^{-13} - 5.38458 \times 10^{-15} \, i, \\
& \quad 9.13158 \times 10^{-13} + 6.28319 \, i, -7.94476 \times 10^{-13} + 6.57696 \times 10^{-13} \, i\}, \\
& \{3.18079 \times 10^{-13} - 1.8095 \times 10^{-12} \, i, -3.84304 \times 10^{-13} - 1.85019 \times 10^{-13} \, i, \\
& \quad 4.54026 \times 10^{-13} + 6.28319 \, i, -7.06546 \times 10^{-13} - 3.66818 \times 10^{-13} \, i\}, \\
& \{3.36176 \times 10^{-13} - 8.32889 \times 10^{-13} \, i, -1.22596 \times 10^{-13} + 7.37188 \times 10^{-14} \, i, \\
& \quad 4.95604 \times 10^{-13} + 6.28319 \, i, 2.12275 \times 10^{-13} + 6.28319 \, i\}, \{-8.68194 \times 10^{-14} + 5.09148 \times 10^{-13} \, i, \\
& \quad 6.42542 \times 10^{-14} - 2.73115 \times 10^{-14} \, i, -5.9508 \times 10^{-14} + 6.28319 \, i, 1.71418 \times 10^{-13} - 6.28319 \, i\}, \\
& \{8.85181 \times 10^{-13} - 6.28319 \, i, 4.21191 \times 10^{-12} + 3.38535 \times 10^{-13} \, i, \\
& \quad -2.76879 \times 10^{-12} + 6.28319 \, i, 5.37792 \times 10^{-13} + 2.71561 \times 10^{-12} \, i\}, \\
& \{-1.0042 \times 10^{-12} + 6.28319 \, i, -1.35442 \times 10^{-12} - 9.68309 \times 10^{-13} \, i, \\
& \quad 1.00209 \times 10^{-12} - 6.28319 \, i, -6.32383 \times 10^{-13} - 8.12461 \times 10^{-13} \, i\}
\end{aligned}$$

```

Im[leqns /. sols] /. arg_Real  $\Rightarrow$  (Abs[arg - 2 Pi] < 1)
{{True, True, False, False}, {False, False, False, False}, {True, True, True, False},
 {False, False, False, False}, {True, False, False, True}, {False, False, True, False},
 {True, False, True, False}, {False, False, True, False}, {False, True, False, False},
 {False, True, False, False}, {False, False, False, False}, {True, True, False, False},
 {True, True, True, False}, {False, False, False, False}, {True, False, False, True},
 {False, False, False, False}, {False, 0, True, False}, {False, True, False, True},
 {False, True, False, True}, {True, True, True, True}, {False, False, False, False},
 {False, False, True, False}, {False, False, True, False}, {False, False, True, True},
 {False, False, True, False}, {False, False, True, False}, {True, False, False, False}}

Position[Im[{2 Pi, 0} /. sols], {(arg_?((Abs[arg - 2 Pi] < 1) &)) .., arg_}]
{}

Position[Im[leqns /. sols] /. arg_Real  $\Rightarrow$  (Abs[arg - 2 Pi] < 1), {True .., False}]
{{3}, {13}}

Position[Apply[And, leqns /. sols /. z_?NumberQ  $\Rightarrow$  (Im[z] == 2 Pi), {1}], True]
{{18}}

Plus @@ Vol /@ (vars /. sols[[13]])
3.33174

```