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<< \ c : / drorbn / projects / KAtlas / KnotTheory.m
Loading KnotTheory`...

<< \ c : / drorbn / projects / KAtlas / KnotTheoryData.m
Loading KnotTheoryData.m...

Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1/(1 - z)})

2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
2.02988

Pachner[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +
  Vol[1/(z * w)] - Vol[z (1 - w) / (1 - z * w)] - Vol[w (1 - z) / (1 - z * w)]}

Pachner[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],
  Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
{0.612218, -0.644866, 3.7132 * 10^-12}

n = 4;

eqns = (# == 1) & /@ (prods = {
  z11 z21 z31 z41, z12 z22 z32 z42, z13 z23 z33 z43,
  z12 z14, z34 z13 z31, z24 z42 z23, z21 z22 z44, z41 z43, z11 z33 z32,
  z32 ' ' z11 ' z41 ' ' z43 ' z13 ' ' z31 ' z21 ' ' z22 ' ,
  z33 ' ' z32 ' z22 ' ' z44 ' z14 ' ' z12 ' z42 ' ' z23 ' ,
  z34 ' ' z13 ' z43 ' ' z41 ' z11 ' ' z33 ' z23 ' ' z24 ' ,
  z24 ' z42 ' ' z32 ' z11 ' ' z21 ' z22 ' ' z12 ' z14 ' ' ,
  z41 ' z43 ' ' z33 ' z32 ' ' z42 ' z23 ' ' z13 ' z31 ' ' ,
  z43 ' z41 ' ' z31 ' z34 ' ' z44 ' z21 ' ' z11 ' z33 ' ' ,
  z21 ' z31 ' ' (z34 ' ' z44 ' ) ^ (-1)
}) / . {(z_) ' => 1 - 1/z, (z_) ' ' => 1/(1 - z)}
{z11 z21 z31 z41 == 1, z12 z22 z32 z42 == 1, z13 z23 z33 z43 == 1,
  z12 z14 == 1, z13 z31 z34 == 1, z23 z24 z42 == 1, z21 z22 z44 == 1,
  z41 z43 == 1, z11 z32 z33 == 1, (1 - 1/z11) (1 - 1/z22) (1 - 1/z31) (1 - 1/z43) /
  (1 - z13) (1 - z21) (1 - z32) (1 - z41) == 1,
  (1 - 1/z12) (1 - 1/z23) (1 - 1/z32) (1 - 1/z44) /
  (1 - z14) (1 - z22) (1 - z33) (1 - z42) == 1, (1 - 1/z13) (1 - 1/z24) (1 - 1/z33) (1 - 1/z41) /
  (1 - z11) (1 - z23) (1 - z34) (1 - z43) == 1,
  (1 - 1/z12) (1 - 1/z21) (1 - 1/z24) (1 - 1/z32) /
  (1 - z11) (1 - z14) (1 - z22) (1 - z42) == 1, (1 - 1/z13) (1 - 1/z33) (1 - 1/z41) (1 - 1/z42) /
  (1 - z23) (1 - z31) (1 - z32) (1 - z43) == 1,
  (1 - 1/z11) (1 - 1/z31) (1 - 1/z43) (1 - 1/z44) /
  (1 - z21) (1 - z33) (1 - z34) (1 - z41) == 1, (1 - 1/z21) (1 - z34) /
  (1 - z31) (1 - 1/z44) == 1}

```

$1 - 1 / (1 - 1 / z)$  // Simplify

$$\frac{1}{1 - z}$$

`vars = Union[Cases[eqns, _Symbol, Infinity]]`

`{z11, z12, z13, z14, z21, z22, z23, z24, z31, z32, z33, z34, z41, z42, z43, z44}`

```
{
  sol =
  FindRoot @@ Join[{eqns}, ({#, Exp[Random[] Pi I]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[({(First /@ eqns) /. sol) - 1}],
  Plus @@ (Vol /@ (vars /. sol))
}
```

FindRoot::cvnwt: Newton's method failed to converge to the prescribed accuracy after 1000 iterations.

```
{z11 -> -1.00742 + 0.767799 i, z12 -> 0.766545 - 0.249402 i,
 z13 -> 485.501 + 1035.78 i, z14 -> 1.19301 + 0.374679 i, z21 -> -6.76537 - 0.119906 i,
 z22 -> 0.0109656 + 0.0185418 i, z23 -> 0.00967572 - 0.00529854 i, z24 -> 2.95393 + 1.42726 i,
 z31 -> -0.00465828 + 0.61902 i, z32 -> 0.862055 + 0.433171 i, z33 -> 1.44361 - 3.39701 i,
 z34 -> -0.0234347 + 0.00894835 i, z41 -> 0.750247 + 0.0598691 i, z42 -> -706.264 - 1848.57 i,
 z43 -> 1.40048 + 0.168355 i, z44 -> -620.472 - 657.217 i}, 132.04, 2.36117}
```

`SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat}]`

`im4[a_] := a + 1 /. 5 -> 1; dm4[a_] := a - 1 /. 0 -> 4`

Expected volume for 5\_2: 2.8281220883.

`L = PD[Knot[4, 1]]`

`PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]]`

`n = Length[L]`

4

`dt = EQs @@ Flatten[`

```
{
  Table[
    t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
    {v, n}, {j, 4}
  ],
  Table[e @@ Position[L, i], {i, 2 n}] /. e[x_, y_] -> {e[x, y], e[y, x]} /. {
    e[{v_, a_}, {u_, b_}] =>
      t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
    e[{v_, a_}, {u_, b_}] => t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
      t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
  }
} /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] => {
  t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
}
] /.
t[z[v_, j_], a_, b_] => t[z[v, j]] ~ Join ~ (t[a, b] /. {3 -> 4, 4 -> 3}) /; EvenQ[j];
```

```

rels = List @@ Union[
  dt /. p[t[z_, a_, b_], t[w_, c_, d_]] =>
    EQs[t[z, a, b] ~to~ t[w, d, c], t[w, c, d] ~to~ t[z, b, a]] /. (
    EQs[xx_. * (t[x_, ab_] ~to~ t[y_, cd_]), zz_. * (t[y_, cd_] ~to~ t[z, ef_])] ×
  => EQs[xx * y[cd] * zz * (t[x, ab] ~to~ t[z, ef])]
) /. (t[z_, ab_] ~to~ t[z_, ab_]) => z[ab] /.
  (w_z)[ab_] => Switch[Sort[{ab}],
    {1, 2} | {3, 4}, w[0],
    {1, 3} | {2, 4}, w[1],
    {1, 4} | {2, 3}, w[2]
  ]
] /. eq_Times => (EQ @@ eq);
merge = Position[Count[#, z[n, _][__]] & /@ rels, 4];
rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
rels = Append[rels, EQ @@ Product[
  z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
  If[L[[k, 4]] > L[[k, 2]] || L[[k, 2]] - L[[k, 4]] > 1,
    z[k, 1][1] * z[k, 4][2],
    z[k, 3][1] * z[k, 2][2]
  ],
  {k, n}
] /. {EQ[z_[0], z_[1], z_[2]] -> EQ[-1], EQ[-1, -1] -> EQ[]}
] /. z[i_, j_] => ToExpression["z" <> ToString[i] <> ToString[j]]
{EQ[z11[0], z12[0], z13[0], z14[0]],
EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]],
EQ[z12[1], z13[2], z21[1], z22[2], z23[1], z24[2], z31[2], z34[1]],
EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2], z34[1]],
EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]],
EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]],
EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2],
z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]],
EQ[z11[1], z12[2], z13[1], z14[2], z22[1], z23[2], z41[2], z44[1]],
EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2], z43[2], z44[1]],
EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1],
z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[z12[2],
z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]]}

```

**Length /@ rels**

```
{4, 2, 4, 3, 4, 8, 8, 3, 2, 8, 3, 8, 3, 8, 8, 20, 12}
```

**Plus @@ len /@ Length /@ rels**

```
2 len[2] + 4 len[3] + 3 len[4] + 6 len[8] + len[12] + len[20]
```

```
pr[k_] := (k + 1) ~Mod~ 3; prpr[k_] := (k + 2) ~Mod~ 3;
```

```

s = Module[{s = rels, t, at1, at2},
  Label[Start]; s = DeleteCases[s, EQ[]];
  For[k = 1, k ≤ Length[s], ++k, Replace[s[[k]], EQ[z1_[p1_], z2_[p2_]] ⇒ (
    Print[StringForm["Doing bigon: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
    t = Delete[s, k] /. {EQ[z1[p1 // pr], z2[p2 // prpr]] → EQ[],
      EQ[z1[p1 // prpr], z2[p2 // pr]] → EQ[]};
    If[Length[at1 = Position[t, z1]] ≠ 1 || Length[at2 = Position[t, z2]] ≠ 1,
      Print[StringForm["Bad bigon: at1=`; at2=`; t=`", at1, at2, t]],
      at1 = at1[[1, 1]]; at2 = at2[[1, 1]];
      s = If[at1 == at2,
        Print["Equal ats", t[[at1]]]; t /. EQ[z1[p1], z2[p2]] → EQ[],
        Append[Delete[t, {{at1}, {at2}}],
          EQ[t[[at1]], t[[at2]]] /. EQ[z1[p1], z2[p2]] → EQ[]
      ]];
    Goto[Start]
  ]];
)]];
For[k = 1, k ≤ Length[s], ++k, Replace[s[[k]], EQ[z1_[p1_], z2_[p2_], z3_[p3_]] ⇒ (
  Print[StringForm["Doing triangle: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
  {w1, w2} = Unique[{w, w}];
  t = Delete[s, k] /. {
    z3[p3] → EQ[w1[0], w2[0]],
    z2[p2] → EQ[w1[2], w2[1]], z1[p1] → EQ[w1[1], w2[2]],
    EQ[z3[p3 // pr], z3[p3 // prpr]] → EQ[-1, w1[2], w2[1], w1[1], w2[2]],
    EQ[z2[p2 // pr], z2[p2 // prpr]] → EQ[-1, w1[0], w2[0], w1[1], w2[2]],
    EQ[z1[p1 // pr], z1[p1 // prpr]] → EQ[-1, w1[0], w2[0], w1[2], w2[1]],
    EQ[z1[p1 // prpr], z2[p2 // pr]] → w1[0],
    EQ[z1[p1 // pr], z2[p2 // prpr]] → w2[0],
    EQ[z2[p2 // prpr], z3[p3 // pr]] → w1[1],
    EQ[z3[p3 // pr], z1[p1 // prpr]] → w2[1],
    EQ[z3[p3 // prpr], z1[p1 // pr]] → w1[2],
    EQ[z2[p2 // pr], z3[p3 // prpr]] → w2[2],
    EQ[z_[0], z_[1], z_[2]] → EQ[-1], EQ[-1, -1] → EQ[]
  };
  If[FreeQ[t, z1 | z2 | z3],
    s = t; Goto[Start],
    Print[StringForm["Failed triangle; t=`; problems at `",
      t, Select[t, !FreeQ[#, z1 | z2 | z3] &]]]
  ]];
)]];
s
]

```

Doing bigon: k=2; s[[k]]=EQ[z13[0], z23[0]];

s={EQ[z11[0], z12[0], z13[0], z14[0]], EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]], EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z13[2], z21[1], z22[2], z23[1], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2], z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2], z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z13[1], z14[2], z22[1], z23[2], z41[2], z44[1]], EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2], z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[z12[2], z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]]}

Doing bigon: k=6; s[[k]]=EQ[z32[0], z42[0]];

s={EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z21[1], z22[2], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z14[2], z24[1], z33[2], z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2], z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z14[2], z22[1], z41[2], z44[1]], EQ[z14[1], z21[1], z22[2], z24[2], z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[z12[2], z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]], EQ[z11[0], z12[0], z14[0], z21[0], z22[0], z24[0]]}

Doing triangle: k=1; s[[k]]=EQ[z14[0], z22[0], z34[0]];

s={EQ[z14[0], z22[0], z34[0]], EQ[z12[1], z21[1], z22[2], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z14[2], z24[1], z33[2], z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z21[2], z22[1], z31[1], z33[1], z34[2], z41[2]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z33[1], z34[2], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z14[2], z22[1], z41[2], z44[1]], EQ[z14[1], z21[1], z22[2], z24[2], z43[2], z44[1]], EQ[z12[2], z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]], EQ[z11[0], z12[0], z14[0], z21[0], z22[0], z24[0]], EQ[z11[2], z12[1], z21[2], z24[1], z31[0], z31[2], z33[0], z33[2], z34[0], z41[0], z41[1], z41[1], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]]}

Doing triangle: k=3; s[[k]]=EQ[z11[0], z33[0], z41[0]];

s={EQ[w\$13[1], z12[1], z21[1], z24[2], z31[2]], EQ[w\$14[1], z11[1], z12[2], z24[1], z33[2]], EQ[z11[0], z33[0], z41[0]], EQ[w\$14[2], z21[2], z31[1], z33[1], z41[2]], EQ[z21[0], z31[0], z43[0]], EQ[w\$13[2], z11[2], z31[1], z33[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$13[0], z11[1], z12[2], z41[2], z44[1]], EQ[w\$14[0], z21[1], z24[2], z43[2], z44[1]], EQ[w\$13[1], w\$14[0], w\$14[2], z12[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]], EQ[w\$13[1], w\$13[2], w\$14[1], w\$14[2], z11[0], z12[0], z21[0], z24[0]], EQ[w\$13[0], w\$14[0], z11[2], z12[1], z21[2], z24[1], z31[0], z31[2], z33[0], z33[2], z41[0], z41[1], z41[1], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]]}

Doing triangle: k=4; s[[k]]=EQ[z21[0], z31[0], z43[0]];

s={EQ[w\$13[1], z12[1], z21[1], z24[2], z31[2]], EQ[w\$14[1], w\$16[0], z12[2], z24[1]], EQ[w\$14[2], w\$16[2], z21[2], z31[1]], EQ[z21[0], z31[0], z43[0]], EQ[w\$13[2], w\$15[0], z31[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$13[0], w\$15[2], z12[2], z44[1]], EQ[w\$14[0], z21[1], z24[2], z43[2], z44[1]], EQ[w\$13[1], w\$14[0], w\$14[2], w\$15[1], w\$15[2], w\$16[1], z12[2], z24[0], z24[1], z31[1], z43[0], z43[2]], EQ[w\$13[1], w\$13[2], w\$14[1], w\$14[2], w\$15[1], w\$16[2], z12[0], z21[0], z24[0]], EQ[-1, w\$13[0], w\$14[0], w\$16[0], w\$16[1], w\$16[1], z12[1], z21[2], z24[1], z31[0], z31[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]]}

Doing triangle:  $k=3$ ;  $s[[k]] = \text{EQ}[w\$14[2], w\$16[2], w\$17[0]]$ ;

$s = \{ \text{EQ}[w\$13[1], w\$18[0], z12[1], z24[2]], \text{EQ}[w\$14[1], w\$16[0], z12[2], z24[1]], \text{EQ}[w\$14[2], w\$16[2], w\$17[0]], \text{EQ}[w\$13[2], w\$15[0], w\$18[2]], \text{EQ}[z12[0], z24[0], z44[0]], \text{EQ}[w\$13[0], w\$15[2], z12[2], z44[1]], \text{EQ}[w\$14[0], w\$17[2], z24[2], z44[1]], \text{EQ}[w\$13[1], w\$14[0], w\$14[2], w\$15[1], w\$15[2], w\$16[1], w\$17[0], w\$18[0], w\$18[2], z12[2], z24[0], z24[1]], \text{EQ}[w\$13[1], w\$13[2], w\$14[1], w\$14[2], w\$15[1], w\$16[2], w\$17[1], w\$18[2], z12[0], z24[0]], \text{EQ}[w\$13[0], w\$14[0], w\$16[0], w\$16[1], w\$16[1], w\$18[0], w\$18[1], w\$18[1], z12[1], z24[1], z44[0], z44[2], z44[2]] \}$

Doing triangle:  $k=2$ ;  $s[[k]] = \text{EQ}[w\$19[0], z12[2], z24[1]]$ ;

$s = \{ \text{EQ}[w\$13[1], w\$18[0], z12[1], z24[2]], \text{EQ}[w\$19[0], z12[2], z24[1]], \text{EQ}[w\$13[2], w\$15[0], w\$18[2]], \text{EQ}[z12[0], z24[0], z44[0]], \text{EQ}[w\$13[0], w\$15[2], z12[2], z44[1]], \text{EQ}[w\$19[2], z24[2], z44[1]], \text{EQ}[w\$13[1], w\$15[1], w\$15[2], w\$18[0], w\$18[2], w\$19[0], w\$19[1], w\$20[0], w\$20[0], w\$20[2], z12[2], z24[0], z24[1]], \text{EQ}[w\$13[1], w\$13[2], w\$15[1], w\$18[2], w\$19[1], w\$19[2], w\$20[1], w\$20[1], w\$20[2], z12[0], z24[0]], \text{EQ}[-1, w\$13[0], w\$18[0], w\$18[1], w\$18[1], w\$19[0], w\$19[1], w\$20[0], w\$20[0], w\$20[2], z12[1], z24[1], z44[0], z44[2], z44[2]] \}$

Doing bigon:  $k=3$ ;  $s[[k]] = \text{EQ}[w\$22[2], z44[0]]$ ;

$s = \{ \text{EQ}[w\$13[1], w\$18[0], w\$21[1]], \text{EQ}[w\$13[2], w\$15[0], w\$18[2]], \text{EQ}[w\$22[2], z44[0]], \text{EQ}[w\$13[0], w\$15[2], w\$21[2], w\$22[1], z44[1]], \text{EQ}[w\$22[1], z44[1]], \text{EQ}[w\$13[1], w\$15[1], w\$15[2], w\$18[0], w\$18[2], w\$20[0], w\$20[0], w\$20[2], w\$21[2]], \text{EQ}[w\$13[1], w\$13[2], w\$15[1], w\$18[2], w\$20[1], w\$20[1], w\$20[2], w\$21[0], w\$21[2]], \text{EQ}[-1, w\$13[0], w\$18[0], w\$18[1], w\$18[1], w\$20[0], w\$20[0], w\$20[2], w\$21[0], w\$21[1], w\$22[0], w\$22[0], w\$22[2], z44[0], z44[2], z44[2]] \}$

Equal at  $s \text{EQ}[-1, w\$13[0], w\$18[0], w\$18[1], w\$18[1],$

$w\$20[0], w\$20[0], w\$20[2], w\$21[0], w\$21[1], w\$22[2], z44[0]]$

Doing triangle:  $k=1$ ;  $s[[k]] = \text{EQ}[w\$13[1], w\$18[0], w\$21[1]]$ ;

$s = \{ \text{EQ}[w\$13[1], w\$18[0], w\$21[1]], \text{EQ}[w\$13[2], w\$15[0], w\$18[2]], \text{EQ}[w\$13[0], w\$15[2], w\$21[2]], \text{EQ}[w\$13[1], w\$15[1], w\$15[2], w\$18[0], w\$18[2], w\$20[0], w\$20[0], w\$20[2], w\$21[2]], \text{EQ}[w\$13[1], w\$13[2], w\$15[1], w\$18[2], w\$20[1], w\$20[1], w\$20[2], w\$21[0], w\$21[2]], \text{EQ}[-1, w\$13[0], w\$18[0], w\$18[1], w\$18[1], w\$20[0], w\$20[0], w\$20[2], w\$21[0], w\$21[1]] \}$

Doing bigon:  $k=1$ ;  $s[[k]] = \text{EQ}[w\$15[0], w\$24[0]]$ ;

$s = \{ \text{EQ}[w\$15[0], w\$24[0]], \text{EQ}[w\$15[2], w\$24[1]], \text{EQ}[w\$15[1], w\$15[2], w\$20[0], w\$20[0], w\$20[2], w\$23[1], w\$23[1], w\$23[2], w\$24[1], w\$24[2]], \text{EQ}[w\$15[1], w\$20[1], w\$20[1], w\$20[2], w\$23[1], w\$23[1], w\$23[2], w\$24[2]], \text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[0], w\$23[0], w\$23[2]] \}$

Bad bigon:  $at1 = \{ \}$ ;  $at2 = 7$ ;

$t = \{ \text{EQ}[], \text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[1], w\$23[1], w\$23[2]], \text{EQ}[w\$20[1], w\$20[1], w\$20[2], w\$23[1], w\$23[1], w\$23[2]], \text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[0], w\$23[0], w\$23[2]] \}$

Doing bigon:  $k=2$ ;  $s[[k]] = \text{EQ}[w\$15[2], w\$24[1]]$ ;

$s = \{ \text{EQ}[w\$15[0], w\$24[0]], \text{EQ}[w\$15[2], w\$24[1]], \text{EQ}[w\$15[1], w\$15[2], w\$20[0], w\$20[0], w\$20[2], w\$23[1], w\$23[1], w\$23[2], w\$24[1], w\$24[2]], \text{EQ}[w\$15[1], w\$20[1], w\$20[1], w\$20[2], w\$23[1], w\$23[1], w\$23[2], w\$24[2]], \text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[0], w\$23[0], w\$23[2]] \}$

Equal at  $s \text{EQ}[w\$15[2], w\$20[0], w\$20[0], w\$20[2], w\$23[1], w\$23[1], w\$23[2], w\$24[1]]$

$\{ \text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[1], w\$23[1], w\$23[2]],$

$\text{EQ}[w\$20[1], w\$20[1], w\$20[2], w\$23[1], w\$23[1], w\$23[2]],$

$\text{EQ}[w\$20[0], w\$20[0], w\$20[2], w\$23[0], w\$23[0], w\$23[2]] \}$

**Length** /@ s

{6, 6, 6}

```
eqns = (Times @@ # == 1) & /@ Rest[Reverse[s]] /.
  {w_[0] -> w, w_[1] -> 1 - 1/w, w_[2] -> 1/(1 - w)}
{ (1 - 1/w$20)^2 (1 - 1/w$23)^2 == 1, w$20^2 (1 - 1/w$23)^2 == 1 }
```

**vars** = Union[Cases[eqns, \_Symbol, Infinity]]

{w\$20, w\$23}

```
{
  sol = FindRoot @@
    Join[{eqns}, ({#, Exp[Random[] Pi I/2]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[(First /@ eqns) /. sol] - 1],
  Plus @@ (Vol /@ (vars /. sol))
}
{w$20 -> 0.5 + 0.866025 i, w$23 -> 0.5 + 0.866025 i}, 7.98034 x 10^-10, 2.02988}
```

**sols** = vars /. Solve[eqns, vars]

General::spell1: Possible spelling error: new symbol name "sols" is similar to existing symbol "sol".

```
{ {(-1)^(1/3), (-1)^(1/3)}, {-(-1)^(2/3), (-1)^(1/3)}, {(-1)^(1/3), -(-1)^(2/3)},
  {-(-1)^(2/3), -(-1)^(2/3)}, {1/2 (-1 - sqrt(5)), -1/2 - sqrt(5)/2}, {1/2 (-1 + sqrt(5)), -1/2 - sqrt(5)/2},
  {1/2 (-1 - sqrt(5)), -1/2 + sqrt(5)/2}, {1/2 (-1 + sqrt(5)), -1/2 + sqrt(5)/2} }
```

**N[sols]**

```
{ {0.5 + 0.866025 i, 0.5 + 0.866025 i}, {0.5 - 0.866025 i, 0.5 + 0.866025 i},
  {0.5 + 0.866025 i, 0.5 - 0.866025 i}, {0.5 - 0.866025 i, 0.5 - 0.866025 i},
  {-1.61803, -1.61803}, {0.618034, -1.61803}, {-1.61803, 0.618034}, {0.618034, 0.618034} }
```

**(Plus @@ Vol /@ #) & /@ sols**

NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u near u = 3.141592647481223`.

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General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation.

```
{ 2.02988, 0., 0., -2.02988, -7.41074 x 10^-15,
  -7.41074 x 10^-15, -7.41074 x 10^-15, -7.41074 x 10^-15 }
```