```
<< \c : /drorbn/projects/KAtlas/KnotTheory.m
```

```
Loading KnotTheory`...
```

```
<< \c : /drorbn/projects/KAtlas/KnotTheoryData.m
```

```
Loading KnotTheoryData.m...
```

```
Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1/(1 - z)})
```

```
2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
```

```
2.02988
```

```
Pachner[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +
  Vol[1/(z * w)] - Vol[z (1 - w)/(1 - z * w)] - Vol[w (1 - z)/(1 - z * w)]}
```

```
Pachner[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],
 Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
```

$$\{0.673587,\ -0.856627,\ -4.67726\times 10^{-11}\}$$

```
n = 4;
```

```
eqns = (# == 1) & /@ (prods = {
       z11 z21 z31 z41, z12 z22 z32 z42, z13 z23 z33 z43,
       z12 z14, z34 z13 z31, z24 z42 z23, z21 z22 z44, z41 z43, z11 z33 z32,
       z32'' z11' z41'' z43' z13'' z31' z21'' z22',
       z33'' z32' z22'' z44' z14'' z12' z42'' z23',
       z34'' z13' z43'' z41' z11'' z33' z23'' z24',
       z24' z42'' z32' z11'' z21' z22'' z12' z14'',
       z41' z43'' z33' z32'' z42' z23'' z13' z31'',
       z43' z41'' z31' z34'' z44' z21'' z11' z33'',
       z21' z31'' (z34'' z44')^(-1)
      }) /. {(z_)' :> 1 - 1/z, (z_)'' :> 1/(1 - z)}
```

$$\{z11\ z21\ z31\ z41 == 1,\ z12\ z22\ z32\ z42 == 1,\ z13\ z23\ z33\ z43 == 1,$$
$$z12\ z14 == 1,\ z13\ z31\ z34 == 1,\ z23\ z24\ z42 == 1,\ z21\ z22\ z44 == 1,$$
$$z41\ z43 == 1,\ z11\ z32\ z33 == 1,\ \frac{\left(1-\frac{1}{z11}\right)\left(1-\frac{1}{z22}\right)\left(1-\frac{1}{z31}\right)\left(1-\frac{1}{z43}\right)}{(1-z13)\ (1-z21)\ (1-z32)\ (1-z41)} == 1,$$
$$\frac{\left(1-\frac{1}{z12}\right)\left(1-\frac{1}{z23}\right)\left(1-\frac{1}{z32}\right)\left(1-\frac{1}{z44}\right)}{(1-z14)\ (1-z22)\ (1-z33)\ (1-z42)} == 1,\ \frac{\left(1-\frac{1}{z13}\right)\left(1-\frac{1}{z24}\right)\left(1-\frac{1}{z33}\right)\left(1-\frac{1}{z41}\right)}{(1-z11)\ (1-z23)\ (1-z34)\ (1-z43)} == 1,$$
$$\frac{\left(1-\frac{1}{z12}\right)\left(1-\frac{1}{z21}\right)\left(1-\frac{1}{z24}\right)\left(1-\frac{1}{z32}\right)}{(1-z11)\ (1-z14)\ (1-z22)\ (1-z42)} == 1,\ \frac{\left(1-\frac{1}{z13}\right)\left(1-\frac{1}{z33}\right)\left(1-\frac{1}{z41}\right)\left(1-\frac{1}{z42}\right)}{(1-z23)\ (1-z31)\ (1-z32)\ (1-z43)} == 1,$$
$$\frac{\left(1-\frac{1}{z11}\right)\left(1-\frac{1}{z31}\right)\left(1-\frac{1}{z43}\right)\left(1-\frac{1}{z44}\right)}{(1-z21)\ (1-z33)\ (1-z34)\ (1-z41)} == 1,\ \frac{\left(1-\frac{1}{z21}\right)(1-z34)}{(1-z31)\left(1-\frac{1}{z44}\right)} == 1\}$$

```
1 - 1 / (1 - 1 / z) // Simplify
```

$$\frac{1}{1-z}$$

```
vars = Union[Cases[eqns, _Symbol, Infinity]]
```

```
{z11, z12, z13, z14, z21, z22, z23, z24, z31, z32, z33, z34, z41, z42, z43, z44}
```

```
{
 sol =
  FindRoot @@ Join[{eqns}, ({#, Exp[Random[] Pi I]} & /@ vars), {MaxIterations → 1000}],
 Max[Abs[((First /@ eqns) /. sol) - 1]],
 Plus @@ (Vol /@ (vars /. sol))
}
```

```
FindRoot::cvnwt : Newton's method failed to converge to the prescribed accuracy after 1000 iterations.
```

```
{{z11 → -0.576747 + 1.82035 i, z12 → 1.00022 + 0.0000746186 i,
  z13 → 0.471464 + 0.269314 i, z14 → 0.874124 + 0.317071 i, z21 → 34.3421 + 18.831 i,
  z22 → 0.992115 + 0.644332 i, z23 → 0.918688 + 0.174357 i, z24 → -0.108107 + 0.0336574 i,
  z31 → -0.00298284 + 0.00619616 i, z32 → -0.0408985 + 0.219476 i, z33 → -1.94015 + 2.11268 i,
  z34 → -13.9632 + 36.6605 i, z41 → -0.422997 + 0.231038 i, z42 → 0.959666 + 0.0524925 i,
  z43 → -1.3272 + 0.376884 i, z44 → -0.0022327 - 0.0123671 i}, 3.93296, 5.34157}
```

```
SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat}]
```

```
im4[a_] := a + 1 /. 5 → 1; dm4[a_] := a - 1 /. 0 → 4
```

Expected volume for 5_2: 2.8281220883.

```
L = PD[Knot[4, 1]]
```

```
PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]]
```

```
n = Length[L]
```

```
4
```

```
dt = EQs @@ Flatten[
      {
        Table[
         t[z[v, j], 3, {1, 2, 4}] ~e~ t[z[v, im4@j], 4, {2, 1, 3}],
         {v, n}, {j, 4}
        ],
        Table[e @@ Position[L, i], {i, 2 n}] /. e[x_, y_] → {e[x, y], e[y, x]} /. {
          e[{v_, a_}, {u_, b_}] :>
           t[z[v, dm4@a], 2, {1, 3, 4}] ~e~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
          e[{v_, a_}, {u_, b_}] :> t[z[v, dm4@a], 2, {1, 3, 4}] ~e~
             t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
         }
       } /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] :> {
         t[z, a, c1] ~p~ t[w, b, d1], t[z, a, c2] ~p~ t[w, b, d2], t[z, a, c3] ~p~ t[w, b, d3]
        }
    ] /.
   t[z[v_, j_], a_, b_] :> t[z[v, j]] ~Join~ (t[a, b] /. {3 → 4, 4 → 3}) /; EvenQ[j];
```

```
rels = List @@ Union[
     dt /. p[t[z_, a_, b_], t[w_, c_, d_]] :>
          EQs[t[z, a, b]~to~t[w, d, c], t[w, c, d]~to~t[z, b, a]] //. (
         EQs[xx_. * (t[x_, ab__]~to~t[y_, cd__]), zz_. * (t[y_, cd__]~to~t[z_, ef__])] ×
:> EQs[xx * y[cd] * zz * (t[x, ab]~to~t[z, ef])]
        ) /. (t[z_, ab__]~to~t[z_, ab__]) :> z[ab] /.
      (w_z)[ab__] :> Switch[Sort[{ab}],
        {1, 2} | {3, 4}, w[0],
        {1, 3} | {2, 4},  w[1],
        {1, 4} | {2, 3}, w[2]
       ]
    ] /. eq_Times :> (EQ @@ eq);
merge = Position[Count[#, z[n, _][__]] & /@ rels, 4];
rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
rels = Append[rels, EQ @@ Product[
      z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
       If[L[[k, 4]] > L[[k, 2]] || L[[k, 2]] - L[[k, 4]] > 1,
        z[k, 1][1] * z[k, 4][2],
        z[k, 3][1] * z[k, 2][2]
       ],
      {k, n}
     ] //. {EQ[z_[0], z_[1], z_[2]] → EQ[-1], EQ[-1, -1] → EQ[]}
  ] /. z[i_, j_] :> ToExpression["z" <> ToString[i] <> ToString[j]]
```

```
{EQ[z11[0], z12[0], z13[0], z14[0]],
 EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
 EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]],
 EQ[z12[1], z13[2], z21[1], z22[2], z23[1], z24[2], z31[2], z34[1]],
 EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2], z34[1]],
 EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]],
 EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]],
 EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2],
  z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]],
 EQ[z11[1], z12[2], z13[1], z14[2], z22[1], z23[2], z41[2], z44[1]],
 EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2], z43[2], z44[1]],
 EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1],
  z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[z12[2],
  z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]]}
```

```
Length /@ rels
```

```
{4, 2, 4, 3, 4, 8, 8, 3, 2, 8, 3, 8, 3, 8, 8, 20, 12}
```

```
Plus @@ len /@ Length  /@ rels
```

$$2\,\text{len}[2] + 4\,\text{len}[3] + 3\,\text{len}[4] + 6\,\text{len}[8] + \text{len}[12] + \text{len}[20]$$

```
pr[k_] := (k + 1)~Mod~3; prpr[k_] := (k + 2)~Mod~3;
```

```
Module[{t, at1, at2},
 For[s = Drop[rels, 0]; k = 1, k ≤ Length[s], ++k,
  Replace[s[[k]], {
    EQ[] :> (s = Delete[s, k--];),
```

```
    EQ[z1_[p1_], z2_[p2_]] :> (
      Print[StringForm["Doing bigon: k=``; s[[k]]=``; s=``", k, s[[k]], s]];
      t = Delete[s, k] //. {
          EQ[z1[p1 // pr], z2[p2 // prpr]] → EQ[], EQ[z1[p1 // prpr], z2[p2 // pr]] → EQ[]
        };
      If[Length[at1 = Position[t, z1]] ≠ 1 || Length[at2 = Position[t, z2]] ≠ 1,
       Print[StringForm["Bad bigon: at1=``; at2=``; t=``", at1, at2, t]];
       Continue[],
       at1 = at1[[1, 1]]; at2 = at2[[1, 1]];
       If[at1 == at2,
        Print["Equal ats", t[[at1]]];
        k = 0;
        s = t /. EQ[z1[p1], z2[p2]] → EQ[];
        Continue[],
        s = Append[Delete[t, {{at1}, {at2}}],
          EQ[t[[at1]], t[[at2]]] /. EQ[z1[p1], z2[p2]] → EQ[]];
        k = 0; Continue[]
       ]
      ]
     ),
    EQ[z1_[p1_], z2_[p2_], z3_[p3_]] :> (
      Print[StringForm["Doing triangle: k=``; s[[k]]=``; s=``", k, s[[k]], s]];
      {w1, w2} = Unique[{w, w}];
      t = Delete[s, k] //. {
          z3[p3] → EQ[w1[0], w2[0]],
          z2[p2] → EQ[w1[2], w2[1]], z1[p1] → EQ[w1[1], w2[2]],
          EQ[z3[p3 // pr], z3[p3 // prpr]] → EQ[-1, w1[2], w2[1], w1[1], w2[2]],
          EQ[z2[p2 // pr], z2[p2 // prpr]] → EQ[-1, w1[0], w2[0], w1[1], w2[2]],
          EQ[z1[p1 // pr], z1[p1 // prpr]] → EQ[-1, w1[0], w2[0], w1[2], w2[1]],
          EQ[z1[p1 // prpr], z2[p2 // pr]] → w1[0],
          EQ[z1[p1 // pr], z2[p2 // prpr]] → w2[0],
          EQ[z2[p2 // prpr], z3[p3 // pr]] → w1[1],
          EQ[z3[p3 // pr], z1[p1 // prpr]] → w2[1],
          EQ[z3[p3 // prpr], z1[p1 // pr]] → w1[2],
          EQ[z2[p2 // pr], z3[p3 // prpr]] → w2[2],
          EQ[z_[0], z_[1], z_[2]] → EQ[-1], EQ[-1, -1] → EQ[]
        };
      If[! FreeQ[t, z1 | z2 | z3],
       Print[StringForm["Failed triangle; t=``; problems at ``",
         t, Select[t, ! FreeQ[#, z1 | z2 | z3] &]]]; Break[],
       s = t; k = 0
      ];
      Continue[]
     )
   }]
 ]; s
]
```

```
Doing bigon: k=2; s[[k]]=EQ[z13[0], z23[0]];
  s={EQ[z11[0], z12[0], z13[0], z14[0]], EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
   EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z13[2], z21[1], z22[
     2], z23[1], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2
     ], z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2]
    , z33[1], z34[2], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32
     [2], z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z13[1],
    z14[2], z22[1], z23[2], z41[2], z44[1]], EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2],
    z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0],
    z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[
    z12[2], z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]
    ]}
Doing triangle: k=1; s[[k]]=EQ[z14[0], z22[0], z34[0]];
  s={EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z21[1], z22[2],
    z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z14[2], z24[1], z33[2], z34[1]], EQ[z11[0], z33[0]
    , z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42
     [1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2], z33[1], z34[2], z42[1],
    z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z14[2], z22[1], z41[2], z44[1]], EQ[
    z14[1], z21[1], z22[2], z24[2], z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2],
    z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1]
    , z44[0], z44[2], z44[2]], EQ[z12[2], z14[0], z14[1], z22[2], z24[0], z24[1], z31[1], z33[0],
    z33[2], z41[1], z43[0], z43[2]], EQ[z11[0], z12[0], z14[0], z21[0], z22[0], z24[0]]}
Doing triangle: k=4; s[[k]]=EQ[z11[0], z33[0], z41[0]];
  s={EQ[w$13[0], w$14[0], z31[0], z32[0], z33[0]], EQ[w$13[1], z12[1], z21[1], z24[2], z31[2]], EQ
    [w$14[1], z11[1], z12[2], z24[1], z33[2]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[
    w$14[2], z21[2], z31[1], z32[2], z33[1], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[w$13[
     2], z11[2], z31[1], z32[2], z33[1], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w$13[0],
    z11[1], z12[2], z41[2], z44[1]], EQ[w$14[0], z21[1], z24[2], z43[2], z44[1]], EQ[z11[2], z12[1
    ], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0], z42[2], z42
     [2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[w$13[1], w$14[0], w$14[2], z12[2],
    z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]], EQ[w$13[1], w$13[2], w$14[1],
    w$14[2], z11[0], z12[0], z21[0], z24[0]]}
Doing bigon: k=4; s[[k]]=EQ[z32[0], z42[0]];
  s={EQ[w$13[0], w$14[0], w$15[2], w$16[1], z31[0], z32[0]], EQ[w$13[1], z12[1], z21[1], z24[2],
    z31[2]], EQ[w$14[1], w$16[0], z12[2], z24[1]], EQ[z32[0], z42[0]], EQ[w$14[2], w$16[2], z21[2
     ], z31[1], z32[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[w$13[2], w$15[0], z31[1], z32[2],
    z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w$13[0], w$15[2], z12[2], z44[1]], EQ[w$14[0]
    , z21[1], z24[2], z43[2], z44[1]], EQ[w$15[0], w$15[1], w$16[0], w$16[1], z12[1], z21[2], z24
     [1], z31[2], z32[1], z32[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2],
    z44[2]], EQ[w$13[1], w$14[0], w$14[2], w$15[1], w$15[2], w$16[1], z12[2], z24[0], z24[1], z31[
     1], z43[0], z43[2]], EQ[w$13[1], w$13[2], w$14[1], w$14[2], w$15[1], w$16[2], z12[0], z21[0],
    z24[0]]}
```

```
Doing triangle: k=4; s[[k]]=EQ[z21[0], z31[0], z43[0]];
  s={EQ[w$13[1], z12[1], z21[1], z24[2], z31[2]], EQ[w$14[1], w$16[0], z12[2], z24[1]], EQ[w$14[2
     ], w$16[2], z21[2], z31[1]], EQ[z21[0], z31[0], z43[0]], EQ[w$13[2], w$15[0], z31[1], z43[2]]
    , EQ[z12[0], z24[0], z44[0]], EQ[w$13[0], w$15[2], z12[2], z44[1]], EQ[w$14[0], z21[1], z24[2]
    , z43[2], z44[1]], EQ[w$13[1], w$14[0], w$14[2], w$15[1], w$15[2], w$16[1], z12[2], z24[0],
    z24[1], z31[1], z43[0], z43[2]], EQ[w$13[1], w$13[2], w$14[1], w$14[2], w$15[1], w$16[2], z12[
     0], z21[0], z24[0]], EQ[w$13[0], w$14[0], w$15[0], w$15[1], w$15[2], w$16[0], w$16[1], w$16[1
     ], z12[1], z21[2], z24[1], z31[0], z31[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]]}

Doing triangle: k=3; s[[k]]=EQ[w$14[2], w$16[2], w$17[0]];
  s={EQ[w$13[1], w$18[0], z12[1], z24[2]], EQ[w$14[1], w$16[0], z12[2], z24[1]], EQ[w$14[2], w$16
     [2], w$17[0]], EQ[w$13[2], w$15[0], w$18[2]], EQ[z12[0], z24[0], z44[0]], EQ[w$13[0], w$15[2
     ], z12[2], z44[1]], EQ[w$14[0], w$17[2], z24[2], z44[1]], EQ[w$13[1], w$14[0], w$14[2], w$15[
     1], w$15[2], w$16[1], w$17[0], w$18[0], w$18[2], z12[2], z24[0], z24[1]], EQ[w$13[1], w$13[2]
    , w$14[1], w$14[2], w$15[1], w$16[2], w$17[1], w$18[2], z12[0], z24[0]], EQ[w$13[0], w$14[0],
    w$16[0], w$16[1], w$16[1], w$18[0], w$18[1], w$18[1], z12[1], z24[1], z44[0], z44[2], z44[2]]}

Doing triangle: k=2; s[[k]]=EQ[w$19[0], z12[2], z24[1]];
  s={EQ[w$13[1], w$18[0], z12[1], z24[2]], EQ[w$19[0], z12[2], z24[1]], EQ[w$13[2], w$15[0], w$18
     [2]], EQ[z12[0], z24[0], z44[0]], EQ[w$13[0], w$15[2], z12[2], z44[1]], EQ[w$19[2], z24[2],
    z44[1]], EQ[w$13[1], w$15[1], w$15[2], w$18[0], w$18[2], w$19[0], w$19[1], w$20[0], w$20[0],
    w$20[2], z12[2], z24[0], z24[1]], EQ[w$13[1], w$13[2], w$15[1], w$18[2], w$19[1], w$19[2],
    w$20[1], w$20[1], w$20[2], z12[0], z24[0]], EQ[-1, w$13[0], w$18[0], w$18[1], w$18[1], w$19[0]
    , w$19[1], w$20[0], w$20[0], w$20[2], z12[1], z24[1], z44[0], z44[2], z44[2]]}

Doing triangle: k=1; s[[k]]=EQ[w$13[1], w$18[0], w$21[1]];
  s={EQ[w$13[1], w$18[0], w$21[1]], EQ[w$13[2], w$15[0], w$18[2]], EQ[w$22[2], z44[0]], EQ[w$13[0
     ], w$15[2], w$21[2], w$22[1], z44[1]], EQ[w$22[1], z44[1]], EQ[w$13[1], w$15[1], w$15[2],
    w$18[0], w$18[2], w$20[0], w$20[0], w$20[2], w$21[2]], EQ[w$13[1], w$13[2], w$15[1], w$18[2],
    w$20[1], w$20[1], w$20[2], w$21[0], w$21[2]], EQ[-1, w$13[0], w$18[0], w$18[1], w$18[1], w$20[
     0], w$20[0], w$20[2], w$21[0], w$21[1], w$22[0], w$22[0], w$22[2], z44[0], z44[2], z44[2]]}

Doing bigon: k=1; s[[k]]=EQ[w$15[0], w$24[0]];
  s={EQ[w$15[0], w$24[0]], EQ[w$22[2], z44[0]], EQ[w$15[2], w$22[1], w$24[1], z44[1]], EQ[w$22[1]
    , z44[1]], EQ[w$15[1], w$15[2], w$20[0], w$20[0], w$20[2], w$23[1], w$23[1], w$23[2], w$24[1]
    , w$24[2]], EQ[w$15[1], w$20[1], w$20[1], w$20[2], w$23[1], w$23[1], w$23[2], w$24[2]], EQ[
    w$20[0], w$20[0], w$20[2], w$22[0], w$22[0], w$22[2], w$23[0], w$23[0], w$23[2], z44[0], z44[2
    ], z44[2]]}

Bad bigon: at1={}; at2=10;
  t={EQ[w$22[2], z44[0]], EQ[w$22[1], z44[1]], EQ[w$22[1], z44[1]], EQ[w$20[0], w$20[0], w$20[2],
    w$23[1], w$23[1], w$23[2]], EQ[w$20[1], w$20[1], w$20[2], w$23[1], w$23[1], w$23[2]], EQ[w$20[
     0], w$20[0], w$20[2], w$22[0], w$22[0], w$22[2], w$23[0], w$23[0], w$23[2], z44[0], z44[2],
    z44[2]]}

Doing bigon: k=2; s[[k]]=EQ[w$22[2], z44[0]];
  s={EQ[w$15[0], w$24[0]], EQ[w$22[2], z44[0]], EQ[w$15[2], w$22[1], w$24[1], z44[1]], EQ[w$22[1]
    , z44[1]], EQ[w$15[1], w$15[2], w$20[0], w$20[0], w$20[2], w$23[1], w$23[1], w$23[2], w$24[1]
    , w$24[2]], EQ[w$15[1], w$20[1], w$20[1], w$20[2], w$23[1], w$23[1], w$23[2], w$24[2]], EQ[
    w$20[0], w$20[0], w$20[2], w$22[0], w$22[0], w$22[2], w$23[0], w$23[0], w$23[2], z44[0], z44[2
    ], z44[2]]}
```

```
Equal atsEQ[w$20[0], w$20[0], w$20[2], w$22[2], w$23[0], w$23[0], w$23[2], z44[0]]
```

```
Doing bigon: k=1; s[[k]]=EQ[w$15[0], w$24[0]];
  s={EQ[w$15[0], w$24[0]], EQ[w$15[2], w$24[1]], EQ[], EQ[w$15[1], w$15[2], w$20[0], w$20[0],
    w$20[2], w$23[1], w$23[1], w$23[2], w$24[1], w$24[2]], EQ[w$15[1], w$20[1], w$20[1], w$20[2],
    w$23[1], w$23[1], w$23[2], w$24[2]], EQ[w$20[0], w$20[0], w$20[2], w$23[0], w$23[0], w$23[2]]}
```

```
Bad bigon: at1={}; at2=6;
  t={EQ[], EQ[], EQ[w$20[0], w$20[0], w$20[2], w$23[1], w$23[1], w$23[2]], EQ[w$20[1], w$20[1],
    w$20[2], w$23[1], w$23[1], w$23[2]], EQ[w$20[0], w$20[0], w$20[2], w$23[0], w$23[0], w$23[2]]}
```

```
Doing bigon: k=2; s[[k]]=EQ[w$15[2], w$24[1]];
  s={EQ[w$15[0], w$24[0]], EQ[w$15[2], w$24[1]], EQ[], EQ[w$15[1], w$15[2], w$20[0], w$20[0],
    w$20[2], w$23[1], w$23[1], w$23[2], w$24[1], w$24[2]], EQ[w$15[1], w$20[1], w$20[1], w$20[2],
    w$23[1], w$23[1], w$23[2], w$24[2]], EQ[w$20[0], w$20[0], w$20[2], w$23[0], w$23[0], w$23[2]]}
```

```
Equal atsEQ[w$15[2], w$20[0], w$20[0], w$20[2], w$23[1], w$23[1], w$23[2], w$24[1]]
```

```
{EQ[w$20[0], w$20[0], w$20[2], w$23[1], w$23[1], w$23[2]],
 EQ[w$20[1], w$20[1], w$20[2], w$23[1], w$23[1], w$23[2]],
 EQ[w$20[0], w$20[0], w$20[2], w$23[0], w$23[0], w$23[2]]}
```

```
Length /@ s
```

```
{6, 6, 6}
```

```
eqns = (Times @@ # == 1) & /@ Rest[Reverse[s]] /.
  {w_[0] :> w, w_[1] :> 1 - 1/w, w_[2] :> 1/(1 - w)}
```

$$\left\{\frac{\left(1-\frac{1}{\text{w\$20}}\right)^2\left(1-\frac{1}{\text{w\$23}}\right)^2}{(1-\text{w\$20})\ (1-\text{w\$23})}==1,\ \frac{\text{w\$20}^2\left(1-\frac{1}{\text{w\$23}}\right)^2}{(1-\text{w\$20})\ (1-\text{w\$23})}==1\right\}$$

```
vars = Union[Cases[eqns, _Symbol, Infinity]]
```

```
{w$20, w$23}
```

```
{
 sol = FindRoot @@
   Join[{eqns}, ({#, Exp[Random[] Pi I/2]} & /@ vars), {MaxIterations → 1000}],
 Max[Abs[((First /@ eqns) /. sol) - 1]],
 Plus @@ (Vol /@ (vars /. sol))
}
```

$$\left\{\{\text{w\$20}\to 0.5+0.866025\,i,\ \text{w\$23}\to 0.5+0.866025\,i\},\ 6.63988\times 10^{-9},\ 2.02988\right\}$$

```
sols = vars /. Solve[eqns, vars]
```

```
General::spell1 : Possible spelling error: new symbol name "sols" is similar to existing symbol "sol".
```

$$\left\{\left\{(-1)^{1/3},\ (-1)^{1/3}\right\},\ \left\{-(-1)^{2/3},\ (-1)^{1/3}\right\},\ \left\{(-1)^{1/3},\ -(-1)^{2/3}\right\},\right.$$
$$\left\{-(-1)^{2/3},\ -(-1)^{2/3}\right\},\ \left\{\frac{1}{2}\left(-1-\sqrt{5}\right),\ -\frac{1}{2}-\frac{\sqrt{5}}{2}\right\},\ \left\{\frac{1}{2}\left(-1+\sqrt{5}\right),\ -\frac{1}{2}-\frac{\sqrt{5}}{2}\right\},$$
$$\left.\left\{\frac{1}{2}\left(-1-\sqrt{5}\right),\ -\frac{1}{2}+\frac{\sqrt{5}}{2}\right\},\ \left\{\frac{1}{2}\left(-1+\sqrt{5}\right),\ -\frac{1}{2}+\frac{\sqrt{5}}{2}\right\}\right\}$$

```
N[sols]
```

```
{{0.5+ 0.866025 i, 0.5+ 0.866025 i}, {0.5- 0.866025 i, 0.5+ 0.866025 i},
 {0.5+ 0.866025 i, 0.5- 0.866025 i}, {0.5- 0.866025 i, 0.5- 0.866025 i},
 {-1.61803, -1.61803}, {0.618034, -1.61803}, {-1.61803, 0.618034}, {0.618034, 0.618034}}
```

```
(Plus @@ Vol /@ #) & /@ sols
```

```
NIntegrate::ncvb : NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u
   near u = 3.141592647481223`.

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   near u = 3.141592647481223`.

NIntegrate::ncvb : NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u
   near u = 3.141592647481223`.

General::stop : Further output of NIntegrate::ncvb will be suppressed during this calculation.
```

$$\{2.02988,\ 0.,\ 0.,\ -2.02988,\ -7.41074\times 10^{-15},\ -7.41074\times 10^{-15},\ -7.41074\times 10^{-15},\ -7.41074\times 10^{-15}\}$$