

Expected volume for 5\_2: 2.8281220883.

```
<< \ c : / drorbn / projects / KAtlas / KnotTheory.m
```

```
Loading KnotTheory` ...
```

```
<< \ c : / drorbn / projects / KAtlas / KnotTheoryData.m
```

```
Loading KnotTheoryData.m...
```

```
Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
```

```
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1 / (1 - z)})
```

```
2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
```

```
2.02988
```

```
PachnerTest[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +  
  Vol[1 / (z * w)] - Vol[z (1 - w) / (1 - z * w)] - Vol[w (1 - z) / (1 - z * w)]}
```

```
PachnerTest[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],  
  Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
```

```
{0.131181, 0.096889, -1.57714 × 10-12}
```

```
SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat, OneIdentity}]
```

```

Rels[K_] := Rels[K] = Rels[PD[K]];
Rels[pd_PD] := Module[
  {n, dt, z, e, u, v, j, im4, dm4, w, zz,
   xx, yy, ab, merge, rels, x, y, a, b, c, d, d1, d2, d3, eq},
  im4[a_] := a + 1 /. 5 -> 1; dm4[a_] := a - 1 /. 0 -> 4;
  n = Length[pd];
  dt = EQs @@ Flatten[
    {
      Table[
        t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
        {v, n}, {j, 4}
      ],
      Table[e @@ Position[pd, i], {i, 2 n}] /. e[x_, y_] -> {e[x, y], e[y, x]} /. {
        e[{v_, a_}, {u_, b_}] ->
          t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
        e[{v_, a_}, {u_, b_}] -> t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
          t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
      }
    ] /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] -> {
      t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
    }
  ] /.
  t[z[v_, j_], a_, b_] -> t[z[v, j]] ~ Join ~ (t[a, b] /. {3 -> 4, 4 -> 3}) /; EvenQ[j];
  rels = List @@ Union[
    dt /. p[t[z_, a_, b_], t[w_, c_, d_]] ->
      EQs[t[z, a, b] ~ to ~ t[w, d, c], t[w, c, d] ~ to ~ t[z, b, a]] // . (
        EQs[xx_. * (t[x_, ab_] ~ to ~ t[y_, cd_]), zz_. *
          (t[y_, cd_] ~ to ~ t[z_, ef_])] ->
          EQs[xx * y[cd] * zz * (t[x, ab] ~ to ~ t[z, ef])]
      ) /. (t[z_, ab_] ~ to ~ t[z_, ab_]) -> z[ab] /.
      (w_z)[ab_] -> Switch[Sort[{ab}],
        {1, 2} | {3, 4}, w[0],
        {1, 3} | {2, 4}, w[1],
        {1, 4} | {2, 3}, w[2]
      ]
    ] /. eq_Times -> (EQ @@ eq);
  merge = Position[Count[#, z[n, _][_] & /@ rels, 4];
  rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
  Append[rels, z1 ~ EQ ~ (EQ @@ Product[
    z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
      If[pd[[k, 4]] > pd[[k, 2]] || pd[[k, 2]] - pd[[k, 4]] > 1,
        z[k, 1][1] * z[k, 4][2],
        z[k, 3][1] * z[k, 2][2]
      ],
    {k, n}
  ])] // . {EQ[z_[0], z_[1], z_[2]] -> EQ[-1], EQ[-1, -1] -> EQ[]}
  ] /. z[i_, j_][k_] -> m[ToExpression["z" <> ToString[i] <> ToString[j]], k]
]

```

General::spell1: Possible spelling error: new symbol name "Rels" is similar to existing symbol "Reals".

General::spell1: Possible spelling error: new symbol name "rels" is similar to existing symbol "Rels".

**rels = Rels[Knot[7, 2]]**

```
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]],
EQ[m[z14, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1], m[z23, 2], m[z31, 2], m[z34, 1]],
EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]],
EQ[m[z42, 0], m[z52, 0]], EQ[m[z51, 0], m[z52, 0], m[z53, 0], m[z54, 0]],
EQ[m[z23, 2], m[z24, 1], m[z41, 2], m[z42, 1], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2]],
EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z43, 0], m[z51, 0], m[z61, 0]],
EQ[m[z32, 0], m[z62, 0]],
EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z61, 2], m[z62, 1]],
EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z62, 1], m[z63, 2]],
EQ[m[z44, 0], m[z64, 0]], EQ[m[z61, 0], m[z62, 0], m[z63, 0], m[z64, 0]],
EQ[m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z52, 1], m[z53, 2], m[z61, 2], m[z64, 1]],
EQ[m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z52, 1], m[z63, 2], m[z64, 1]],
EQ[m[z32, 1], m[z33, 2], m[z41, 2], m[z44, 1], m[z61, 1], m[z62, 2], m[z63, 1], m[z64, 2]],
EQ[m[z31, 2], m[z32, 1], m[z43, 2], m[z44, 1], m[z61, 1], m[z62, 2], m[z63, 1], m[z64, 2]],
EQ[m[z13, 0], m[z31, 0], m[z41, 0], m[z53, 0], m[z63, 0], m[z71, 0]],
EQ[m[z12, 0], m[z22, 0], m[z72, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z33, 2], m[z34, 1], m[z71, 2], m[z72, 1]],
EQ[m[z21, 0], m[z73, 0]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2],
m[z51, 2], m[z54, 1], m[z72, 1], m[z73, 2]], EQ[m[z24, 0], m[z54, 0], m[z74, 0]],
EQ[m[z42, 1], m[z43, 2], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2], m[z71, 2], m[z74, 1]],
EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z73, 2], m[z74, 1]],
EQ[m[z12, 1], m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1],
m[z53, 2], m[z54, 1], m[z71, 0], m[z71, 1], m[z71, 1], m[z72, 0], m[z72, 2],
m[z72, 2], m[z73, 0], m[z73, 1], m[z73, 1], m[z74, 0], m[z74, 2], m[z74, 2]],
EQ[-1, z1, m[z11, 1], m[z13, 0], m[z13, 2], m[z21, 1], m[z23, 0], m[z23, 2], m[z31, 1],
m[z33, 0], m[z33, 2], m[z41, 1], m[z43, 0], m[z43, 2], m[z51, 1], m[z53, 0],
m[z53, 2], m[z61, 1], m[z63, 0], m[z63, 2], m[z71, 1], m[z73, 0], m[z73, 2]]}
```

```

rels = Delete[rels, Position[rels, z1][[1, 1]]]
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
 EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]],
 EQ[m[z14, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]],
 EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1], m[z23, 2], m[z31, 2], m[z34, 1]],
 EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]],
 EQ[m[z42, 0], m[z52, 0]], EQ[m[z51, 0], m[z52, 0], m[z53, 0], m[z54, 0]],
 EQ[m[z23, 2], m[z24, 1], m[z41, 2], m[z42, 1], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2]],
 EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z43, 0], m[z51, 0], m[z61, 0]],
 EQ[m[z32, 0], m[z62, 0]],
 EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z61, 2], m[z62, 1]],
 EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z62, 1], m[z63, 2]],
 EQ[m[z44, 0], m[z64, 0]], EQ[m[z61, 0], m[z62, 0], m[z63, 0], m[z64, 0]],
 EQ[m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z52, 1], m[z53, 2], m[z61, 2], m[z64, 1]],
 EQ[m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z52, 1], m[z63, 2], m[z64, 1]],
 EQ[m[z32, 1], m[z33, 2], m[z41, 2], m[z44, 1], m[z61, 1], m[z62, 2], m[z63, 1], m[z64, 2]],
 EQ[m[z31, 2], m[z32, 1], m[z43, 2], m[z44, 1], m[z61, 1], m[z62, 2], m[z63, 1], m[z64, 2]],
 EQ[m[z13, 0], m[z31, 0], m[z41, 0], m[z53, 0], m[z63, 0], m[z71, 0]],
 EQ[m[z12, 0], m[z22, 0], m[z72, 0]],
 EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z33, 2], m[z34, 1], m[z71, 2], m[z72, 1]],
 EQ[m[z21, 0], m[z73, 0]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2],
 m[z51, 2], m[z54, 1], m[z72, 1], m[z73, 2]], EQ[m[z24, 0], m[z54, 0], m[z74, 0]],
 EQ[m[z42, 1], m[z43, 2], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2], m[z71, 2], m[z74, 1]],
 EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z73, 2], m[z74, 1]],
 EQ[m[z12, 1], m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1],
 m[z53, 2], m[z54, 1], m[z71, 0], m[z71, 1], m[z71, 1], m[z72, 0], m[z72, 2],
 m[z72, 2], m[z73, 0], m[z73, 1], m[z73, 1], m[z74, 0], m[z74, 2], m[z74, 2]]}

```

Length /@ rels

```
{4, 4, 2, 4, 8, 4, 2, 4, 8, 6, 2, 8, 8, 2, 4, 8, 8, 8, 8, 6, 3, 8, 2, 8, 3, 8, 8, 20}
```

Plus @@ len /@ Length /@ rels

\$Log = 100;

RandomPermutation[l\_List] := Last /@ Sort[{Random[], #} & /@ l];

m[z\_, k1\_][k2\_] := m[z, (k1 + k2) ~Mod~ 3];

```

PachnerRules[z1_, z2_, z3_, w1_, w2_] := {
  z3 → EQ[w1, w2], z2 → EQ[w1[2], w2[1]], z1 → EQ[w1[1], w2[2]],
  EQ[z1[2], z2[1]] → w1, EQ[z1[1], z2[2]] → w2, EQ[z2[2], z3[1]] → w1[1],
  EQ[z3[1], z1[2]] → w2[1], EQ[z3[2], z1[1]] → w1[2], EQ[z2[1], z3[2]] → w2[2]
};

```

```

RemoveBigon[s_, k_Integer] := (Replace[List@@s[[k]], {z1_m, z2_m} :-> (
  ($Log > 50) && Print[StringForm["Doing bigon: k=``; s[[k]]=``; s=``", k, s[[k]], s]];
  t = s //. {EQ[z1, z2] → EQ[], EQ[z1[1], z2[2]] → EQ[], EQ[z1[2], z2[1]] → EQ[]};
  at = Position[t, # // First] & /@ {z1, z2};
  If[(Length /@ at) === {0, 0},
    ($Log > 25) && Print["Annular face removed..."]; Return[t];
  If[(Length /@ at) != {1, 1},
    ($Log > 25) &&
    Print[StringForm["Bad bigon: at=``; t=``", at, t]]; Return[Failed],

```

```

    at1 = at[[1, 1, 1]]; at2 = at[[2, 1, 1]];
    Return[If[at1 == at2,
      ($Log > 25) && Print["Equal ats ", t[[at1]]; t /. EQ[z1, z2] → EQ[],
      Append[Delete[t, {{at1}, {at2}}],
      EQ[t[[at1]], t[[at2]]] /. EQ[z1, z2] → EQ[]
    ]]]];
  Return[Failed]
);
RemoveTriangle[Failed, _] = Failed;
RemoveTriangle[s_, k_Integer] := (Replace[List@s[[k]], {z1_m, z2_m, z3_m} => (
  ($Log > 50) &&
  Print[StringForm["Doing triangle: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
  {w1, w2} = m[#, 0] & /@ Unique[{w, w}];
  t = Delete[s, k] /. PachnerRules[z1, z2, z3, w1, w2];
  If[FreeQ[t, z123 = First[z1] | First[z2] | First[z3]],
    Return[t],
    ($Log > 25) && Print[StringForm[
      "Failed triangle; t=`; problems at `", t, Select[t, !FreeQ[#, z123] &]]]
  ]
)];
Return[Failed]
);
ContractDomino[s_, k1_Integer, k2_Integer] := (
  ($Log > 70) && Print[StringForm[
    "k1, k2 = {`, `}; s[{{k1,k2}}]={`, `}", k1, k2, s[[k1]], s[[k2]]];
  Replace[s[{{k1, k2}}],
  {EQ[_m, _m, w1_m, w2_m], EQ[wa_m, wb_m, _m, _m]} => (
    ($Log > 50) && Print[StringForm["Found domino at k1=` , k2=`", k1, k2]];
    {z1, z2, z3} = m[#, 0] & /@ Unique[{z, z, z}];
    t = Append[s /. (Reverse /@ PachnerRules[z1, z2, z3, w1, w2]), EQ[z1, z2, z3]];
    t = RemoveTriangle[RemoveTriangle[t, k2], k1];
    ($Log > 50) && Print["t=", t];
    Return[t]
  ) /; (w1 == wa[1] && w2 == wb[2])
  ];
Return[Failed]
);
Red[rels_List] := Module[
  {s = rels, ks, k, t, at, at1, at2, k1, k2},
  Label[Start]; s = DeleteCases[s, EQ[]];
  Do[If[Failed != (t = RemoveBigon[s, k]), s = t; Goto[Start]], {k, Length[s]}];
  Do[If[Failed != (t = RemoveTriangle[s, k]), s = t; Goto[Start]], {k, Length[s]}];
  ($Log > 50) && Print["Contracting dominos; Lengths=", Length /@ s, " s=", s];
  Do[
    If[{4, 4} == (Length /@ s[{{k1, k2}}]) &&
      Failed != (t = ContractDomino[s, k1, k2]), s = t; Goto[Start]],

```

```

    {k2, 2, Length[s]}, {k1, k2 - 1}];
  s
]

{rels = Rels[Knot[4, 1]];
 s = Red[rels = Delete[rels, Position[rels, z1][[1, 1]]], Length /@ s, Length[s]}

Doing bigon: k=2; s[[k]]=EQ[m[z13, 0], m[z23, 0]];
s={EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]], EQ[m[z13, 0], m[z23, 0]], EQ[m[z21, 0], m[z22,
0], m[z23, 0], m[z24, 0]], EQ[m[z14, 0], m[z22, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33,
0], m[z34, 0]], EQ[m[z12, 1], m[z13, 2], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z31, 2],
m[z34, 1]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z23, 2], m[z24, 1], m[z33, 2], m[z34
, 1]], EQ[m[z11, 0], m[z33, 0], m[z41, 0]], EQ[m[z32, 0], m[z42, 0]], EQ[m[z21, 2], m[z22, 1],
m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]], EQ[m[z21, 0], m[z31, 0], m[z43
, 0]], EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z42, 1], m[z43, 2]
], EQ[m[z12, 0], m[z24, 0], m[z44, 0]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1
], m[z23, 2], m[z41, 2], m[z44, 1]], EQ[m[z13, 2], m[z14, 1], m[z21, 1], m[z22, 2], m[z23, 1], m
[z24, 2], m[z43, 2], m[z44, 1]], EQ[m[z11, 2], m[z12, 1], m[z21, 2], m[z24, 1], m[z31, 2], m[z32
, 1], m[z32, 1], m[z33, 2], m[z41, 0], m[z41, 1], m[z41, 1], m[z42, 0], m[z42, 2], m[z42, 2], m[
z43, 0], m[z43, 1], m[z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]]}

Doing bigon: k=6; s[[k]]=EQ[m[z32, 0], m[z42, 0]];
s={EQ[m[z14, 0], m[z22, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]], EQ[m[z12,
1], m[z21, 1], m[z22, 2], m[z24, 2], m[z31, 2], m[z34, 1]], EQ[m[z11, 1], m[z12, 2], m[z14, 2],
m[z24, 1], m[z33, 2], m[z34, 1]], EQ[m[z11, 0], m[z33, 0], m[z41, 0]], EQ[m[z32, 0], m[z42, 0]],
EQ[m[z21, 2], m[z22, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]], EQ[m[
z21, 0], m[z31, 0], m[z43, 0]], EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34,
2], m[z42, 1], m[z43, 2]], EQ[m[z12, 0], m[z24, 0], m[z44, 0]], EQ[m[z11, 1], m[z12, 2], m[z14,
2], m[z22, 1], m[z41, 2], m[z44, 1]], EQ[m[z14, 1], m[z21, 1], m[z22, 2], m[z24, 2], m[z43, 2],
m[z44, 1]], EQ[m[z11, 2], m[z12, 1], m[z21, 2], m[z24, 1], m[z31, 2], m[z32, 1], m[z32, 1], m[z33
, 2], m[z41, 0], m[z41, 1], m[z41, 1], m[z42, 0], m[z42, 2], m[z42, 2], m[z43, 0], m[z43, 1], m[
z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]], EQ[m[z11, 0], m[z12, 0], m[z14, 0], m[z21, 0], m[z22,
0], m[z24, 0]]}

Doing triangle: k=1; s[[k]]=EQ[m[z14, 0], m[z22, 0], m[z34, 0]];
s={EQ[m[z14, 0], m[z22, 0], m[z34, 0]], EQ[m[z12, 1], m[z21, 1], m[z22, 2], m[z24, 2], m[z31, 2],
m[z34, 1]], EQ[m[z11, 1], m[z12, 2], m[z14, 2], m[z24, 1], m[z33, 2], m[z34, 1]], EQ[m[z11, 0], m
[z33, 0], m[z41, 0]], EQ[m[z21, 2], m[z22, 1], m[z31, 1], m[z33, 1], m[z34, 2], m[z41, 2]], EQ[
m[z21, 0], m[z31, 0], m[z43, 0]], EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z33, 1], m[z34, 2], m[z43
, 2]], EQ[m[z12, 0], m[z24, 0], m[z44, 0]], EQ[m[z11, 1], m[z12, 2], m[z14, 2], m[z22, 1], m[
z41, 2], m[z44, 1]], EQ[m[z14, 1], m[z21, 1], m[z22, 2], m[z24, 2], m[z43, 2], m[z44, 1]], EQ[m
[z11, 0], m[z12, 0], m[z14, 0], m[z21, 0], m[z22, 0], m[z24, 0]], EQ[m[z11, 2], m[z12, 1], m[z21
, 2], m[z24, 1], m[z31, 0], m[z31, 2], m[z33, 0], m[z33, 2], m[z34, 0], m[z41, 0], m[z41, 1], m[
z41, 1], m[z43, 0], m[z43, 1], m[z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]]}

```

Doing triangle: k=3; s[[k]]=EQ[m[z11, 0], m[z33, 0], m[z41, 0]];

$$s = \{ \text{EQ}[m[w\$876, 1], m[z12, 1], m[z21, 1], m[z24, 2], m[z31, 2]], \text{EQ}[m[w\$877, 1], m[z11, 1], m[z12, 2], m[z24, 1], m[z33, 2]], \text{EQ}[m[z11, 0], m[z33, 0], m[z41, 0]], \text{EQ}[m[w\$877, 2], m[z21, 2], m[z31, 1], m[z33, 1], m[z41, 2]], \text{EQ}[m[z21, 0], m[z31, 0], m[z43, 0]], \text{EQ}[m[w\$876, 2], m[z11, 2], m[z31, 1], m[z33, 1], m[z43, 2]], \text{EQ}[m[z12, 0], m[z24, 0], m[z44, 0]], \text{EQ}[m[w\$876, 0], m[z11, 1], m[z12, 2], m[z41, 2], m[z44, 1]], \text{EQ}[m[w\$877, 0], m[z21, 1], m[z24, 2], m[z43, 2], m[z44, 1]], \text{EQ}[m[w\$876, 1], m[w\$876, 2], m[w\$877, 1], m[w\$877, 2], m[z11, 0], m[z12, 0], m[z21, 0], m[z24, 0]], \text{EQ}[m[w\$876, 0], m[w\$877, 0], m[z11, 2], m[z12, 1], m[z21, 2], m[z24, 1], m[z31, 0], m[z31, 2], m[z33, 0], m[z33, 2], m[z41, 0], m[z41, 1], m[z41, 1], m[z43, 0], m[z43, 1], m[z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]] \}$$

Doing triangle: k=4; s[[k]]=EQ[m[z21, 0], m[z31, 0], m[z43, 0]];

$$s = \{ \text{EQ}[m[w\$876, 1], m[z12, 1], m[z21, 1], m[z24, 2], m[z31, 2]], \text{EQ}[m[w\$877, 1], m[w\$879, 0], m[z12, 2], m[z24, 1]], \text{EQ}[m[w\$877, 2], m[w\$879, 2], m[z21, 2], m[z31, 1]], \text{EQ}[m[z21, 0], m[z31, 0], m[z43, 0]], \text{EQ}[m[w\$876, 2], m[w\$878, 0], m[z31, 1], m[z43, 2]], \text{EQ}[m[z12, 0], m[z24, 0], m[z44, 0]], \text{EQ}[m[w\$876, 0], m[w\$878, 2], m[z12, 2], m[z44, 1]], \text{EQ}[m[w\$877, 0], m[z21, 1], m[z24, 2], m[z43, 2], m[z44, 1]], \text{EQ}[m[w\$876, 1], m[w\$876, 2], m[w\$877, 1], m[w\$877, 2], m[w\$878, 1], m[w\$879, 2], m[z12, 0], m[z21, 0], m[z24, 0]], \text{EQ}[m[w\$876, 0], m[w\$877, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$879, 0], m[w\$879, 1], m[w\$879, 1], m[z12, 1], m[z21, 2], m[z24, 1], m[z31, 0], m[z31, 2], m[z43, 0], m[z43, 1], m[z43, 1], m[z44, 0], m[z44, 2], m[z44, 2]] \}$$

Doing triangle: k=3; s[[k]]=EQ[m[w\\$877, 2], m[w\\$879, 2], m[w\\$880, 0]];

$$s = \{ \text{EQ}[m[w\$876, 1], m[w\$881, 0], m[z12, 1], m[z24, 2]], \text{EQ}[m[w\$877, 1], m[w\$879, 0], m[z12, 2], m[z24, 1]], \text{EQ}[m[w\$877, 2], m[w\$879, 2], m[w\$880, 0]], \text{EQ}[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], \text{EQ}[m[z12, 0], m[z24, 0], m[z44, 0]], \text{EQ}[m[w\$876, 0], m[w\$878, 2], m[z12, 2], m[z44, 1]], \text{EQ}[m[w\$877, 0], m[w\$880, 2], m[z24, 2], m[z44, 1]], \text{EQ}[m[w\$876, 1], m[w\$876, 2], m[w\$877, 1], m[w\$877, 2], m[w\$878, 1], m[w\$879, 2], m[w\$880, 1], m[w\$881, 2], m[z12, 0], m[z24, 0]], \text{EQ}[m[w\$876, 0], m[w\$877, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$879, 0], m[w\$879, 1], m[w\$879, 1], m[w\$880, 0], m[w\$880, 1], m[w\$880, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[z12, 1], m[z24, 1], m[z44, 0], m[z44, 2], m[z44, 2]] \}$$

Doing triangle: k=2; s[[k]]=EQ[m[w\\$882, 0], m[z12, 2], m[z24, 1]];

$$s = \{ \text{EQ}[m[w\$876, 1], m[w\$881, 0], m[z12, 1], m[z24, 2]], \text{EQ}[m[w\$882, 0], m[z12, 2], m[z24, 1]], \text{EQ}[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], \text{EQ}[m[z12, 0], m[z24, 0], m[z44, 0]], \text{EQ}[m[w\$876, 0], m[w\$878, 2], m[z12, 2], m[z44, 1]], \text{EQ}[m[w\$882, 2], m[z24, 2], m[z44, 1]], \text{EQ}[m[w\$876, 1], m[w\$876, 2], m[w\$878, 1], m[w\$881, 2], m[w\$882, 1], m[w\$882, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[z12, 0], m[z24, 0]], \text{EQ}[m[w\$876, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$882, 0], m[w\$882, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[z12, 1], m[z24, 1], m[z44, 0], m[z44, 2], m[z44, 2]] \}$$

Doing bigon: k=3; s[[k]]=EQ[m[w\\$885, 2], m[z44, 0]];

$$s = \{ \text{EQ}[m[w\$876, 1], m[w\$881, 0], m[w\$884, 1]], \text{EQ}[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], \text{EQ}[m[w\$885, 2], m[z44, 0]], \text{EQ}[m[w\$876, 0], m[w\$878, 2], m[w\$884, 2], m[w\$885, 1], m[z44, 1]], \text{EQ}[m[w\$885, 1], m[z44, 1]], \text{EQ}[m[w\$876, 1], m[w\$876, 2], m[w\$878, 1], m[w\$881, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2]], \text{EQ}[m[w\$876, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1], m[w\$885, 0], m[w\$885, 0], m[w\$885, 2], m[z44, 0], m[z44, 2], m[z44, 2]] \}$$

Annular face removed...

Doing triangle: k=1; s[[k]]=EQ[m[w\$876, 1], m[w\$881, 0], m[w\$884, 1]];

s={EQ[m[w\$876, 1], m[w\$881, 0], m[w\$884, 1]], EQ[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], EQ[m[w\$876, 0], m[w\$878, 2], m[w\$884, 2]], EQ[m[w\$876, 1], m[w\$876, 2], m[w\$878, 1], m[w\$881, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2]], EQ[m[w\$876, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1]]}

Failed triangle;

t={EQ[m[w\$878, 0], m[w\$887, 0]], EQ[m[w\$878, 2], m[w\$887, 1]], EQ[m[w\$878, 1], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2], m[w\$886, 1], m[w\$887, 0], m[w\$887, 2]], EQ[m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$886, 0], m[w\$886, 0], m[w\$886, 2], m[w\$887, 0], m[w\$887, 1], m[w\$887, 2]]}; problems at  
{EQ[m[w\$878, 1], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2], m[w\$886, 1], m[w\$887, 0], m[w\$887, 2]]}

Doing triangle: k=2; s[[k]]=EQ[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]];

s={EQ[m[w\$876, 1], m[w\$881, 0], m[w\$884, 1]], EQ[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], EQ[m[w\$876, 0], m[w\$878, 2], m[w\$884, 2]], EQ[m[w\$876, 1], m[w\$876, 2], m[w\$878, 1], m[w\$881, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2]], EQ[m[w\$876, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1]]}

Failed triangle;

t={EQ[m[w\$884, 1], m[w\$889, 1]], EQ[m[w\$884, 2], m[w\$889, 0]], EQ[m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2], m[w\$888, 0], m[w\$888, 0], m[w\$888, 1], m[w\$889, 0], m[w\$889, 2]], EQ[m[w\$881, 0], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1], m[w\$888, 2], m[w\$889, 0], m[w\$889, 1], m[w\$889, 2]]}; problems at  
{EQ[m[w\$881, 0], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1], m[w\$888, 2], m[w\$889, 0], m[w\$889, 1], m[w\$889, 2]]}

Doing triangle: k=3; s[[k]]=EQ[m[w\$876, 0], m[w\$878, 2], m[w\$884, 2]];

s={EQ[m[w\$876, 1], m[w\$881, 0], m[w\$884, 1]], EQ[m[w\$876, 2], m[w\$878, 0], m[w\$881, 2]], EQ[m[w\$876, 0], m[w\$878, 2], m[w\$884, 2]], EQ[m[w\$876, 1], m[w\$876, 2], m[w\$878, 1], m[w\$881, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$884, 0], m[w\$884, 2]], EQ[m[w\$876, 0], m[w\$878, 0], m[w\$878, 1], m[w\$878, 2], m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$884, 0], m[w\$884, 1]]}

Doing bigon: k=1; s[[k]]=EQ[m[w\$881, 0], m[w\$890, 2]];

s={EQ[m[w\$881, 0], m[w\$890, 2]], EQ[m[w\$881, 2], m[w\$890, 0]], EQ[m[w\$881, 2], m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$890, 0], m[w\$891, 0], m[w\$891, 0], m[w\$891, 1]], EQ[m[w\$881, 0], m[w\$881, 1], m[w\$881, 1], m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$890, 1], m[w\$890, 1], m[w\$890, 2], m[w\$891, 1], m[w\$891, 2], m[w\$891, 2]]}

Annular face removed...

Contracting dominos; Lengths={6, 6} s=

{EQ[m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$891, 0], m[w\$891, 0], m[w\$891, 1]], EQ[m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$891, 1], m[w\$891, 2], m[w\$891, 2]]}  
{EQ[m[w\$883, 1], m[w\$883, 1], m[w\$883, 2], m[w\$891, 0], m[w\$891, 0], m[w\$891, 1]], EQ[m[w\$883, 0], m[w\$883, 0], m[w\$883, 2], m[w\$891, 1], m[w\$891, 2], m[w\$891, 2]]}, {6, 6}, 2}



**rels**

```
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]], EQ[m[z14, 0], m[z22, 0], m[z32, 0]],
EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]], EQ[m[z13, 0], m[z33, 0], m[z41, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z21, 2], m[z22, 1], m[z41, 2], m[z42, 1]],
EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]], EQ[m[z22, 1], m[z23, 2], m[z31, 1],
m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z44, 1]], EQ[m[z43, 0], m[z51, 0]],
EQ[m[z33, 2], m[z34, 1], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z52, 1]],
EQ[m[z34, 0], m[z44, 0], m[z54, 0]], EQ[m[z51, 0], m[z52, 0], m[z53, 0], m[z54, 0]],
EQ[m[z12, 1], m[z13, 2], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]],
EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z53, 2], m[z54, 1]],
EQ[m[z31, 2], m[z34, 1], m[z42, 1], m[z43, 2], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2]],
EQ[m[z11, 0], m[z21, 0], m[z61, 0]], EQ[m[z24, 0], m[z62, 0]],
EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z31, 2], m[z32, 1], m[z61, 2], m[z62, 1]],
EQ[m[z23, 0], m[z31, 0], m[z53, 0], m[z63, 0]],
EQ[m[z11, 2], m[z14, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z62, 1], m[z63, 2]],
EQ[m[z12, 0], m[z42, 0], m[z52, 0], m[z64, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z32, 1], m[z33, 2], m[z61, 2], m[z64, 1]],
EQ[m[z43, 2], m[z44, 1], m[z51, 1], m[z52, 2], m[z53, 1], m[z54, 2], m[z63, 2], m[z64, 1]],
EQ[m[z11, 2], m[z12, 1], m[z21, 2], m[z23, 2], m[z24, 1], m[z24, 1],
m[z52, 1], m[z53, 2], m[z61, 0], m[z61, 1], m[z61, 1], m[z62, 0], m[z62, 2],
m[z62, 2], m[z63, 0], m[z63, 1], m[z63, 1], m[z64, 0], m[z64, 2], m[z64, 2]]}
```

```
eqns = (Times @@ # == 1) & /@ Rest[Reverse[s]] /.
{m[w_, 0] => w, m[w_, 1] => 1 - 1/w, m[w_, 2] => 1 / (1 - w)}
{ (1 - 1/w$27) (1 - 1/w$29)^2 w$29 == 1, w$23 (1 - 1/w$27) w$29 == 1 }
```

```
vars = Complement[Union[Cases[eqns, _Symbol, Infinity]], {z1}]
```

```
{w$23, w$27, w$29}
```

```
{
sol = FindRoot @@ Join[{eqns /. z1 -> 1},
({#, Exp[Random[] Pi I/2]} & /@ vars), {MaxIterations -> 1000}],
Max[Abs[(First /@ (eqns /. z1 -> 1)) /. sol] - 1],
Plus @@ (Vol /@ (vars /. sol))
}
```

FindRoot::frnum: Function {-0.813583 - 0.0306575 i, 2.36227 + 0.312227 i} is not a length 3 list of numbers at {w\$23, w\$27, w\$29} = {0.956149 + 0.292879 i, 0.999586 + 0.0287548 i, 0.0272597 + 0.999628 i}.

FindRoot::frnum: Function {-0.813583 - 0.0306575 i, 2.36227 + 0.312227 i} is not a length 3 list of numbers at {w\$23, w\$27, w\$29} = {0.956149 + 0.292879 i, 0.999586 + 0.0287548 i, 0.0272597 + 0.999628 i}.

FindRoot::frnum: Function {-0.813583 - 0.0306575 i, 2.36227 + 0.312227 i} is not a length 3 list of numbers at {w\$23, w\$27, w\$29} = {0.956149 + 0.292879 i, 0.999586 + 0.0287548 i, 0.0272597 + 0.999628 i}.

General::stop: Further output of FindRoot::frnum will be suppressed during this calculation.

ReplaceAll::reps : {FindRoot[ $\left\{\frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1\right\}$ , {w\$23, 0.956149 + 0.292879 i}, <<1>>, <<1>>, MaxIterations -> 1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

ReplaceAll::reps : {FindRoot[ $\left\{\frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1\right\}$ , {w\$23, 0.956149 + 0.292879 i}, <<1>>, <<1>>, MaxIterations -> 1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

ReplaceAll::reps : {FindRoot[ $\left\{\frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1\right\}$ , {w\$23, 0.956149 + 0.292879 i}, <<1>>, <<1>>, MaxIterations -> 1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

General::stop : Further output of ReplaceAll::reps will be suppressed during this calculation.

NIntegrate::nlim : u = {Arg[w\$23], Arg[w\$27], Arg[w\$29]} is not a valid limit of integration.

NIntegrate::nlim : u = {Arg[ $1 - \frac{1}{w\$23}$ ], Arg[ $1 - \frac{1}{w\$27}$ ], Arg[ $1 - \frac{1}{w\$29}$ ]} is not a valid limit of integration.

NIntegrate::nlim : u = {Arg[ $\frac{1}{1 - 1. w\$23}$ ], Arg[ $\frac{1}{1 - 1. w\$27}$ ], Arg[ $\frac{1}{1 - 1. w\$29}$ ]} is not a valid limit of integration.

General::stop : Further output of NIntegrate::nlim will be suppressed during this calculation.

$$\left\{ \text{FindRoot} \left[ \left\{ \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1 \right\}, \right. \right.$$

$$\left. \left. \left\{ w\$23, 0.956149 + 0.292879 i \right\}, \left\{ w\$27, 0.999586 + 0.0287548 i \right\}, \left\{ w\$29, 0.0272597 + 0.999628 i \right\}, \text{MaxIterations} \rightarrow 1000 \right\}, \right.$$

$$\left. \text{Abs} \left[ -1 + \left( \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23}, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} \right) \right] / . \right.$$

$$\left. \left. \left. \text{FindRoot} \left[ \left\{ \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1 \right\}, \right. \right. \right.$$

$$\left. \left. \left. \left\{ w\$23, 0.956149 + 0.292879 i \right\}, \left\{ w\$27, 0.999586 + 0.0287548 i \right\}, \left\{ w\$29, 0.0272597 + 0.999628 i \right\}, \text{MaxIterations} \rightarrow 1000 \right\} \right] \right] ,$$

$$-\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \text{Arg} \left[ 1 - \frac{1}{\text{FindRoot} \left[ \left\{ \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \right. \right.} \right. \right. \right.$$

$$\left. \left. \left. \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1 \right\}, \left\{ w\$23, 0.956149 + 0.292879 i \right\}, \left\{ w\$27, 0.999586 + 0.0287548 i \right\}, \left\{ w\$29, 0.0272597 + 0.999628 i \right\}, \text{MaxIterations} \rightarrow 1000 \right\} \right] \right] -$$

$$\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \text{Arg} \left[ 1 / \left( 1 - \text{FindRoot} \left[ \left\{ \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \right. \right. \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1 \right\}, \left\{ w\$23, 0.956149 + 0.292879 i \right\}, \left\{ w\$27, 0.999586 + 0.0287548 i \right\}, \left\{ w\$29, 0.0272597 + 0.999628 i \right\}, \text{MaxIterations} \rightarrow 1000 \right\} \right] \right] -$$

$$\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \text{Arg} \left[ \text{FindRoot} \left[ \left\{ \frac{\left(1 - \frac{1}{w\$27}\right) \left(1 - \frac{1}{w\$29}\right)^2 w\$29}{1 - w\$23} == 1, \right. \right. \right. \right. \right. \right. \right.$$

$$\left. \left. \left. \frac{w\$23 \left(1 - \frac{1}{w\$27}\right) w\$29}{(1 - w\$23) (1 - w\$27)} == 1 \right\}, \left\{ w\$23, 0.956149 + 0.292879 i \right\}, \left\{ w\$27, 0.999586 + 0.0287548 i \right\}, \left\{ w\$29, 0.0272597 + 0.999628 i \right\}, \text{MaxIterations} \rightarrow 1000 \right\} \right] \right] -$$

$$\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \left\{ \text{Arg} \left[ 1 - \frac{1}{w\$23} \right], \text{Arg} \left[ 1 - \frac{1}{w\$27} \right], \text{Arg} \left[ 1 - \frac{1}{w\$29} \right] \right\} \right] \right] -$$

$$\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \left\{ \text{Arg} \left[ \frac{1}{1 - w\$23} \right], \text{Arg} \left[ \frac{1}{1 - w\$27} \right], \text{Arg} \left[ \frac{1}{1 - w\$29} \right] \right\} \right] \right] -$$

$$\text{NIntegrate} \left[ \text{Log} \left[ \text{Abs} \left[ 2 \text{Sin} \left[ u \right] \right] \right], \left\{ u, 0, \left\{ \text{Arg} \left[ w\$23 \right], \text{Arg} \left[ w\$27 \right], \text{Arg} \left[ w\$29 \right] \right\} \right] \right]$$

**sols = vars /. Solve[eqns /. z1 -> 1, vars]**

General::spell1: Possible spelling error: new symbol name "sols" is similar to existing symbol "sol".

Solve::svars: Equations may not give solutions for all "solve" variables.

$$\left\{ \left\{ 1 + \frac{1}{w^2} - \frac{2}{w} - \frac{1}{2(1-2w+w^2)} - \frac{1}{2w^2(1-2w+w^2)} + \frac{3}{2w(1-2w+w^2)} - \frac{2w}{1-2w+w^2} + \frac{2w^2}{1-2w+w^2} - \frac{w^3}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2w^2(1-2w+w^2)} - \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{w(1-2w+w^2)}, \right. \right.$$

$$\left. \left. \frac{1-w-2w^2+w^3-\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)}, w \right\} \right\},$$

$$\left\{ \left\{ 1 + \frac{1}{w^2} - \frac{2}{w} - \frac{1}{2(1-2w+w^2)} - \frac{1}{2w^2(1-2w+w^2)} + \frac{3}{2w(1-2w+w^2)} - \frac{2w}{1-2w+w^2} + \frac{2w^2}{1-2w+w^2} - \frac{w^3}{2(1-2w+w^2)} - \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2w^2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{w(1-2w+w^2)}, \right. \right.$$

$$\left. \left. \frac{1-w-2w^2+w^3+\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)}, w \right\} \right\}$$

**N[sols]**

$$\left\{ \left\{ 1 + \frac{1}{w\$29^2} - \frac{2}{w\$29} - \frac{0.5}{1 - 2 \cdot w\$29 + w\$29^2} - \frac{0.5}{w\$29^2 (1 - 2 \cdot w\$29 + w\$29^2)} + \right. \right.$$

$$\frac{1.5}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)} - \frac{2 \cdot w\$29}{1 - 2 \cdot w\$29 + w\$29^2} + \frac{2 \cdot w\$29^2}{1 - 2 \cdot w\$29 + w\$29^2} - \frac{0.5 w\$29^3}{1 - 2 \cdot w\$29 + w\$29^2} +$$

$$\frac{0.5 \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{1 - 2 \cdot w\$29 + w\$29^2} +$$

$$\frac{0.5 \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{w\$29^2 (1 - 2 \cdot w\$29 + w\$29^2)} -$$

$$\left. \frac{1 \cdot \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)} \right\},$$

$$\frac{1}{1 - 2 \cdot w\$29 + w\$29^2} \left( 0.5 (1 - 1 \cdot w\$29 - 2 \cdot w\$29^2 + w\$29^3 - \right.$$

$$\left. 1 \cdot \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6} \right), w\$29 \left. \right\},$$

$$\left\{ 1 + \frac{1}{w\$29^2} - \frac{2}{w\$29} - \frac{0.5}{1 - 2 \cdot w\$29 + w\$29^2} - \frac{0.5}{w\$29^2 (1 - 2 \cdot w\$29 + w\$29^2)} + \right.$$

$$\frac{1.5}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)} - \frac{2 \cdot w\$29}{1 - 2 \cdot w\$29 + w\$29^2} + \frac{2 \cdot w\$29^2}{1 - 2 \cdot w\$29 + w\$29^2} - \frac{0.5 w\$29^3}{1 - 2 \cdot w\$29 + w\$29^2} -$$

$$\frac{0.5 \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{1 - 2 \cdot w\$29 + w\$29^2} -$$

$$\frac{0.5 \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{w\$29^2 (1 - 2 \cdot w\$29 + w\$29^2)} +$$

$$\left. \frac{\sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)} \right\},$$

$$\frac{1}{1 - 2 \cdot w\$29 + w\$29^2} \left( 0.5 (1 - 1 \cdot w\$29 - 2 \cdot w\$29^2 + w\$29^3 + \right.$$

$$\left. \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6} \right), w\$29 \left. \right\}$$

**(Plus @@ Vol /@ #) & /@ sols**

NIntegrate::nlim: u = Arg[1 + <<13>> + <<1>> +  $\frac{0.5 \sqrt{\langle\langle 9 \rangle\rangle + w\$29^6}}{w\$29^2 \langle\langle 1 \rangle\rangle (1 - 2 \cdot w\$29 + w\$29^2)}$  -  $\frac{1 \cdot \sqrt{1 - 6 \cdot w\$29 + 13 \cdot w\$29^2 - 18 \cdot w\$29^3 + 18 \cdot w\$29^4 - 8 \cdot w\$29^5 + w\$29^6}}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)}$ ] is not a valid limit of integration.

NIntegrate::nlim: u = Arg[1 -  $\frac{1}{1 + \langle\langle 14 \rangle\rangle + \frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle} - \frac{1 \cdot \sqrt{1 - 6 \cdot w\$29 + \langle\langle 6 \rangle\rangle + w\$29^6}}{w\$29 (1 - 2 \cdot w\$29 + w\$29^2)}}$ ] is not a valid limit of integration.

NIntegrate::nlim: u = Arg[ $\frac{1}{-\frac{1}{w^2 29^2} + \frac{2}{w 29} + \frac{1}{2(1-2w 29 + w^2 29^2)} + \frac{\sqrt{1-6w 29 + 13w^2 29^2 - 18w^3 29^3 + 18w^4 29^4 - 8w^5 29^5 + w^6 29^6}}{w 29(1-2w 29 + w^2 29^2)}}$ ] is not a valid limit of integration.

General::stop: Further output of NIntegrate::nlim will be suppressed during this calculation.

$$\left\{ -\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[1 - \frac{1}{w 29}\right]\right\}\right] - \right.$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{1}{1 - w 29}\right]\right\}\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[w 29\right]\right\}\right] - \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \right.$$

$$\text{Arg}\left[\frac{1 - w 29 - 2 w 29^2 + w 29^3 - \sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{1 - 2 w 29 + w 29^2}\right]\right\}\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \right.$$

$$\left\{u, 0, \text{Arg}\left[1 + \frac{1}{w 29^2} - \frac{2}{w 29} - \frac{1}{2(1 - 2 w 29 + w^2 29^2)} - \frac{1}{2 w 29^2(1 - 2 w 29 + w^2 29^2)} + \right.$$

$$\frac{3}{2 w 29(1 - 2 w 29 + w^2 29^2)} - \frac{2 w 29}{1 - 2 w 29 + w^2 29^2} + \frac{2 w 29^2}{1 - 2 w 29 + w^2 29^2} - \right.$$

$$\frac{w 29^3}{2(1 - 2 w 29 + w^2 29^2)} + \frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{2(1 - 2 w 29 + w^2 29^2)} +$$

$$\frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{2 w 29^2(1 - 2 w 29 + w^2 29^2)} -$$

$$\left. \left. \frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{w 29(1 - 2 w 29 + w^2 29^2)}\right]\right\}\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \right.$$

$$\text{Arg}\left[1 / \left(-\frac{1}{w 29^2} + \frac{2}{w 29} + \frac{1}{2(1 - 2 w 29 + w^2 29^2)} + \frac{1}{2 w 29^2(1 - 2 w 29 + w^2 29^2)} - \right.$$

$$\frac{3}{2 w 29(1 - 2 w 29 + w^2 29^2)} + \frac{2 w 29}{1 - 2 w 29 + w^2 29^2} - \frac{2 w 29^2}{1 - 2 w 29 + w^2 29^2} + \right.$$

$$\frac{w 29^3}{2(1 - 2 w 29 + w^2 29^2)} - \frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{2(1 - 2 w 29 + w^2 29^2)} -$$

$$\frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{2 w 29^2(1 - 2 w 29 + w^2 29^2)} +$$

$$\left. \left. \frac{\sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{w 29(1 - 2 w 29 + w^2 29^2)}\right]\right\}\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\right.$$

$$\frac{1}{2(1 - 2 w 29 + w^2 29^2)}\right]\right\}\right] -$$

$$\frac{1 - w 29 - 2 w 29^2 + w 29^3 - \sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{1 - 2 w 29 + w^2 29^2}\right]\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \right.$$

$$\left\{u, 0, \text{Arg}\left[\frac{1}{1 - \frac{1 - w 29 - 2 w 29^2 + w 29^3 - \sqrt{1 - 6 w 29 + 13 w^2 29^2 - 18 w^3 29^3 + 18 w^4 29^4 - 8 w^5 29^5 + w^6 29^6}}{2(1 - 2 w 29 + w^2 29^2)}}\right]\right\}\right] -$$

$$\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \right.$$

$$\left\{ u, \theta, \text{Arg} \left[ 1 - 1 / \left( 1 + \frac{1}{w^2} - \frac{2}{w} - \frac{1}{2(1-2w+w^2)} - \frac{1}{2w^2(1-2w+w^2)} + \frac{3}{2w(1-2w+w^2)} - \frac{2w}{1-2w+w^2} + \frac{2w^2}{1-2w+w^2} - \frac{w^3}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2w^2(1-2w+w^2)} - \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{w(1-2w+w^2)} \right) \right] \right\} -$$

$$\text{NIntegrate} \left[ \text{Log} [\text{Abs} [2 \text{Sin} [u]]], \left\{ u, \theta, \text{Arg} \left[ 1 - \frac{1}{w} \right] \right\} \right] -$$

$$\text{NIntegrate} \left[ \text{Log} [\text{Abs} [2 \text{Sin} [u]]], \left\{ u, \theta, \text{Arg} \left[ \frac{1}{1-w} \right] \right\} \right] -$$

$$\text{NIntegrate} [\text{Log} [\text{Abs} [2 \text{Sin} [u]]], \{u, \theta, \text{Arg} [w]\}] -$$

$$\text{NIntegrate} [\text{Log} [\text{Abs} [2 \text{Sin} [u]]], \{u, \theta, \text{Arg} \left[ \frac{1-w-2w^2+w^3+\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{1-2w+w^2} \right]\}] -$$

$$\text{NIntegrate} [\text{Log} [\text{Abs} [2 \text{Sin} [u]]], \left\{ u, \theta, \text{Arg} \left[ 1 / \left( -\frac{1}{w^2} + \frac{2}{w} + \frac{1}{2(1-2w+w^2)} + \frac{1}{2w^2(1-2w+w^2)} - \frac{3}{2w(1-2w+w^2)} + \frac{2w}{1-2w+w^2} - \frac{2w^2}{1-2w+w^2} + \frac{w^3}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2w^2(1-2w+w^2)} - \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{w(1-2w+w^2)} \right) \right] \right\} \right] -$$

$$\text{NIntegrate} [\text{Log} [\text{Abs} [2 \text{Sin} [u]]], \left\{ u, \theta, \text{Arg} \left[ 1 + \frac{1}{w^2} - \frac{2}{w} - \frac{1}{2(1-2w+w^2)} - \frac{1}{2w^2(1-2w+w^2)} + \frac{3}{2w(1-2w+w^2)} + \frac{2w}{1-2w+w^2} - \frac{2w^2}{2(1-2w+w^2)} - \frac{w^3}{2(1-2w+w^2)} + \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} - \frac{\sqrt{1-6w+13w^2-18w^3+18w^4-8w^5+w^6}}{2(1-2w+w^2)} \right] \right\} \right] -$$

$$\begin{aligned}
 & \frac{\sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{2 w\$29^2 (1 - 2 w\$29 + w\$29^2)} + \\
 & \frac{\sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{w\$29 (1 - 2 w\$29 + w\$29^2)} \Big] \Big] - \\
 & \text{NIntegrate} \left[ \text{Log}[\text{Abs}[2 \text{Sin}[u]]], \{u, 0, \text{Arg} \left[ \frac{1 - \frac{1}{2(1 - 2 w\$29 + w\$29^2)}}{1 - w\$29 - 2 w\$29^2 + w\$29^3 + \sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}} \right]} \right] \Big] - \\
 & \text{NIntegrate} \left[ \text{Log}[\text{Abs}[2 \text{Sin}[u]]], \{u, 0, \text{Arg} \left[ \frac{1}{1 - \frac{1 - w\$29 - 2 w\$29^2 + w\$29^3 + \sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{2(1 - 2 w\$29 + w\$29^2)}} \right]} \right] \Big] - \\
 & \text{NIntegrate} \left[ \text{Log}[\text{Abs}[2 \text{Sin}[u]]], \{u, 0, \text{Arg} \left[ 1 - 1 / \left( 1 + \frac{1}{w\$29^2} - \frac{2}{w\$29} - \frac{1}{2(1 - 2 w\$29 + w\$29^2)} - \frac{1}{2 w\$29^2 (1 - 2 w\$29 + w\$29^2)} + \right. \right. \right. \\
 & \left. \left. \frac{3}{2 w\$29 (1 - 2 w\$29 + w\$29^2)} - \frac{2 w\$29}{1 - 2 w\$29 + w\$29^2} + \frac{2 w\$29^2}{1 - 2 w\$29 + w\$29^2} - \frac{w\$29^3}{2(1 - 2 w\$29 + w\$29^2)} - \frac{\sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{2(1 - 2 w\$29 + w\$29^2)} \right. \right. \\
 & \left. \left. \frac{\sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{2 w\$29^2 (1 - 2 w\$29 + w\$29^2)} + \frac{\sqrt{1 - 6 w\$29 + 13 w\$29^2 - 18 w\$29^3 + 18 w\$29^4 - 8 w\$29^5 + w\$29^6}}{w\$29 (1 - 2 w\$29 + w\$29^2)} \right) \right] \Big] \Big] \Big]
 \end{aligned}$$

**RemoveTriangle[Failed, \_Integer] := Failed**