

Pensieve header: A y-only solution, modding out by the g's.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << ../APAI/Rot.m];
TS = T S;
( $\alpha_+$ )+ :=  $\alpha^{++}$ ; (* this is for cosmetic reasons only *)
 $\delta_{i,j}$  := If[i == j, 1, 0];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/APAI> to compute rotation numbers.

Lower Rank 2 Gassner: (Modified from Rank2Gassner.nb)

```
In[*]:= LR2G $_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i$  => T  $e_i$  + (1 - T)  $e_j$ ,  $f_i$  => S  $f_i$  + (1 - S)  $f_j$ ,
 $g_i$  => T S  $g_i$  + (1 - T S)  $g_j$ 
+ S (T - 1)  $e_j f_i$  + (S - 1) T  $e_i f_j$  + (S + T - 2 S T)  $e_j f_j$  }];
 $\overline{\text{LR2G}}_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i$  => T-1  $e_i$  + (1 - T-1)  $e_j$ ,  $f_i$  => S-1  $f_i$  + (1 - S-1)  $f_j$ ,
 $g_i$  => T-1 S-1  $g_i$  + (1 - T-1 S-1)  $g_j$ 
- S-1 (1 - T-1)  $e_j f_i$  - T-1 (1 - S-1)  $e_i f_j$  + (S-1 + T-1 - 2 S-1 T-1)  $e_j f_j$  }];
bas =
{ $e_1$ ,  $e_2$ ,  $e_3$ ,  $f_1$ ,  $f_2$ ,  $f_3$ ,  $e_1 f_1$ ,  $e_1 f_2$ ,  $e_1 f_3$ ,  $e_2 f_1$ ,  $e_2 f_2$ ,  $e_2 f_3$ ,  $e_3 f_1$ ,  $e_3 f_2$ ,  $e_3 f_3$ ,  $g_1$ ,  $g_2$ ,  $g_3$ };
(bas // LR2G $_{1,2}$  //  $\overline{\text{LR2G}}_{1,2}$ ) == bas
(bas // LR2G $_{3,2}$  //  $\overline{\text{LR2G}}_{3,2}$ ) == bas
(1hs = bas // LR2G $_{1,2}$  // LR2G $_{1,3}$  // LR2G $_{2,3}$ ) == (bas // LR2G $_{2,3}$  // LR2G $_{1,3}$  // LR2G $_{1,2}$ )

Out[*]=
True

Out[*]=
True

Out[*]=
True
```

```

In[*]:= R1[s_, i_, j_] := TBD; (* a sign(s)-dependent quadratic polynomial in  $g_{1\alpha\beta}$ ,  $g_{2\alpha\beta}$ ,  $g_{3\alpha\beta}$ ,
and  $y_{\alpha\beta\gamma}$ , where  $\deg y_{\alpha\beta\gamma}=2$  and  $\alpha, \beta, \gamma \in \{i, j\}$ , with coefficients in  $\mathbb{Z}[T^{\pm 1}, S^{\pm 1}]$ . *)
CF[ε_] := Factor@Together[ε];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];
  Δ = T(-Total[φ]-Total[Cs[[All,1]])/2 Det[A];
  G = Inverse[A];
  gEval[ε_] := CF[ε /. α_+ => α + 1 /.
    {g1,α,β_ => G[[α, β]], g2,α,β_ => (G[[α, β]] /. T -> S), g3,α,β_ => (G[[α, β]] /. T -> TS)}];
  Y[α_, β_, γ_] := Sum[{s, i, j} = c;
    g3,α,i (Ss (Ts - 1) g1,j+,β g2,i+,γ + (Ss - 1) Ts g1,i+,β g2,j+,γ + (Ss + Ts - 2 TSs) g1,j+,β g2,j+,γ),
    {c, Cs}];
  yEval[ε_] := CF[ε /. y_α,β,γ_ => Y[α, β, γ]];
  λ1 = Sumk=1n R1@@Cs[[k]] - Sumk=12n φ[[k]] (g1,k,k + g2,k,k + g3,k,k);
  {Δ, Δ (Δ /. T -> S) (Δ /. T -> TS) λ1} // yEval // gEval
];

```

Step-by-step Run-Through

```

In[*]:= CF[ε_] := Factor@Together[ε];

```

```

In[*]:= K = {
  PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
  Knot[6, 2]
}[[1]]

```

```

Out[*]=
PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]

```

```

In[*]:= {Cs, φ} = Rot[K]

```

```

Out[*]=
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}

```

```

In[*]:= n = Length[Cs];

```

```

A = IdentityMatrix[2 n + 1];

```

```

Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];

```



```
In[*]:= Column[MatrixForm/@Table[Y[α, β, γ] // gEval, {α, 3}, {β, 3}, {γ, 3}]]
```

```
Out[*]=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(-1+S)^2 T}{1-S+S^2} \\ 0 & -\frac{S(-1+T)^2}{1-T+T^2} & -\frac{-S+S^2-T+2ST-2S^2T+T^2-2ST^2+2S^2T^2}{(1-S+S^2)(1-T+T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

```
In[*]:= yEval[ε_] := CF[ε /. y_{α,β,γ} -> Y[α, β, γ]]
```

The g-Rules

```
In[*]:= gRules_{s_,i_,j_} := {
  g_{1,i,β} -> δ_{i,β} + T^S g_{1,i^+,β} + (1 - T^S) g_{1,j^+,β}, g_{1,j,β} -> δ_{j,β} + g_{1,j^+,β},
  g_{1,α,i} -> T^{-S} (g_{1,α,i^+} - δ_{α,i^+}), g_{1,α,j} -> g_{1,α,j^+} - (1 - T^S) g_{1,α,i} - δ_{α,j^+},
  g_{2,i,β} -> δ_{i,β} + S^S g_{2,i^+,β} + (1 - S^S) g_{2,j^+,β}, g_{2,j,β} -> δ_{j,β} + g_{2,j^+,β},
  g_{2,α,i} -> S^{-S} (g_{2,α,i^+} - δ_{α,i^+}), g_{2,α,j} -> g_{2,α,j^+} - (1 - S^S) g_{2,α,i} - δ_{α,j^+},
  g_{3,i,β} -> δ_{i,β} + TS^S g_{3,i^+,β} + (1 - TS^S) g_{3,j^+,β}, g_{3,j,β} -> δ_{j,β} + g_{3,j^+,β},
  g_{3,α,i} -> TS^{-S} (g_{3,α,i^+} - δ_{α,i^+}), g_{3,α,j} -> g_{3,α,j^+} - (1 - TS^S) g_{3,α,i} - δ_{α,j^+}
}
```

```
In[*]:= (gs = Table[{g_{1,α,β}, g_{2,α,β}, g_{3,α,β}}, {α, 2n+1}, {β, 2n+1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{{{g_{1,1,1}, g_{2,1,1}}, {g_{1,1,2}, g_{2,1,2}}, {{g_{1,2,1}, g_{2,2,1}}, {g_{1,2,2}, g_{2,2,2}}}}
```

```
In[*]:= Cs
```

```
Out[*]=
```

```
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

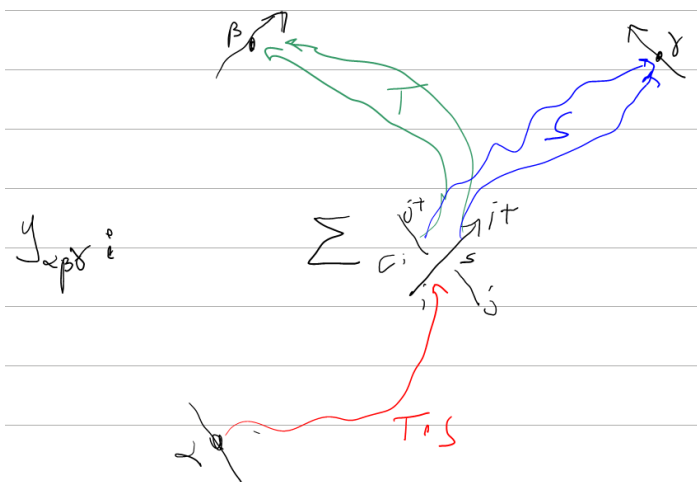
```
In[*]:= (gs /. gRules_{Sequence@@Cs[[1]]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{
  {{(1 + T) g_{1,1^+,1} + (1 - T) g_{1,4^+,1}, (1 + S) g_{2,1^+,1} + (1 - S) g_{2,4^+,1}},
  {T g_{1,1^+,2} + (1 - T) g_{1,4^+,2}, S g_{2,1^+,2} + (1 - S) g_{2,4^+,2}},
  {{-If[2 == 1^+, 1, 0] + g_{1,2,1^+}, -If[2 == 1^+, 1, 0] + g_{2,2,1^+}}, {g_{1,2,2}, g_{2,2,2}}}
}
```

```
In[ ]:= gEval[gs - (gs /. Flatten@Table[gRulesSequence@@c, {c, Cs}]]]
Out[ ]:=
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

The y-Rules



```
In[ ]:= yRuless, i, j := {
  yi, beta, gamma -> TSs yi+, beta, gamma + (1 - TSs) yj+, beta, gamma +
    (Ss (Ts - 1) g1, j+, beta g2, i+, gamma + (Ss - 1) Ts g1, i+, beta g2, j+, gamma + (Ss + Ts - 2 TSs) g1, j+, beta g2, j+, gamma),
  yj, beta, gamma -> yj+, beta, gamma,
  yalpha, i, gamma -> T-s yalpha, i+, gamma + (1 - Ss) g3, alpha, i g2, j+, gamma,
  yalpha, j, gamma -> yalpha, j+, gamma - (1 - Ts) yalpha, i, gamma - g3, alpha, i (Ss (Ts - 1) g2, i+, gamma + (Ss + Ts - 2 TSs) g2, j+, gamma),
  yalpha, beta, i -> S-s yalpha, beta, i+ + (1 - Ts) g3, alpha, i g1, j+, beta,
  yalpha, beta, j -> yalpha, beta, j+ - (1 - Ss) yalpha, beta, i - g3, alpha, i (Ts (Ss - 1) g1, i+, beta + (Ss + Ts - 2 TSs) g1, beta, j+)
}
```

```
In[ ]:= (ys = Table[yalpha, beta, gamma, {alpha, 2 n + 1}, {beta, 2 n + 1}, {gamma, 2 n + 1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[ ]:=
{{{y1,1,1, y1,1,2}, {y1,2,1, y1,2,2}, {{y2,1,1, y2,1,2}, {y2,2,1, y2,2,2}}}
```

```
In[ ]:= Cs
```

```
Out[ ]:=
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

In[*]:= (ys /. yRules_{Sequence@@Cs}[1]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]

Out[*]=

$$\left\{ \left\{ \left\{ S(-1+T)g_{1,4^+,1}g_{2,1^+,1} + (-1+S)Tg_{1,1^+,1}g_{2,4^+,1} + (S+T-2ST)g_{1,4^+,1}g_{2,4^+,1} + STy_{1^+,1,1} + (1-ST)y_{4^+,1,1}, S(-1+T)g_{1,4^+,1}g_{2,1^+,2} + (-1+S)Tg_{1,1^+,1}g_{2,4^+,2} + (S+T-2ST)g_{1,4^+,1}g_{2,4^+,2} + STy_{1^+,1,2} + (1-ST)y_{4^+,1,2} \right\}, \left\{ S(-1+T)g_{1,4^+,2}g_{2,1^+,1} + (-1+S)Tg_{1,1^+,2}g_{2,4^+,1} + (S+T-2ST)g_{1,4^+,2}g_{2,4^+,1} + STy_{1^+,2,1} + (1-ST)y_{4^+,2,1}, S(-1+T)g_{1,4^+,2}g_{2,1^+,2} + (-1+S)Tg_{1,1^+,2}g_{2,4^+,2} + (S+T-2ST)g_{1,4^+,2}g_{2,4^+,2} + STy_{1^+,2,2} + (1-ST)y_{4^+,2,2} \right\} \right\}, \left\{ \left\{ (1-S)g_{2,4^+,1}g_{3,2,1} + \frac{y_{2,1^+,1}}{T}, (1-S)g_{2,4^+,2}g_{3,2,1} + \frac{y_{2,1^+,2}}{T} \right\}, \left\{ (1-T)g_{1,4^+,2}g_{3,2,1} + \frac{y_{2,2,1}}{S}, y_{2,2,2} \right\} \right\} \right\}$$

In[*]:= yEval[ys - (ys /. yRules_{Sequence@@Cs}[1])] [[1, 1, 1]]

Out[*]=

$$\begin{aligned} & Sg_{1,4^+,1}g_{2,1^+,1} - STg_{1,4^+,1}g_{2,1^+,1} + Tg_{1,1^+,1}g_{2,4^+,1} - STg_{1,1^+,1}g_{2,4^+,1} - Sg_{1,4^+,1}g_{2,4^+,1} - \\ & Tg_{1,4^+,1}g_{2,4^+,1} + 2STg_{1,4^+,1}g_{2,4^+,1} - Sg_{1,4^+,1}g_{2,1^+,1}g_{3,1,1} + STg_{1,4^+,1}g_{2,1^+,1}g_{3,1,1} - \\ & Tg_{1,1^+,1}g_{2,4^+,1}g_{3,1,1} + STg_{1,1^+,1}g_{2,4^+,1}g_{3,1,1} + Sg_{1,4^+,1}g_{2,4^+,1}g_{3,1,1} + Tg_{1,4^+,1}g_{2,4^+,1}g_{3,1,1} - \\ & 2STg_{1,4^+,1}g_{2,4^+,1}g_{3,1,1} - Sg_{1,6^+,1}g_{2,3^+,1}g_{3,1,3} + STg_{1,6^+,1}g_{2,3^+,1}g_{3,1,3} - Tg_{1,3^+,1}g_{2,6^+,1}g_{3,1,3} + \\ & STg_{1,3^+,1}g_{2,6^+,1}g_{3,1,3} + Sg_{1,6^+,1}g_{2,6^+,1}g_{3,1,3} + Tg_{1,6^+,1}g_{2,6^+,1}g_{3,1,3} - 2STg_{1,6^+,1}g_{2,6^+,1}g_{3,1,3} + \\ & Sg_{1,2^+,1}g_{2,2^+,1}g_{3,1,5} + Tg_{1,2^+,1}g_{2,2^+,1}g_{3,1,5} - 2STg_{1,2^+,1}g_{2,2^+,1}g_{3,1,5} - Tg_{1,5^+,1}g_{2,2^+,1}g_{3,1,5} + \\ & STg_{1,5^+,1}g_{2,2^+,1}g_{3,1,5} - Sg_{1,2^+,1}g_{2,5^+,1}g_{3,1,5} + STg_{1,2^+,1}g_{2,5^+,1}g_{3,1,5} + S^2Tg_{1,4^+,1}g_{2,1^+,1}g_{3,1^+,1} - \\ & S^2T^2g_{1,4^+,1}g_{2,1^+,1}g_{3,1^+,1} + S^2T^2g_{1,1^+,1}g_{2,4^+,1}g_{3,1^+,1} - S^2T^2g_{1,1^+,1}g_{2,4^+,1}g_{3,1^+,1} - \\ & S^2Tg_{1,4^+,1}g_{2,4^+,1}g_{3,1^+,1} - S^2T^2g_{1,4^+,1}g_{2,4^+,1}g_{3,1^+,1} + 2S^2T^2g_{1,4^+,1}g_{2,4^+,1}g_{3,1^+,1} + \\ & S^2Tg_{1,6^+,1}g_{2,3^+,1}g_{3,1^+,3} - S^2T^2g_{1,6^+,1}g_{2,3^+,1}g_{3,1^+,3} + S^2T^2g_{1,3^+,1}g_{2,6^+,1}g_{3,1^+,3} - \\ & S^2T^2g_{1,3^+,1}g_{2,6^+,1}g_{3,1^+,3} - S^2Tg_{1,6^+,1}g_{2,6^+,1}g_{3,1^+,3} - S^2T^2g_{1,6^+,1}g_{2,6^+,1}g_{3,1^+,3} + \\ & 2S^2T^2g_{1,6^+,1}g_{2,6^+,1}g_{3,1^+,3} - S^2Tg_{1,2^+,1}g_{2,2^+,1}g_{3,1^+,5} - S^2T^2g_{1,2^+,1}g_{2,2^+,1}g_{3,1^+,5} + \\ & 2S^2T^2g_{1,2^+,1}g_{2,2^+,1}g_{3,1^+,5} + S^2T^2g_{1,5^+,1}g_{2,2^+,1}g_{3,1^+,5} - S^2T^2g_{1,5^+,1}g_{2,2^+,1}g_{3,1^+,5} + \\ & S^2Tg_{1,2^+,1}g_{2,5^+,1}g_{3,1^+,5} - S^2T^2g_{1,2^+,1}g_{2,5^+,1}g_{3,1^+,5} + Sg_{1,4^+,1}g_{2,1^+,1}g_{3,4^+,1} - STg_{1,4^+,1}g_{2,1^+,1}g_{3,4^+,1} - \\ & S^2Tg_{1,4^+,1}g_{2,1^+,1}g_{3,4^+,1} + S^2T^2g_{1,4^+,1}g_{2,1^+,1}g_{3,4^+,1} + Tg_{1,1^+,1}g_{2,4^+,1}g_{3,4^+,1} - STg_{1,1^+,1}g_{2,4^+,1}g_{3,4^+,1} - \\ & S^2T^2g_{1,1^+,1}g_{2,4^+,1}g_{3,4^+,1} + S^2T^2g_{1,1^+,1}g_{2,4^+,1}g_{3,4^+,1} - Sg_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} - Tg_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} + \\ & 2STg_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} + S^2Tg_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} + S^2T^2g_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} - \\ & 2S^2T^2g_{1,4^+,1}g_{2,4^+,1}g_{3,4^+,1} + Sg_{1,6^+,1}g_{2,3^+,1}g_{3,4^+,3} - STg_{1,6^+,1}g_{2,3^+,1}g_{3,4^+,3} - S^2Tg_{1,6^+,1}g_{2,3^+,1}g_{3,4^+,3} + \\ & S^2T^2g_{1,6^+,1}g_{2,3^+,1}g_{3,4^+,3} + Tg_{1,3^+,1}g_{2,6^+,1}g_{3,4^+,3} - STg_{1,3^+,1}g_{2,6^+,1}g_{3,4^+,3} - S^2T^2g_{1,3^+,1}g_{2,6^+,1}g_{3,4^+,3} + \\ & S^2T^2g_{1,3^+,1}g_{2,6^+,1}g_{3,4^+,3} - Sg_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} - Tg_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} + 2STg_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} + \\ & S^2Tg_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} + S^2T^2g_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} - 2S^2T^2g_{1,6^+,1}g_{2,6^+,1}g_{3,4^+,3} - \\ & Sg_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} - Tg_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} + 2STg_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} + S^2Tg_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} + \\ & S^2T^2g_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} - 2S^2T^2g_{1,2^+,1}g_{2,2^+,1}g_{3,4^+,5} + Tg_{1,5^+,1}g_{2,2^+,1}g_{3,4^+,5} - \\ & STg_{1,5^+,1}g_{2,2^+,1}g_{3,4^+,5} - S^2T^2g_{1,5^+,1}g_{2,2^+,1}g_{3,4^+,5} + S^2T^2g_{1,5^+,1}g_{2,2^+,1}g_{3,4^+,5} + \\ & Sg_{1,2^+,1}g_{2,5^+,1}g_{3,4^+,5} - STg_{1,2^+,1}g_{2,5^+,1}g_{3,4^+,5} - S^2Tg_{1,2^+,1}g_{2,5^+,1}g_{3,4^+,5} + S^2T^2g_{1,2^+,1}g_{2,5^+,1}g_{3,4^+,5} \end{aligned}$$

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

```

In[*]:= Clear[i, j, k];
Cs = {{1, i, j}, {1, i+, k}, {1, j+, k+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
lhs = Simplify[Z //. gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+ /. _If → 0]

Out[*]=
{{1, i, j}, {1, i+, k}, {1, j+, k+}}

Out[*]=
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i+
(S (-1 + T) g1,k+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+

Out[*]=
1
S^2 T^2 ((S (-1 + T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,i+,γ - (-1 + S) g2,k+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ) +
(S + T - 2 S T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ)) g3,α,i+ +
S T ((-1 + S) T g1,i+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,i+ +
S T ((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,j+)

In[*]:= Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k+}, {1, i+, j+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
rhs = Simplify[Z //. gRules1,j,k ∪ gRules1,i,k+ ∪ gRules1,i+,j+ /. _If → 0]

Out[*]=
{{1, j, k}, {1, i, k+}, {1, i+, j+}}

Out[*]=
((S + T - 2 S T) g1,k+,β g2,k+,γ + (-1 + S) T g1,i+,β g2,k+,γ + S (-1 + T) g1,k+,β g2,i+,γ) g3,α,i+
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i+

Out[*]=
1
S^2 T^2 (S T ((-1 + S) T g1,i+,β g2,j+,γ + g1,j+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,j+,γ)) g3,α,i+ +
(S (-1 + T) g1,k+,β (S g2,i+,γ - (-1 + S) g2,j+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,j+,β) g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+ +
((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ))
((-1 + S T) g3,α,i+ + S T g3,α,j+))

```



```
In[*]:= Simplify[lhs == rhs]
```

```
Out[*]=
True
```

Setting up R_1

```
In[*]:= bas = {yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
len = Length[bas];
vars = Table[cp = 0; cp = .; cp, {p, 2 len}]
R1[1, i_, j_] := Evaluate[Sum[cp bas[[p]], {p, len}]];
R1[-1, i_, j_] := Evaluate[Sum[c1en+p bas[[p]], {p, Length[bas]}]]];
```

```
Out[*]=
{yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
```

```
Out[*]=
{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16}
```

```
In[*]:= R1[-1, i+, 7]
```

```
Out[*]=
c16 y7,7,7 + c15 y7,7,i+ + c14 y7,i+,7 + c13 y7,i+,i+ + c12 yi+,7,7 + c11 yi+,7,i+ + c10 yi+,i+,7 + c9 yi+,i+,i+
```

Solving R2b

In[]:= `Clear[i, j];`

`eqn = Expand[R1[1, i, j] + R1[-1, i+, j+] //.`

`gRules_{1,i,j} U gRules_{-1,i+,j+} U yRules_{1,i,j} U yRules_{-1,i+,j+} /. {_If -> 0, g_{_,_} -> 0}]`

Out[]:=

$$\begin{aligned}
 & c_1 y_{i^{++}, i^{++}, i^{++}} + c_9 y_{i^{++}, i^{++}, i^{++}} - c_{10} y_{i^{++}, i^{++}, i^{++}} + \frac{c_{10} y_{i^{++}, i^{++}, i^{++}}}{S} - c_{11} y_{i^{++}, i^{++}, i^{++}} + \frac{c_{11} y_{i^{++}, i^{++}, i^{++}}}{T} + \\
 & c_{12} y_{i^{++}, i^{++}, i^{++}} - \frac{c_{12} y_{i^{++}, i^{++}, i^{++}}}{S} - \frac{c_{12} y_{i^{++}, i^{++}, i^{++}}}{T} + \frac{c_{12} y_{i^{++}, i^{++}, i^{++}}}{S T} + c_2 y_{i^{++}, i^{++}, j^{++}} + \frac{c_{10} y_{i^{++}, i^{++}, j^{++}}}{S} - \\
 & \frac{c_{12} y_{i^{++}, i^{++}, j^{++}}}{S} + \frac{c_{12} y_{i^{++}, i^{++}, j^{++}}}{S T} + c_3 y_{i^{++}, j^{++}, i^{++}} + \frac{c_{11} y_{i^{++}, j^{++}, i^{++}}}{T} - \frac{c_{12} y_{i^{++}, j^{++}, i^{++}}}{T} + \frac{c_{12} y_{i^{++}, j^{++}, i^{++}}}{S T} + \\
 & c_4 y_{i^{++}, j^{++}, j^{++}} + \frac{c_{12} y_{i^{++}, j^{++}, j^{++}}}{S T} + c_5 y_{j^{++}, i^{++}, i^{++}} - c_9 y_{j^{++}, i^{++}, i^{++}} + S T c_9 y_{j^{++}, i^{++}, i^{++}} + c_{10} y_{j^{++}, i^{++}, i^{++}} - \\
 & \frac{c_{10} y_{j^{++}, i^{++}, i^{++}}}{S} + T c_{10} y_{j^{++}, i^{++}, i^{++}} - S T c_{10} y_{j^{++}, i^{++}, i^{++}} + c_{11} y_{j^{++}, i^{++}, i^{++}} + S c_{11} y_{j^{++}, i^{++}, i^{++}} - \frac{c_{11} y_{j^{++}, i^{++}, i^{++}}}{T} - \\
 & S T c_{11} y_{j^{++}, i^{++}, i^{++}} + \frac{c_{12} y_{j^{++}, i^{++}, i^{++}}}{S} - S c_{12} y_{j^{++}, i^{++}, i^{++}} + \frac{c_{12} y_{j^{++}, i^{++}, i^{++}}}{T} - \frac{c_{12} y_{j^{++}, i^{++}, i^{++}}}{S T} - T c_{12} y_{j^{++}, i^{++}, i^{++}} + \\
 & S T c_{12} y_{j^{++}, i^{++}, i^{++}} + S T c_{13} y_{j^{++}, i^{++}, i^{++}} + T c_{14} y_{j^{++}, i^{++}, i^{++}} - S T c_{14} y_{j^{++}, i^{++}, i^{++}} + S c_{15} y_{j^{++}, i^{++}, i^{++}} - \\
 & S T c_{15} y_{j^{++}, i^{++}, i^{++}} + c_{16} y_{j^{++}, i^{++}, i^{++}} - S c_{16} y_{j^{++}, i^{++}, i^{++}} - T c_{16} y_{j^{++}, i^{++}, i^{++}} + S T c_{16} y_{j^{++}, i^{++}, i^{++}} + \\
 & c_6 y_{j^{++}, i^{++}, j^{++}} - \frac{c_{10} y_{j^{++}, i^{++}, j^{++}}}{S} + T c_{10} y_{j^{++}, i^{++}, j^{++}} + c_{12} y_{j^{++}, i^{++}, j^{++}} + \frac{c_{12} y_{j^{++}, i^{++}, j^{++}}}{S} - \frac{c_{12} y_{j^{++}, i^{++}, j^{++}}}{S T} - \\
 & T c_{12} y_{j^{++}, i^{++}, j^{++}} + T c_{14} y_{j^{++}, i^{++}, j^{++}} + c_{16} y_{j^{++}, i^{++}, j^{++}} - T c_{16} y_{j^{++}, i^{++}, j^{++}} + c_7 y_{j^{++}, j^{++}, i^{++}} + S c_{11} y_{j^{++}, j^{++}, i^{++}} - \\
 & \frac{c_{11} y_{j^{++}, j^{++}, i^{++}}}{T} + c_{12} y_{j^{++}, j^{++}, i^{++}} - S c_{12} y_{j^{++}, j^{++}, i^{++}} + \frac{c_{12} y_{j^{++}, j^{++}, i^{++}}}{T} - \frac{c_{12} y_{j^{++}, j^{++}, i^{++}}}{S T} + S c_{15} y_{j^{++}, j^{++}, i^{++}} + \\
 & c_{16} y_{j^{++}, j^{++}, i^{++}} - S c_{16} y_{j^{++}, j^{++}, i^{++}} + c_8 y_{j^{++}, j^{++}, j^{++}} + c_{12} y_{j^{++}, j^{++}, j^{++}} - \frac{c_{12} y_{j^{++}, j^{++}, j^{++}}}{S T} + c_{16} y_{j^{++}, j^{++}, j^{++}}
 \end{aligned}$$

In[]:= `covars = Cases[eqn, y_., ∞]`

Out[]:=

- `{y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}},`
- `y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, i^{++}}, y_{i^{++}, i^{++}, j^{++}}, y_{i^{++}, i^{++}, j^{++}}, y_{i^{++}, i^{++}, j^{++}}, y_{i^{++}, i^{++}, j^{++}},`
- `y_{i^{++}, j^{++}, i^{++}}, y_{i^{++}, j^{++}, i^{++}}, y_{i^{++}, j^{++}, i^{++}}, y_{i^{++}, j^{++}, i^{++}}, y_{i^{++}, j^{++}, j^{++}}, y_{i^{++}, j^{++}, j^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}},`
- `y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}},`
- `y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}}, y_{j^{++}, i^{++}, i^{++}},`
- `y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}},`
- `y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, i^{++}, j^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}},`
- `y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, i^{++}}, y_{j^{++}, j^{++}, j^{++}}, y_{j^{++}, j^{++}, j^{++}}, y_{j^{++}, j^{++}, j^{++}}, y_{j^{++}, j^{++}, j^{++}}`

```
In[*]:= Short[eqns = {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}], 5]
```

Out[*]//Short=

$$\left\{ c_4 + \frac{c_{12}}{S T} = 0, c_2 + \frac{c_{10}}{S} - \frac{c_{12}}{S} + \frac{c_{12}}{S T} = 0, \ll 5 \gg, \right.$$

$$c_5 - c_9 + S T c_9 + c_{10} - \frac{c_{10}}{S} + T c_{10} - S T c_{10} + \ll 19 \gg + S c_{15} - S T c_{15} + c_{16} - S c_{16} - T c_{16} + S T c_{16} = 0,$$

$$c_1 y_{i^{++}, i^{++}, i^{++}} + c_9 y_{i^{++}, i^{++}, i^{++}} - c_{10} y_{i^{++}, i^{++}, i^{++}} + \frac{c_{10} y_{i^{++}, i^{++}, i^{++}}}{S} - c_{11} y_{i^{++}, i^{++}, i^{++}} +$$

$$\ll 85 \gg + c_8 y_{j^{++}, j^{++}, j^{++}} + c_{12} y_{j^{++}, j^{++}, j^{++}} - \frac{c_{12} y_{j^{++}, j^{++}, j^{++}}}{S T} + c_{16} y_{j^{++}, j^{++}, j^{++}} = 0 \left. \right\}$$

```
In[*]:= Short[{sol} = Solve[eqns, vars], 5]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[*]//Short=

$$\left\{ \left\{ c_2 \rightarrow -\frac{c_1}{-1+S} - \frac{c_9}{-1+S} - \frac{(1-T)c_{11}}{(-1+S)T}, c_3 \rightarrow -\frac{c_1}{-1+T} - \frac{c_9}{-1+T} - \frac{(1-S)c_{10}}{S(-1+T)}, \right. \right.$$

$$c_4 \rightarrow \frac{c_1}{(-1+S)(-1+T)} + \frac{c_9}{(-1+S)(-1+T)} - \frac{c_{10}}{S(-1+T)} - \frac{c_{11}}{(-1+S)T},$$

$$\ll 3 \gg, c_{12} \rightarrow -\frac{S T c_1}{(-1+S)(-1+T)} - \frac{S T c_9}{(-1+S)(-1+T)} + \frac{T c_{10}}{-1+T} + \frac{S c_{11}}{-1+S},$$

$$\left. c_{13} \rightarrow -\frac{(1-S T)c_1}{S T} - \frac{c_5}{S T} - \frac{(1-S)c_{14}}{S} - \frac{(1-T)c_{15}}{T} - \frac{(1-S-T+S T)c_{16}}{S T} \right\}$$

```
In[*]:= Short[sol /. (v_ -> val_) :-> (v = CF[val])]
```

Out[*]//Short=

$$\left\{ -\frac{T c_1 + T c_9 + c_{11} - T c_{11}}{(-1+S) T}, -\frac{S c_1 + S \ll 1 \gg \ll 1 \gg \ll 1 \gg - S c_{10}}{S(-1+T)}, \ll 4 \gg, -\frac{\ll 1 \gg}{\ll 1 \gg}, \frac{-c_1 + \ll 15 \gg}{S T} \right\}$$

```
In[*]:= Short[R1[1, i, j], 5]
```

Out[*]//Short=

$$c_1 y_{i, i, i} - \frac{(T c_1 + T c_9 + c_{11} - T c_{11}) y_{i, i, j}}{(-1+S) T} -$$

$$\frac{(S c_1 + S c_9 + c_{10} - S c_{10}) y_{\ll 1 \gg}}{S(-1+T)} + \frac{\ll 1 \gg}{\ll 1 \gg} + \ll 1 \gg \ll 1 \gg \ll 1 \gg - \frac{(\ll 1 \gg) y_{\ll 1 \gg}}{(-1+S) T} -$$

$$\frac{(-S c_1 + S^2 T c_1 - S c_9 + \ll 12 \gg + S^2 c_{16} + S T c_{16} - S^2 T c_{16}) y_{\ll 1 \gg}}{S(-1+T)} +$$

$$\frac{1}{(-1+S) S(-1+T) T} \left(-S T c_1 + S^2 T^2 c_1 - S T c_9 + S^2 T^2 c_9 - T c_{10} + S T c_{10} + S T^2 c_{10} - S^2 T^2 c_{10} - \right.$$

$$\left. S c_{11} + S T c_{11} + S^2 T c_{11} - S^2 T^2 c_{11} - S T c_{16} + S^2 T c_{16} + S T^2 c_{16} - S^2 T^2 c_{16} \right) y_{j, j, j}$$

Solving R3

```
In[*]:= Clear[i, j, k];
lhs =
  Expand[R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] // . gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+ ∪
    yRules1,i,j ∪ yRules1,i+,k ∪ yRules1,j+,k+ /. {_If → 0, g_>, _ → 0}];
rhs =
  Expand[R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] // . gRules1,j,k ∪ gRules1,i,k+ ∪ gRules1,i+,j+ ∪
    yRules1,j,k ∪ yRules1,i,k+ ∪ yRules1,i+,j+ /. {_If → 0, g_>, _ → 0}];
eqn = lhs - rhs
```

Out[*]=

$$\frac{S c_1}{(-1+S)(-1+T)} - \frac{T c_1}{(-1+S)(-1+T)} + \frac{2 S T c_1}{(-1+S)(-1+T)} - \frac{S c_9}{(-1+S)(-1+T)} - \frac{T c_9}{(-1+S)(-1+T)} + \frac{2 S T c_9}{(-1+S)(-1+T)} - \frac{c_{10}}{(-1+S)(-1+T)} +$$

$$\frac{S c_{10}}{(-1+S)(-1+T)} + \dots 984 \dots + \frac{c_{16} Y_{k^{++},k^{++},j^{++}}}{(-1+S) S (-1+T)} + \frac{S c_{16} Y_{k^{++},k^{++},j^{++}}}{(-1+S)(-1+T)} + \frac{2 T c_{16} Y_{k^{++},k^{++},j^{++}}}{(-1+S)(-1+T)} - \frac{T c_{16} Y_{k^{++},k^{++},j^{++}}}{(-1+S) S (-1+T)} - \frac{S T c_{16} Y_{k^{++},k^{++},j^{++}}}{(-1+S)(-1+T)}$$

Full expression not available (original memory size: 0.5 MB) ⚙

```
In[*]:= vars = Union@Cases[eqn, c_., ∞]
```

Out[*]=

{c₁, c₅, c₉, c₁₀, c₁₁, c₁₄, c₁₅, c₁₆}

```
In[*]:= Short[
  covars = Cases[eqn, c_.*(g1 : g_>, _) (g2 : g_>, _) => g1 g2, ∞] ∪
  Cases[eqn, g_>, ∞] ∪ Cases[eqn, y_>, ∞],
  5]
```

Out[*]//Short=

<<1>>

```
In[*]:= Short[eqns =
  {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y)_>, _ → 0}, 5]
```

Out[*]//Short=

$$\left\{ \frac{c_1}{S} - \frac{c_1}{(-1+S)(-1+T)} + \frac{2 c_1}{S(-1+T)} + \frac{c_1}{(-1+S)S(-1+T)} + \frac{c_1}{T} + \right.$$

$$\frac{2 c_1}{(-1+S)T} - \frac{2 c_1}{S T} - \frac{c_1}{(-1+S)S T} - \frac{S c_1}{(-1+S)T} + \ll 44 \gg + \frac{2 c_{11}}{(-1+S)T^2} - \frac{c_{11}}{(-1+S)S T^2} -$$

$$\frac{S c_{11}}{(-1+S)T^2} + \frac{c_{11}}{(-1+S)(-1+T)T^2} - \frac{c_{11}}{(-1+S)S(-1+T)T^2} - \frac{2 c_{11}}{(-1+S)T} +$$

$$\frac{c_{11}}{(-1+S)S T} + \frac{S c_{11}}{(-1+S)T} - \frac{2 c_{11}}{(-1+S)(-1+T)T} + \frac{2 c_{11}}{(-1+S)S(-1+T)T} = 0,$$

$$\left. - \frac{S c_1}{(-1+S)(\ll 1 \gg)} + \ll 15 \gg = 0, \ll 19 \gg, \ll 160 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} = 0, \ll 1 \gg = 0 \right\}$$

In[*]:= `Short[{sol} = Solve[eqns, vars], 5]`

 **Solve**: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

{ {c₁ → 0, c₅ → S T c₁₄ + S T c₁₅, c₉ → 0, c₁₀ → 0, c₁₁ → 0, c₁₆ → 0} }

In[*]:= `Short[sol /. (v_ → val_) := (v = CF[val])]`

Out[*]//Short=

{0, S T (c₁₄ + c₁₅), 0, 0, 0, 0}

In[*]:= `Short[R1[1, i, j], 5]`

Out[*]//Short=

$$S T (c_{14} + c_{15}) y_{j,i,i} - \frac{(-T^2 c_{14} + S T^2 c_{14}) y_{j,i,j}}{(-1 + S) T} - \frac{(-S^2 c_{15} + S^2 T c_{15}) y_{j,j,i}}{S (-1 + T)}$$

```
In[*]:= Clear[i, j, k];
lhs =
  Expand[R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] /. gRules1,i,j U gRules1,i+,k U gRules1,j+,k+ U
    yRules1,i,j U yRules1,i+,k U yRules1,j+,k+ /. {_If -> 0, g_.,_ -> 0}];
rhs = Expand[R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] /. gRules1,j,k U gRules1,i,k+ U
  gRules1,i+,j+ U yRules1,j,k U yRules1,i,k+ U yRules1,i+,j+ /. {_If -> 0}];
eqn = Simplify[lhs - rhs]
```

```
Out[*]=
```

$$\frac{1}{S^4 T^4} \left(-S c_{15} \left(S^2 T^2 \left((S - ST) g_{1,j+,i++} + T (S - ST + S(-1 + T) g_{1,j+,j++} + S(-1 + T) g_{2,i++,i++} + g_{2,j+,i++} + T g_{2,j+,i++} - 2ST g_{2,j+,i++} \right) g_{3,j+,i++} + \right. \right. \\ \left. \left(S^2 T^2 - S^3 T^2 - S^2 T^3 + 2S^3 T^3 - S^3 T^4 - (-1 + S) S^2 T^3 g_{1,i++,j++} + (-1 + S) S^2 T^4 g_{1,i++,k++} + \right. \right. \\ \left. S^2 T g_{1,j+,i++} - S^2 T^2 g_{1,j+,i++} - S^2 T^2 g_{1,j+,j++} + S^3 T^3 g_{1,j+,j++} - S^3 T^3 g_{1,j+,k++} + \right. \\ \left. S^3 T^4 g_{1,j+,k++} + S^2 g_{1,k++,i++} - S^2 T g_{1,k++,i++} - S^2 T g_{1,k++,j++} + S^2 T^2 g_{1,k++,j++} + \right. \\ \left. S^3 T^2 g_{1,k++,j++} + S^2 T^4 g_{1,k++,j++} - 2S^3 T^4 g_{1,k++,j++} - S^3 T^3 g_{1,k++,k++} + S^3 T^4 g_{1,k++,k++} - \right. \\ \left. S^2 T^2 g_{2,i++,i++} + S^2 T^3 g_{2,i++,i++} + 2S^2 T^3 g_{2,j+,i++} - 2S^2 T^3 g_{2,j+,i++} - S^2 T^4 g_{2,j+,i++} + \right. \\ \left. S^3 T^4 g_{2,j+,i++} - S^2 T^3 g_{2,j+,j++} + S^3 T^4 g_{2,j+,j++} + T^2 g_{2,k++,i++} + 2T^3 g_{2,k++,i++} - \right. \\ \left. 5S T^3 g_{2,k++,i++} + 2S^2 T^3 g_{2,k++,i++} - S T^4 g_{2,k++,i++} + 3S^2 T^4 g_{2,k++,i++} - 2S^3 T^4 g_{2,k++,i++} - \right. \\ \left. S T^3 g_{2,k++,j++} + 2S^2 T^3 g_{2,k++,j++} + S^2 T^4 g_{2,k++,j++} - 2S^3 T^4 g_{2,k++,j++} \right) g_{3,k++,i++} + \\ S T^2 \left(-S^2 (-1 + T) g_{1,k++,j++} + T \left(S^2 - S^2 T + S^2 (-1 + T) g_{1,k++,k++} + (-1 + S) S (-1 + T) g_{2,j+,i++} - \right. \right. \\ \left. S^2 g_{2,j+,j++} + S^2 T g_{2,j+,j++} - g_{2,k++,i++} + S g_{2,k++,i++} - T g_{2,k++,i++} + 3ST g_{2,k++,i++} - \right. \\ \left. 2S^2 T g_{2,k++,i++} + S g_{2,k++,j++} + ST g_{2,k++,j++} - 2S^2 T g_{2,k++,j++} \right) g_{3,k++,j++} \left. \right) + \\ c_{14} \left(S^2 T^3 \left(-ST + S^2 T - (-1 + S) ST g_{1,i++,i++} + ST^2 (-S - T + 2ST) g_{1,i++,j++} - S g_{1,j+,i++} + \right. \right. \\ \left. ST g_{1,j+,i++} - S^2 T g_{1,j+,j++} - ST^2 g_{1,j+,j++} + 3S^2 T^2 g_{1,j+,j++} + S T^3 g_{1,j+,j++} - \right. \\ \left. 2S^2 T^3 g_{1,j+,j++} - T g_{2,j+,i++} + ST g_{2,j+,i++} + ST g_{2,j+,j++} - S^2 T g_{2,j+,j++} \right) g_{3,j+,i++} - \\ \left(S^2 T^3 - S^3 T^3 - S^2 T^4 + 2S^3 T^4 - S^4 T^4 + (-1 + S) S^2 T^3 g_{1,i++,i++} + S^2 T^3 (S + T - 2ST) g_{1,i++,k++} + \right. \\ \left. S^3 g_{1,j+,i++} + S^2 T g_{1,j+,i++} - 3S^3 T g_{1,j+,i++} - S^4 T g_{1,j+,i++} - S^2 T^2 g_{1,j+,i++} + \right. \\ \left. 2S^3 T^2 g_{1,j+,i++} + 3S^4 T^2 g_{1,j+,i++} - S^2 T^3 g_{1,j+,i++} + 2S^3 T^3 g_{1,j+,i++} - 3S^4 T^3 g_{1,j+,i++} + \right. \\ \left. S^2 T^4 g_{1,j+,i++} - 2S^3 T^4 g_{1,j+,i++} + S^4 T^4 g_{1,j+,i++} - S^3 T^3 g_{1,j+,j++} + S^4 T^4 g_{1,j+,j++} + \right. \\ \left. S^3 T^2 g_{1,j+,k++} + S^2 T^3 g_{1,j+,k++} - 3S^3 T^3 g_{1,j+,k++} + S^4 T^3 g_{1,j+,k++} - S^2 T^4 g_{1,j+,k++} + \right. \\ \left. 3S^3 T^4 g_{1,j+,k++} - 2S^4 T^4 g_{1,j+,k++} + S^2 T g_{1,k++,i++} + S^3 T g_{1,k++,i++} - S^2 T^2 g_{1,k++,i++} - \right. \\ \left. 3S^3 T^2 g_{1,k++,i++} + 3S^3 T^3 g_{1,k++,i++} - S^3 T^4 g_{1,k++,i++} - S^3 T^2 g_{1,k++,j++} + 2S^3 T^3 g_{1,k++,j++} - \right. \\ \left. S^3 T^4 g_{1,k++,j++} + S^3 T^3 g_{2,i++,j++} - S^3 T^4 g_{2,i++,j++} - S^4 T^3 g_{2,i++,k++} + S^4 T^4 g_{2,i++,k++} + \right. \\ \left. S T^3 g_{2,j+,i++} - S^2 T^3 g_{2,j+,i++} - S^2 T^3 g_{2,j+,j++} + S^3 T^4 g_{2,j+,j++} + S^4 T^3 g_{2,j+,k++} - \right. \\ \left. S^4 T^4 g_{2,j+,k++} + T^3 g_{2,k++,i++} - S T^3 g_{2,k++,i++} - S T^3 g_{2,k++,j++} + S^2 T^3 g_{2,k++,j++} + \right. \\ \left. S^2 T^4 g_{2,k++,j++} - S^3 T^4 g_{2,k++,j++} - S^3 T^4 g_{2,k++,k++} + S^4 T^4 g_{2,k++,k++} \right) g_{3,k++,i++} + \\ S^2 T \left(-S (S (-1 + T) - T) (-1 + T)^2 g_{1,j+,i++} + T \left(-S T^2 + S^2 T^2 - (-1 + S) S T^2 g_{1,j+,j++} + \right. \right. \\ \left. ST (-S - T + 2ST) g_{1,j+,k++} + S g_{1,k++,i++} - 2ST g_{1,k++,i++} + S T^2 g_{1,k++,i++} - ST g_{1,k++,j++} + \right. \\ \left. S T^2 g_{1,k++,j++} - T^2 g_{2,k++,j++} + S T^2 g_{2,k++,j++} + S T^2 g_{2,k++,k++} - S^2 T^2 g_{2,k++,k++} \right) \left. \right) g_{3,k++,j++} \left. \right)$$