Pensieve header: An attempt on the rank 2 (sl_3) Gassner representation.

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In[*]:= G_{i_,j_} [𝓔_] := Expand[𝓔 /. {
            e_j :> e^{t_i} e_j + (1 - e^{t_i}) e_i
          }];
        bas = {e_1, e_2, e_3};
        lhs = bas // G_{1,2} // G_{1,3} // G_{2,3};
        rhs = bas // G_{2,3} // G_{1,3} // G_{1,2};
        Simplify@Thread[lhs == rhs]
```

Out[*]=
$$\left\{e_1,\ e_1 - e^{t_1} e_1 + e^{t_1} e_2,\ e_1 - e^{t_1} e_1 + e^{t_1} e_2 - e^{t_1+t_2} e_2 + e^{t_1+t_2} e_3\right\}$$

Out[*]=
True

```
In[*]:= R2G_{i_,j_} [𝓔_] := Expand[𝓔 /. {
            e_j :> e^{t_i} e_j + (1 - e^{t_i}) e_i,
            f_j :> e^{s_i} f_j + (1 - e^{s_i}) f_i,
            g_j :> e^{t_i+s_i} g_j + (1 - e^{t_i+s_i}) g_i + (a_i) e_i (f_j + c_i f_i) + (b_i) f_i (e_j + d_i e_i)
          }];
        bas = {e_1, e_2, e_3, f_1, f_2, f_3, e_1 f_1, e_1 f_2, e_1 f_3, e_2 f_1, e_2 f_2, e_2 f_3, e_3 f_1, e_3 f_2, e_3 f_3, g_1, g_2, g_3};
        lhs = bas // R2G_{1,2} // R2G_{1,3} // R2G_{2,3};
        rhs = bas // R2G_{2,3} // R2G_{1,3} // R2G_{1,2};
        Thread[bas → Expand[lhs - rhs]] // DeleteCases[_ → 0] // Column
        diff = Simplify[Last[lhs - rhs] /. {a_{i_} :> e^{s_i} (-1 + e^{t_i}), b_{i_} :> e^{t_i} (-1 + e^{s_i}), (c | d)_ → -1}]
        Table[Factor[Coefficient[diff, β] /. (e | f | g)_ → 0], {β, bas}] // DeleteCases[0] // Column
```

Out[○]=

$\{e_1,\ e_1 - e^{t_1} e_1 + e^{t_1} e_2,\ e_1 - e^{t_1} e_1 + e^{t_1} e_2 - e^{t_1+t_2} e_2 + e^{t_1+t_2} e_3,\ f_1,\ f_1 - e^{s_1} f_1 + e^{s_1} f_2,$

$f_1 - e^{s_1} f_1 + e^{s_1} f_2 - e^{s_1+s_2} f_2 + e^{s_1+s_2} f_3,\ e_1 f_1,\ e_1 f_1 - e^{s_1} e_1 f_1 + e^{s_1} e_1 f_2,$

$e_1 f_1 - e^{s_1} e_1 f_1 + e^{s_1} e_1 f_2 - e^{s_1+s_2} e_1 f_2 + e^{s_1+s_2} e_1 f_3,\ e_1 f_1 - e^{t_1} e_1 f_1 + e^{t_1} e_2 f_1,$

$e_1 f_1 - e^{s_1} e_1 f_1 - e^{t_1} e_1 f_1 + e^{s_1+t_1} e_1 f_1 + e^{t_1} e_2 f_1 - e^{s_1+t_1} e_2 f_1 + e^{s_1} e_1 f_2 - e^{s_1+t_1} e_1 f_2 + e^{s_1+t_1} e_2 f_2,$

$e_1 f_1 - e^{s_1} e_1 f_1 - e^{t_1} e_1 f_1 + e^{s_1+t_1} e_1 f_1 + e^{t_1} e_2 f_1 - e^{s_1+t_1} e_2 f_1 + e^{s_1} e_1 f_2 - e^{s_1+s_2} e_1 f_2 - e^{s_1+t_1} e_1 f_2 +$

$e^{s_1+s_2+t_1} e_1 f_2 + e^{s_1+t_1} e_2 f_2 - e^{s_1+s_2+t_1} e_2 f_2 + e^{s_1+s_2} e_1 f_3 - e^{s_1+s_2+t_1} e_1 f_3 + e^{s_1+s_2+t_1} e_2 f_3,$

$e_1 f_1 - e^{t_1} e_1 f_1 + e^{t_1} e_2 f_1 - e^{t_1+t_2} e_2 f_1 + e^{t_1+t_2} e_3 f_1,$

$e_1 f_1 - e^{s_1} e_1 f_1 - e^{t_1} e_1 f_1 + e^{s_1+t_1} e_1 f_1 + e^{t_1} e_2 f_1 - e^{s_1+t_1} e_2 f_1 - e^{t_1+t_2} e_2 f_1 + e^{s_1+t_1+t_2} e_2 f_1 +$

$e^{t_1+t_2} e_3 f_1 - e^{s_1+t_1+t_2} e_3 f_1 + e^{s_1} e_1 f_2 - e^{s_1+t_1} e_1 f_2 + e^{s_1+t_1} e_2 f_2 - e^{s_1+t_1+t_2} e_2 f_2 + e^{s_1+t_1+t_2} e_3 f_2,$

$e_1 f_1 - e^{s_1} e_1 f_1 - e^{t_1} e_1 f_1 + e^{s_1+t_1} e_1 f_1 + e^{t_1} e_2 f_1 - e^{s_1+t_1} e_2 f_1 - e^{t_1+t_2} e_2 f_1 + e^{s_1+t_1+t_2} e_2 f_1 +$

$e^{t_1+t_2} e_3 f_1 - e^{s_1+t_1+t_2} e_3 f_1 + e^{s_1} e_1 f_2 - e^{s_1+s_2} e_1 f_2 - e^{s_1+t_1} e_1 f_2 + e^{s_1+s_2+t_1} e_1 f_2 + e^{s_1+t_1} e_2 f_2 -$

$e^{s_1+s_2+t_1} e_2 f_2 - e^{s_1+t_1+t_2} e_2 f_2 + e^{s_1+s_2+t_1+t_2} e_2 f_2 + e^{s_1+t_1+t_2} e_3 f_2 - e^{s_1+s_2+t_1+t_2} e_3 f_2 +$

$e^{s_1+s_2} e_1 f_3 - e^{s_1+s_2+t_1} e_1 f_3 + e^{s_1+s_2+t_1} e_2 f_3 - e^{s_1+s_2+t_1+t_2} e_2 f_3 + e^{s_1+s_2+t_1+t_2} e_3 f_3,$

$g_1,\ a_1 c_1 e_1 f_1 + b_1 d_1 e_1 f_1 + b_1 e_2 f_1 + a_1 e_1 f_2 + g_1 - e^{s_1+t_1} g_1 + e^{s_1+t_1} g_2,$

$a_2 e_1 f_1 - e^{s_1} a_2 e_1 f_1 - e^{t_1} a_2 e_1 f_1 + e^{s_1+t_1} a_2 e_1 f_1 + b_2 e_1 f_1 - e^{s_1} b_2 e_1 f_1 - e^{t_1} b_2 e_1 f_1 + e^{s_1+t_1} b_2 e_1 f_1 +$

$a_1 c_1 e_1 f_1 + a_2 c_2 e_1 f_1 - e^{s_1} a_2 c_2 e_1 f_1 - e^{t_1} a_2 c_2 e_1 f_1 + e^{s_1+t_1} a_2 c_2 e_1 f_1 + b_1 d_1 e_1 f_1 + b_2 d_2 e_1 f_1 -$

$e^{s_1} b_2 d_2 e_1 f_1 - e^{t_1} b_2 d_2 e_1 f_1 + e^{s_1+t_1} b_2 d_2 e_1 f_1 + e^{t_1} a_2 e_2 f_1 - e^{s_1+t_1} a_2 e_2 f_1 + b_1 e_2 f_1 - e^{s_2+t_2} b_1 e_2 f_1 +$

$e^{t_1} a_2 c_2 e_2 f_1 - e^{s_1+t_1} a_2 c_2 e_2 f_1 + e^{t_1} b_2 d_2 e_2 f_1 - e^{s_1+t_1} b_2 d_2 e_2 f_1 + e^{s_2+t_2} b_1 e_3 f_1 + e^{t_1} b_2 e_3 f_1 -$

$e^{s_1+t_1} b_2 e_3 f_1 + a_1 e_1 f_2 - e^{s_2+t_2} a_1 e_1 f_2 + e^{s_1} b_2 e_1 f_2 - e^{s_1+t_1} b_2 e_1 f_2 + e^{s_1} a_2 c_2 e_1 f_2 - e^{s_1+t_1} a_2 c_2 e_1 f_2 +$

$e^{s_1} b_2 d_2 e_1 f_2 - e^{s_1+t_1} b_2 d_2 e_1 f_2 + e^{s_1+t_1} a_2 c_2 e_2 f_2 + e^{s_1+t_1} b_2 d_2 e_2 f_2 + e^{s_1+t_1} b_2 e_3 f_2 + e^{s_2+t_2} a_1 e_1 f_3 +$

$e^{s_1} a_2 e_1 f_3 - e^{s_1+t_1} a_2 e_1 f_3 + e^{s_1+t_1} a_2 e_2 f_3 + g_1 - e^{s_1+t_1} g_1 + e^{s_1+t_1} g_2 - e^{s_1+s_2+t_1+t_2} g_2 + e^{s_1+s_2+t_1+t_2} g_3\}$

Out[○]=

$g_3 \to - a_2 e_1 f_1 + e^{s_1} a_2 e_1 f_1 + e^{t_1} a_2 e_1 f_1 - e^{s_1+t_1} a_2 e_1 f_1 - b_2 e_1 f_1 + e^{s_1} b_2 e_1 f_1 +$

$e^{t_1} b_2 e_1 f_1 - e^{s_1+t_1} b_2 e_1 f_1 - a_2 c_2 e_1 f_1 + e^{s_1} a_2 c_2 e_1 f_1 + e^{t_1} a_2 c_2 e_1 f_1 - e^{s_1+t_1} a_2 c_2 e_1 f_1 -$

$b_2 d_2 e_1 f_1 + e^{s_1} b_2 d_2 e_1 f_1 + e^{t_1} b_2 d_2 e_1 f_1 - e^{s_1+t_1} b_2 d_2 e_1 f_1 - e^{t_1} a_2 e_2 f_1 + e^{s_1+t_1} a_2 e_2 f_1 -$

$e^{t_2} b_1 e_2 f_1 + e^{s_2+t_2} b_1 e_2 f_1 - e^{t_1} a_2 c_2 e_2 f_1 + e^{s_1+t_1} a_2 c_2 e_2 f_1 - e^{t_1} b_2 d_2 e_2 f_1 +$

$e^{s_1+t_1} b_2 d_2 e_2 f_1 + e^{t_2} b_1 e_3 f_1 - e^{s_2+t_2} b_1 e_3 f_1 - e^{t_1} b_2 e_3 f_1 + e^{s_1+t_1} b_2 e_3 f_1 - e^{s_2} a_1 e_1 f_2 +$

$e^{s_2+t_2} a_1 e_1 f_2 - e^{s_1} b_2 e_1 f_2 + e^{s_1+t_1} b_2 e_1 f_2 - e^{s_1} a_2 c_2 e_1 f_2 + e^{s_1+t_1} a_2 c_2 e_1 f_2 -$

$e^{s_1} b_2 d_2 e_1 f_2 + e^{s_1+t_1} b_2 d_2 e_1 f_2 + e^{s_2} a_1 e_1 f_3 - e^{s_2+t_2} a_1 e_1 f_3 - e^{s_1} a_2 e_1 f_3 + e^{s_1+t_1} a_2 e_1 f_3$

Out[○]=

0

Out[○]=

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R2G_{i_,j_}[ε_] := Expand[ε /. {
    e_j ↦ T_i e_j + (1 - T_i) e_i, f_j ↦ S_i f_j + (1 - S_i) f_i,
    g_j ↦ T_i S_i g_j + (1 - T_i S_i) g_i
        + (S_i + T_i - 2 S_i T_i) e_i f_i + S_i (T_i - 1) e_i f_j + T_i (S_i - 1) e_j f_i}]
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```
bas = {e₁, e₂, e₃, f₁, f₂, f₃, e₁ f₁, e₁ f₂, e₁ f₃, e₂ f₁, e₂ f₂, e₂ f₃, e₃ f₁, e₃ f₂, e₃ f₃, g₁, g₂, g₃};
lhs = bas // R2G₁,₂ // R2G₁,₃ // R2G₂,₃;
rhs = bas // R2G₂,₃ // R2G₁,₃ // R2G₁,₂
lhs == rhs
```

$Out[\bullet]=$

$\{e_1, \ e_1 - e_1 T_1 + e_2 T_1, \ e_1 - e_1 T_1 + e_2 T_1 - e_2 T_1 T_2 + e_3 T_1 T_2, \ f_1,$

$f_1 - f_1 S_1 + f_2 S_1, \ f_1 - f_1 S_1 + f_2 S_1 - f_2 S_1 S_2 + f_3 S_1 S_2, \ e_1 f_1, \ e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1,$

$e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1 - e_1 f_2 S_1 S_2 + e_1 f_3 S_1 S_2, \ e_1 f_1 - e_1 f_1 T_1 + e_2 f_1 T_1,$

$e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1 - e_1 f_1 T_1 + e_2 f_1 T_1 + e_1 f_1 S_1 T_1 - e_2 f_1 S_1 T_1 - e_1 f_2 S_1 T_1 + e_2 f_2 S_1 T_1,$

$e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1 - e_1 f_2 S_1 S_2 + e_1 f_3 S_1 S_2 - e_1 f_1 T_1 + e_2 f_1 T_1 + e_1 f_1 S_1 T_1 - e_2 f_1 S_1 T_1 -$
$\quad e_1 f_2 S_1 T_1 + e_2 f_2 S_1 T_1 + e_1 f_2 S_1 S_2 T_1 - e_2 f_2 S_1 S_2 T_1 - e_1 f_3 S_1 S_2 T_1 + e_2 f_3 S_1 S_2 T_1,$

$e_1 f_1 - e_1 f_1 T_1 + e_2 f_1 T_1 - e_2 f_1 T_1 T_2 + e_3 f_1 T_1 T_2,$

$e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1 - e_1 f_1 T_1 + e_2 f_1 T_1 + e_1 f_1 S_1 T_1 - e_2 f_1 S_1 T_1 - e_1 f_2 S_1 T_1 + e_2 f_2 S_1 T_1 -$
$\quad e_2 f_1 T_1 T_2 + e_3 f_1 T_1 T_2 + e_2 f_1 S_1 T_1 T_2 - e_3 f_1 S_1 T_1 T_2 - e_2 f_2 S_1 T_1 T_2 + e_3 f_2 S_1 T_1 T_2,$

$e_1 f_1 - e_1 f_1 S_1 + e_1 f_2 S_1 - e_1 f_2 S_1 S_2 + e_1 f_3 S_1 S_2 - e_1 f_1 T_1 + e_2 f_1 T_1 + e_1 f_1 S_1 T_1 -$
$\quad e_2 f_1 S_1 T_1 - e_1 f_2 S_1 T_1 + e_2 f_2 S_1 T_1 + e_1 f_2 S_1 S_2 T_1 - e_2 f_2 S_1 S_2 T_1 - e_1 f_3 S_1 S_2 T_1 +$
$\quad e_2 f_3 S_1 S_2 T_1 - e_2 f_1 T_1 T_2 + e_3 f_1 T_1 T_2 + e_2 f_1 S_1 T_1 T_2 - e_3 f_1 S_1 T_1 T_2 - e_2 f_2 S_1 T_1 T_2 +$
$\quad e_3 f_2 S_1 T_1 T_2 + e_2 f_2 S_1 S_2 T_1 T_2 - e_3 f_2 S_1 S_2 T_1 T_2 - e_2 f_3 S_1 S_2 T_1 T_2 + e_3 f_3 S_1 S_2 T_1 T_2, \ g_1,$

$g_1 + e_1 f_1 S_1 - e_1 f_2 S_1 + e_1 f_1 T_1 - e_2 f_1 T_1 - 2 e_1 f_1 S_1 T_1 + e_2 f_1 S_1 T_1 + e_1 f_2 S_1 T_1 - g_1 S_1 T_1 + g_2 S_1 T_1,$

$g_1 + e_1 f_1 S_1 - e_1 f_2 S_1 + e_1 f_2 S_1 S_2 - e_1 f_3 S_1 S_2 + e_1 f_1 T_1 - e_2 f_1 T_1 - 2 e_1 f_1 S_1 T_1 + e_2 f_1 S_1 T_1 +$
$\quad e_1 f_2 S_1 T_1 - g_1 S_1 T_1 + g_2 S_1 T_1 - e_1 f_2 S_1 S_2 T_1 + e_2 f_2 S_1 S_2 T_1 + e_1 f_3 S_1 S_2 T_1 - e_2 f_3 S_1 S_2 T_1 +$
$\quad e_2 f_1 T_1 T_2 - e_3 f_1 T_1 T_2 - e_2 f_1 S_1 T_1 T_2 + e_3 f_1 S_1 T_1 T_2 + e_2 f_2 S_1 T_1 T_2 - e_3 f_2 S_1 T_1 T_2 -$
$\quad 2 e_2 f_2 S_1 S_2 T_1 T_2 + e_3 f_2 S_1 S_2 T_1 T_2 + e_2 f_3 S_1 S_2 T_1 T_2 - g_2 S_1 S_2 T_1 T_2 + g_3 S_1 S_2 T_1 T_2\}$

$Out[\bullet]=$

True

Lower Rank 2 Gassner:

$In[\bullet]:=$ `{eⱼ ⇸ Tᵢ eⱼ + (1 - Tᵢ) eᵢ, fⱼ ⇸ Sᵢ fⱼ + (1 - Sᵢ) fᵢ,`
`    gⱼ ⇸ Tᵢ Sᵢ gⱼ + (1 - Tᵢ Sᵢ) gᵢ + Sᵢ (Tᵢ - 1) eᵢ (fⱼ - fᵢ) + Tᵢ (Sᵢ - 1) fᵢ (eⱼ - eᵢ) } /. {i → j, j → i}`

$Out[\bullet]=$

$\{e_i \rightsquigarrow T_j \, e_i + (1 - T_j) \, e_j, \ f_i \rightsquigarrow S_j \, f_i + (1 - S_j) \, f_j,$
$\ g_i \rightsquigarrow T_j \, S_j \, g_i + (1 - T_j \, S_j) \, g_j + S_j \, (T_j - 1) \, e_j \, (f_i - f_j) + T_j \, (S_j - 1) \, f_j \, (e_i - e_j)\}$

```
In[•]:=  LR2G_{i_,j_} [ℰ_] := Expand[ℰ /. {e_i :→ T_j e_i + (1 - T_j) e_j, f_i :→ S_j f_i + (1 - S_j) f_j,
            g_i :→ T_j S_j g_i + (1 - T_j S_j) g_j
                + e_j f_j (S_j + T_j - 2 S_j T_j) + e_j f_i (-S_j + S_j T_j) + e_i f_j (-T_j + S_j T_j)}];
        LR2G‾_{i_,j_} [ℰ_] := Expand[ℰ /. {e_i :→ T_j^{-1} e_i + (1 - T_j^{-1}) e_j, f_i :→ S_j^{-1} f_i + (1 - S_j^{-1}) f_j,
            g_i :→ T_j^{-1} S_j^{-1} g_i + (1 - T_j^{-1} S_j^{-1}) g_j +
                e_j f_j (S_j^{-1} + T_j^{-1} - 2 T_j^{-1} S_j^{-1}) + e_j f_i (-S_j^{-1} + T_j^{-1} S_j^{-1}) + e_i f_j (-T_j^{-1} + S_j^{-1} T_j^{-1})}];
        bas = {e_1, e_2, e_3, f_1, f_2, f_3, e_1 f_1, e_1 f_2, e_1 f_3, e_2 f_1, e_2 f_2, e_2 f_3, e_3 f_1, e_3 f_2, e_3 f_3, g_1, g_2, g_3};
        (bas // LR2G_{1,2} // LR2G‾_{1,2}) == bas
        (bas // LR2G_{3,2} // LR2G‾_{3,2}) == bas
        lhs = bas // LR2G_{1,2} // LR2G_{1,3} // LR2G_{2,3};
        rhs = bas // LR2G_{2,3} // LR2G_{1,3} // LR2G_{1,2};
        lhs == rhs

Out[•]=
        True

Out[•]=
        True

Out[•]=
        True
```