Pensieve header: A first implementation of nilpotent integration.

Initialization

```
In[•]:=  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
         Once[<< KnotTheory`; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/icbs24 to compute rotation numbers.

```
In[•]:=  CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
         CCF[ℰ_] := Factor[ℰ];
         CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
         CF[ℰ_List] := CF /@ ℰ;
         CF[sd_SeriesData] := MapAt[CF, sd, 3];
         CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p)__, ∞] ⋃ {x, p, ϵ}, ps, c},
            Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)] ];
```

Integration

Using Picard Iteration!

```
In[•]:=  𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B]
```

```
In[•]:=  $π = Identity;
```

```
In[•]:= Unprotect[Integrate];
        Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
        ∫ ω_. 𝔼[L_] đ(vs_List) := Module[{n, L0, Q, Δ, G, Z, e, λ, a, b},
            n = Length@vs; L0 = L /. ε → 0;
            Q = Table[(-∂_{vs[[a]],vs[[b]]} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
            If[(Δ = CF@Det[Q]) == 0, Message[Integrate::sing]; Return[]];
            Z = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
            While[
              e = CF@$π[(∂_λ Z) - 1/2 ∑_{a=1}^{n} ∑_{b=1}^{n} G[[a, b]] ((∂_{vs[[a]],vs[[b]]} Z) + (∂_{vs[[a]]} Z) (∂_{vs[[b]]} Z))];
              0 =!= e, Z -= ∫_0^λ e đλ
            ];
            PowerExpand@Factor[ω (Δ (2 π)^n)^{-1/2}] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]
          ];
        Protect[Integrate];
```

```
In[•]:= ∫ 𝔼[i λ x_1^2 / 2] đ{x_1}
```
Out[•]=
$$\frac{(-1)^{1/4} \, 𝔼[0]}{\sqrt{2\pi} \, \sqrt{λ}}$$

```
In[•]:= ∫ 𝔼[-i λ x_1^2 / 2] đ{x_1}
```
Out[•]=
$$-\frac{(-1)^{3/4} \, 𝔼[0]}{\sqrt{2\pi} \, \sqrt{λ}}$$

```
In[•]:= ∫ 𝔼[i/2 {x_1, x_2} . (a b / b c) . {x_1, x_2}] đ{x_1, x_2}
```
Out[•]=
$$\frac{𝔼[0]}{2\sqrt{b^2 - a\,c}\,\pi}$$

```
In[•]:= ∫ 𝔼[-λ x_1^2 / 2] đ{x_1}
```
Out[•]=
$$\frac{𝔼[0]}{\sqrt{2\pi}\,\sqrt{λ}}$$

$In[\bullet]:=$ $\int \mathbb{E}\left[-x_1^2 / 2 + \xi\, x_1\right] \mathbb{d}\{x_1\}$

$Out[\bullet]=$

$$\frac{\mathbb{E}\left[\frac{\xi^2}{2}\right]}{\sqrt{2\,\pi}}$$

$In[\bullet]:=$ $\int \mathbb{E}\left[-\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\}\right] \mathbb{d}\{x_1, x_2\}$

$Out[\bullet]=$

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2 - 2\,b\,\xi_1\,\xi_2 + a\,\xi_2^2}{2\,(-b^2 + a\,c)}\right]}{2\,\sqrt{-b^2 + a\,c}\,\pi}$$

$In[\bullet]:=$ $\mathbf{I1} = \int \mathbb{E}\left[-\frac{1}{2}\{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\}\right] \mathbb{d}\{x_1\}$

$Out[\bullet]=$

$$\frac{\mathbb{E}\left[-\frac{(-b^2 + a\,c)\,x_2^2}{2\,a} + \frac{\xi_1^2}{2\,a} + \frac{x_2\,(-b\,\xi_1 + a\,\xi_2)}{a}\right]}{\sqrt{a}\,\sqrt{2\,\pi}}$$

$In[\bullet]:=$ $\int \mathbf{I1}\, \mathbb{d}\{x_2\}$

$Out[\bullet]=$

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2 - 2\,b\,\xi_1\,\xi_2 + a\,\xi_2^2}{2\,(-b^2 + a\,c)}\right]}{2\,\sqrt{-b^2 + a\,c}\,\pi}$$

$In[\bullet]:=$ $\int \mathbb{E}\left[\xi\, x + \eta\, y + z\,(x - y) + x^2\right] \mathbb{d}\{x, z\}$

$Out[\bullet]=$

$$-\frac{\mathbb{i}\,\mathbb{E}\left[y\,(y + \eta + \xi)\right]}{2\,\pi}$$

Integration of ϵ-Series

$In[\bullet]:=$ $\mathbf{Block}\left[\{\$\pi = \mathbf{Normal}\left[\# + \mathbf{O}[\epsilon]^7\right]\, \&\},\right.$
$\left.\int \mathbb{E}\left[-x^2 / 2 + \epsilon\, x^3 / 6\right] \mathbb{d}\{x\}\right]$

$Out[\bullet]=$

$$\frac{\mathbb{E}\left[\frac{5\,\epsilon^2}{24} + \frac{5\,\epsilon^4}{16} + \frac{1105\,\epsilon^6}{1152}\right]}{\sqrt{2\,\pi}}$$

```mathematica
In[ ]:= Block[{$π = Normal[# + O[ϵ]^7] &},
          ∫ 𝔼[-ϕ^2 / 2 + ϵ ϕ^4 / 24] d{ϕ}]
```

Out[ ]=

$$\frac{\mathbb{E}\left[\frac{\epsilon}{8} + \frac{\epsilon^2}{12} + \frac{11\,\epsilon^3}{96} + \frac{17\,\epsilon^4}{72} + \frac{619\,\epsilon^5}{960} + \frac{709\,\epsilon^6}{324}\right]}{\sqrt{2\,\pi}}$$

```mathematica
In[ ]:= Block[{$π = Normal[# + O[ϵ]^5] &},
          ∫ 𝔼[p x + ϵ p^2 x] d{p, x}]
```

Out[ ]=

$$-\frac{i\,\mathbb{E}[0]}{2\,\pi}$$

```mathematica
In[ ]:= Block[{$π = Total@Select[MonomialList[#, {ϵ, x, p}],
                mon ↦ And[
                  Exponent[mon, ϵ] ≤ 2,
                  Exponent[mon, x] == Exponent[mon, p]
                ]
              ] &},
          ∫ 𝔼[p x + a x^2 p + ϵ b x^3 p^3] d{p, x}]
```

Out[ ]=

$$-\frac{i\,\mathbb{E}\left[-6\,b\,\epsilon + 342\,b^2\,\epsilon^2\right]}{2\,\pi}$$

```mathematica
In[ ]:= Block[{$π = Total@Select[MonomialList[#, {ϵ, x, p}],
                mon ↦ And[
                  Exponent[mon, ϵ] < 4,
                  Exponent[mon, x] - Exponent[mon, p] ≤ 3
                ]
              ] &},
          ∫ 𝔼[p x + a x^2 p + ϵ b p^2 x] d{p, x}]
```

Out[ ]=

$$-\frac{i\,\mathbb{E}\left[-6\,a\,b\,\epsilon + 162\,a^2\,b^2\,\epsilon^2 - 9072\,a^3\,b^3\,\epsilon^3\right]}{2\,\pi}$$

```mathematica
In[ ]:= Block[{$π = Total@Select[MonomialList[#, {ϵ, x, p}],
            mon ↦ And[
               Exponent[mon, ϵ] < 4,
               Exponent[mon, x] - Exponent[mon, p] ≤ 3 - Exponent[mon, ϵ]
            ]
         ] &},
      ∫ 𝔼[p x + a x² p + ϵ b p² x] ⅆ{p, x}]
```

$$Out[ ]= -\frac{i\ \mathbb{E}\left[-6\ a\ b\ \epsilon + 162\ a^2\ b^2\ \epsilon^2 - 9072\ a^3\ b^3\ \epsilon^3\right]}{2\ \pi}$$

```mathematica
In[ ]:= MatrixForm@Table[
      ∫ 𝔼[x₁ p₂ + x₂ p₃ + x₃ p₁ + ξᵢ xᵢ + πⱼ pⱼ] ⅆ{x₁, x₂, x₃, p₁, p₂, p₃},
      {i, 3}, {j, 3}]
```

$$Out[ ]//MatrixForm=
\begin{pmatrix}
-\frac{i\ \mathbb{E}[0]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[-\pi_2\ \xi_1]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[0]}{8\ \pi^3} \\
-\frac{i\ \mathbb{E}[0]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[0]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[-\pi_3\ \xi_2]}{8\ \pi^3} \\
-\frac{i\ \mathbb{E}[-\pi_1\ \xi_3]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[0]}{8\ \pi^3} & -\frac{i\ \mathbb{E}[0]}{8\ \pi^3}
\end{pmatrix}$$

The ρ_1 Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

```mathematica
In[ ]:= q[s_, i_, j_] := xᵢ ((1 - Tˢ) pᵢ₊₁ + (Tˢ - 1) pⱼ₊₁);
     r₁[s_, i_, j_] :=
        s (-1 + 2 pᵢ xᵢ - 2 pⱼ xᵢ + (Tˢ - 1) pᵢ pⱼ xᵢ² + (1 - Tˢ) pⱼ² xᵢ² - 2 pᵢ pⱼ xᵢ xⱼ + 2 pⱼ² xᵢ xⱼ) / 2;
     γ₁[φ_, k_] := ϵ φ (1/2 - xₖ pₖ);
     ρ1i[s_, i_, j_] := Tˢ/² 𝔼[-q[s, i, j] + ϵ r₁[s, i, j]];
     ρ1i[φ_, k_] := Tᵠ/² 𝔼[-xₖ (pₖ - pₖ₊₁) + γ₁[φ, k]];
     ρ1i[End, k_] := 𝔼[-xₖ pₖ];
     ρ1i[K_] := Module[{Cs, φ, n, c, k, ℰ},
        {Cs, φ} = Rot[K]; n = Length[Cs];
        ℰ = ρ1i[End, 2 n + 1];
        Do[ℰ *= ρ1i @@ c, {c, Cs}];
        Do[ℰ *= ρ1i[φ[[k]], k], {k, 2 n}];
        CF@ℰ
     ];
     ρ1vs[K_] := Union @@ Table[{xᵢ, pᵢ}, {i, 2 Crossings[K] + 1}]
```

In[ ]:= ρ1i[Knot[3, 1]]

Out[ ]=

$$T^2 \, \mathbb{E}\Big[-\big(\,(p_1 - p_2)\, x_1\big) - (p_2 - p_3)\, x_2 - \Big(\Big(1 - \frac{1}{T}\Big)\, p_3 + \Big(-1 + \frac{1}{T}\Big)\, p_6\Big)\, x_2 -$$

$$(p_3 - p_4)\, x_3 - (p_4 - p_5)\, x_4 - \Big(\Big(-1 + \frac{1}{T}\Big)\, p_2 + \Big(1 - \frac{1}{T}\Big)\, p_5\Big)\, x_4 - \epsilon\Big(\frac{1}{2} - p_4\, x_4\Big) +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 + 2\, p_1\, x_4 - 2\, p_4\, x_4 - 2\, p_1^2\, x_1\, x_4 + 2\, p_1\, p_4\, x_1\, x_4 - \Big(1 - \frac{1}{T}\Big)\, p_1^2\, x_4^2 - \Big(-1 + \frac{1}{T}\Big)\, p_1\, p_4\, x_4^2\Big) - (p_5 - p_6)\, x_5 +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 - 2\, p_2\, x_2 + 2\, p_5\, x_2 - \Big(-1 + \frac{1}{T}\Big)\, p_2\, p_5\, x_2^2 - \Big(1 - \frac{1}{T}\Big)\, p_5^2\, x_2^2 + 2\, p_2\, p_5\, x_2\, x_5 - 2\, p_5^2\, x_2\, x_5\Big) -$$

$$(p_6 - p_7)\, x_6 - \Big(\Big(-1 + \frac{1}{T}\Big)\, p_4 + \Big(1 - \frac{1}{T}\Big)\, p_7\Big)\, x_6 +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 + 2\, p_3\, x_6 - 2\, p_6\, x_6 - 2\, p_3^2\, x_3\, x_6 + 2\, p_3\, p_6\, x_3\, x_6 - \Big(1 - \frac{1}{T}\Big)\, p_3^2\, x_6^2 - \Big(-1 + \frac{1}{T}\Big)\, p_3\, p_6\, x_6^2\Big) - p_7\, x_7\Big]$$

In[ ]:= ρ1vs[Knot[3, 1]]

Out[ ]=

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

In[ ]:=
```
K = Knot[5, 2];
Block[{$π = Normal[# + O[ϵ]^2] &},
 ρ1i[K]]
```

Out[ ]=

$$T^3 \, \mathbb{E}\Big[-\big(\,(p_1 - p_2)\, x_1\big) - (p_2 - p_3)\, x_2 - \Big(\Big(1 - \frac{1}{T}\Big)\, p_3 + \Big(-1 + \frac{1}{T}\Big)\, p_8\Big)\, x_2 -$$

$$(p_3 - p_4)\, x_3 - (p_4 - p_5)\, x_4 - \Big(\Big(-1 + \frac{1}{T}\Big)\, p_2 + \Big(1 - \frac{1}{T}\Big)\, p_5\Big)\, x_4 - \epsilon\Big(\frac{1}{2} - p_4\, x_4\Big) +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 + 2\, p_1\, x_4 - 2\, p_4\, x_4 - 2\, p_1^2\, x_1\, x_4 + 2\, p_1\, p_4\, x_1\, x_4 - \Big(1 - \frac{1}{T}\Big)\, p_1^2\, x_4^2 - \Big(-1 + \frac{1}{T}\Big)\, p_1\, p_4\, x_4^2\Big) -$$

$$(p_5 - p_6)\, x_5 - (p_6 - p_7)\, x_6 - \Big(\Big(1 - \frac{1}{T}\Big)\, p_7 + \Big(-1 + \frac{1}{T}\Big)\, p_{10}\Big)\, x_6 - (p_7 - p_8)\, x_7 +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 - 2\, p_2\, x_2 + 2\, p_7\, x_2 - \Big(-1 + \frac{1}{T}\Big)\, p_2\, p_7\, x_2^2 - \Big(1 - \frac{1}{T}\Big)\, p_7^2\, x_2^2 + 2\, p_2\, p_7\, x_2\, x_7 - 2\, p_7^2\, x_2\, x_7\Big) -$$

$$(p_8 - p_9)\, x_8 - \Big(\Big(-1 + \frac{1}{T}\Big)\, p_4 + \Big(1 - \frac{1}{T}\Big)\, p_9\Big)\, x_8 +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 + 2\, p_3\, x_8 - 2\, p_8\, x_8 - 2\, p_3^2\, x_3\, x_8 + 2\, p_3\, p_8\, x_3\, x_8 - \Big(1 - \frac{1}{T}\Big)\, p_3^2\, x_8^2 - \Big(-1 + \frac{1}{T}\Big)\, p_3\, p_8\, x_8^2\Big) -$$

$$(p_9 - p_{10})\, x_9 + \epsilon\Big(\frac{1}{2} - p_9\, x_9\Big) +$$

$$\frac{1}{2}\,\epsilon\, \Big(1 - 2\, p_6\, x_6 + 2\, p_9\, x_6 - \Big(-1 + \frac{1}{T}\Big)\, p_6\, p_9\, x_6^2 - \Big(1 - \frac{1}{T}\Big)\, p_9^2\, x_6^2 + 2\, p_6\, p_9\, x_6\, x_9 - 2\, p_9^2\, x_6\, x_9\Big) -$$

$$(p_{10} - p_{11})\, x_{10} - \Big(\Big(-1 + \frac{1}{T}\Big)\, p_6 + \Big(1 - \frac{1}{T}\Big)\, p_{11}\Big)\, x_{10} - \epsilon\Big(\frac{1}{2} - p_{10}\, x_{10}\Big) + \frac{1}{2}\,\epsilon$$

$$\Big(1 + 2\, p_5\, x_{10} - 2\, p_{10}\, x_{10} - 2\, p_5^2\, x_5\, x_{10} + 2\, p_5\, p_{10}\, x_5\, x_{10} - \Big(1 - \frac{1}{T}\Big)\, p_5^2\, x_{10}^2 - \Big(-1 + \frac{1}{T}\Big)\, p_5\, p_{10}\, x_{10}^2\Big) - p_{11}\, x_{11}\Big]$$

$In[\circ]:=$ **K = Knot[5, 2];**

Block$\left[\{\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right] \&\},\right.$

$\left.\int \rho 1i[K] \, dl \, (\rho 1vs@K)\right]$

$Out[\circ]=$

$$-\frac{i\, T^7 \, \mathbb{E}\left[\frac{(-1+T)^2 \, (5-4\, T+5\, T^2)\, \epsilon}{(2-3\, T+2\, T^2)^2}\right]}{2048\, \pi^{11}\, (2-3\, T+2\, T^2)}$$

$In[\circ]:=$ **K = Knot[8, 19];**

Block$\left[\{\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right] \&\},\right.$

$\left.\int \rho 1i@K \, dl \, (\rho 1vs@K)\right]$

$Out[\circ]=$

$$-\frac{i\, \mathbb{E}\left[-\frac{(-1+T)^2 \, (1+T^4) \, (3+4\, T^3+3\, T^6)\, \epsilon}{(1-T+T^2)^2 \, (1-T^2+T^4)^2}\right]}{131\,072\, \pi^{17}\, T^5 \, (1-T+T^3-T^5+T^6)}$$

Concatenating edges

$In[\circ]:=$ **Block$\left[\{\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right] \&\},\right.$**

$\left. \text{lhs} = \int (\mathbb{E}[\pi_i \, p_i] \, \rho 1i[\varphi 1, i] \, \rho 1i[\varphi 2, i+1]) \, dl\{x_i, p_i, x_{i+1}, p_{i+1}\}\right]$

$Out[\circ]=$

$$\frac{T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}} \, \mathbb{E}\left[\frac{1}{2} \, \epsilon \, (-\varphi 1 - \varphi 2) + p_{2+i} \, \pi_i - \epsilon \, (\varphi 1 + \varphi 2) \, p_{2+i} \, \pi_i\right]}{4\, \pi^2}$$

$In[\circ]:=$ **Block$\left[\{\$\pi = \text{Normal}\left[\# + O[\epsilon]^2\right] \&\},\right.$**

$\left. \text{lhs} = \int (\mathbb{E}[\pi_i \, p_i] \, \rho 1i[\varphi 1 + \varphi 2, i]) \, dl\{x_i, p_i\}\right]$

$Out[\circ]=$

$$-\frac{i\, T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}} \, \mathbb{E}\left[\frac{1}{2} \, \epsilon \, (-\varphi 1 - \varphi 2) + p_{1+i} \, \pi_i - \epsilon \, (\varphi 1 + \varphi 2) \, p_{1+i} \, \pi_i\right]}{2\, \pi}$$

Invariance Under Reidemeister 3b

In[ ]:= `Block[{$π = Normal[# + O[ε]²] &},`

  `lhs = ∫ (𝔼[π_i p_i + π_j p_j + π_k p_k] ρ1i[1, i, j] ρ1i[1, i + 1, k] ρ1i[1, j + 1, k + 1]`

     `ρ1i[0, i] ρ1i[0, j] ρ1i[0, k] ρ1i[0, i + 1] ρ1i[0, j + 1] ρ1i[0, k + 1])`

    `ⅆ{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}}]`

Out[ ]=

$$\frac{1}{64\,\pi^6}$$

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2} + T^2\,p_{2+i}\,\pi_i + \frac{1}{2}\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$

$$T\,\epsilon\,p_{2+j}\,(T\,\pi_i - \pi_j) - T\,p_{2+j}\,(-\pi_i + T\,\pi_i - \pi_j) - \frac{1}{2}\,T\,\epsilon\,p_{2+k}^2\,(\pi_i + \pi_j)\,(-\pi_i + T\,\pi_i - \pi_j + T\,\pi_j - 2\,\pi_k) +$$

$$\frac{1}{2}\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j + 2\,T\,\pi_j - 2\,\pi_k) +$$

$$\epsilon\,p_{2+k}\,(T\,\pi_i - \pi_j + 2\,T\,\pi_j - 2\,\pi_k) + p_{2+k}\,(\pi_i - T\,\pi_i + \pi_j - T\,\pi_j + \pi_k) - \frac{1}{2}\,T\,\epsilon\,p_{2+j}\,p_{2+k}$$

$$\Big(\pi_i^2 - 2\,T\,\pi_i^2 + T^2\,\pi_i^2 + 2\,\pi_i\,\pi_j - 4\,T\,\pi_i\,\pi_j + 2\,T^2\,\pi_i\,\pi_j + \pi_j^2 - T\,\pi_j^2 + 2\,\pi_i\,\pi_k - 2\,T\,\pi_i\,\pi_k + 2\,\pi_j\,\pi_k\Big)\Big]$$

In[ ]:= `Block[{$π = Normal[# + O[ε]²] &},`

  `rhs = ∫ (𝔼[π_i p_i + π_j p_j + π_k p_k] ρ1i[1, j, k] ρ1i[1, i, k + 1] ρ1i[1, i + 1, j + 1]`

     `ρ1i[0, i] ρ1i[0, j] ρ1i[0, k] ρ1i[0, i + 1] ρ1i[0, j + 1] ρ1i[0, k + 1])`

    `ⅆ{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}}]`

Out[ ]=

$$\frac{1}{64\,\pi^6}$$

$$T^{3/2}\,\mathbb{E}\Big[-\frac{3\,\epsilon}{2} + T^2\,p_{2+i}\,\pi_i + \frac{1}{2}\,T^3\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T^3\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$

$$T\,\epsilon\,p_{2+j}\,(T\,\pi_i - \pi_j) - T\,p_{2+j}\,(-\pi_i + T\,\pi_i - \pi_j) - \frac{1}{2}\,T\,\epsilon\,p_{2+k}^2\,(\pi_i + \pi_j)\,(-\pi_i + T\,\pi_i - \pi_j + T\,\pi_j - 2\,\pi_k) +$$

$$\frac{1}{2}\,T^2\,\epsilon\,p_{2+i}\,p_{2+k}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j + 2\,T\,\pi_j - 2\,\pi_k) +$$

$$\epsilon\,p_{2+k}\,(T\,\pi_i - \pi_j + 2\,T\,\pi_j - 2\,\pi_k) + p_{2+k}\,(\pi_i - T\,\pi_i + \pi_j - T\,\pi_j + \pi_k) - \frac{1}{2}\,T\,\epsilon\,p_{2+j}\,p_{2+k}$$

$$\Big(\pi_i^2 - 2\,T\,\pi_i^2 + T^2\,\pi_i^2 + 2\,\pi_i\,\pi_j - 4\,T\,\pi_i\,\pi_j + 2\,T^2\,\pi_i\,\pi_j + \pi_j^2 - T\,\pi_j^2 + 2\,\pi_i\,\pi_k - 2\,T\,\pi_i\,\pi_k + 2\,\pi_j\,\pi_k\Big)\Big]$$

In[ ]:= `lhs == rhs`

Out[ ]=

`True`

Invariance Under Reidemeister 2b

$In[\bullet]:=$ `Block[{$\pi = Normal[# + O[\epsilon]^2] &},`

`lhs =`

$\int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \xi_{i+1}\, x_{i+1} + \xi_{j+1}\, x_{j+1}]\, \rho 1i[1, i, j]\, \rho 1i[-1, i+1, j+1]\, \rho 1i[0, i]\, \rho 1i[0, j]$

$\qquad \rho 1i[0, i+1]\, \rho 1i[0, j+1])\, d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}]$

$Out[\bullet]=$

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j + T\,\pi_i\,\xi_{1+i} + \pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j}]}{16\,\pi^4}$$

$In[\bullet]:=$ `Block[{$\pi = Normal[# + O[\epsilon]^2] &},`

`rhs =` $\int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \xi_{i+1}\, x_{i+1} + \xi_{j+1}\, x_{j+1}]\, \rho 1i[0, i]\, \rho 1i[0, j]\, \rho 1i[0, i+1]\, \rho 1i[0, j+1])$

$\qquad d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}]$

$Out[\bullet]=$

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j + \pi_i\,\xi_{1+i} + \pi_j\,\xi_{1+j}]}{16\,\pi^4}$$

$In[\bullet]:=$ `lhs == rhs`

$Out[\bullet]=$

$$\frac{\mathbb{E}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j + T\,\pi_i\,\xi_{1+i} + \pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j}]}{16\,\pi^4} == \frac{\mathbb{E}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j + \pi_i\,\xi_{1+i} + \pi_j\,\xi_{1+j}]}{16\,\pi^4}$$

Invariance Under R2c
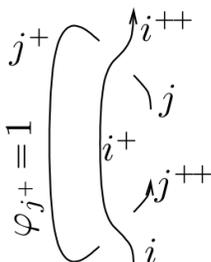
$In[\bullet]:=$ `Block[{$\pi = Normal[# + O[\epsilon]^2] &},`

`lhs =` $\int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\, \rho 1i[-1, i, j+1]\, \rho 1i[1, i+1, j]\, \rho 1i[0, i]$

$\qquad \rho 1i[0, j]\, \rho 1i[0, i+1]\, \rho 1i[1, j+1])\, d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}]$

$Out[\bullet]=$

$$\frac{\sqrt{T}\, \mathbb{E}\left[-\frac{\epsilon}{2} + p_{2+i}\,\pi_i + p_{2+j}\,\pi_j - \epsilon\, p_{2+j}\,\pi_j\right]}{16\,\pi^4}$$

$In[\bullet]:=$ `Block[ {$\pi = Normal[ # + O[\epsilon]^2] &},`

$\quad$ `rhs = ` $\int (\mathbb{E} [\pi_i \, p_i + \pi_j \, p_j] \, \rho 1i[0, i] \, \rho 1i[0, j] \, \rho 1i[0, i + 1] \, \rho 1i[1, j + 1])$

$\qquad$ `𝕕` $\{x_i, \, x_j, \, p_i, \, p_j, \, x_{i+1}, \, x_{j+1}, \, p_{i+1}, \, p_{j+1}\}]$

$Out[\bullet]=$

$$\frac{\sqrt{T} \;\; \mathbb{E}\left[-\frac{\epsilon}{2} + p_{2+i} \, \pi_i + p_{2+j} \, \pi_j - \epsilon \, p_{2+j} \, \pi_j \right]}{16 \, \pi^4}$$

$In[\bullet]:=$ `lhs == rhs`

$Out[\bullet]=$

$\quad$ True

Invariance Under R1l
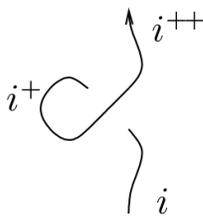
$In[\bullet]:=$ `Block[ {$\pi = Normal[ # + O[\epsilon]^2] &},`

$\quad$ `lhs = ` $\int (\mathbb{E} [\pi_i \, p_i] \, \rho 1i[1, i + 1, i] \, \rho 1i[0, i] \, \rho 1i[1, i + 1])$ `𝕕` $\{x_i, \, p_i, \, x_{i+1}, \, p_{i+1}\}]$

$Out[\bullet]=$

$$\frac{\mathbb{E} [p_{2+i} \, \pi_i]}{4 \, \pi^2}$$

$In[\bullet]:=$ `Block[ {$\pi = Normal[ # + O[\epsilon]^2] &},`

$\quad$ `rhs = ` $\int (\mathbb{E} [\pi_i \, p_i] \, \rho 1i[0, i] \, \rho 1i[0, i + 1])$ `𝕕` $\{x_i, \, p_i, \, x_{i+1}, \, p_{i+1}\}]$

$Out[\bullet]=$

$$\frac{\mathbb{E} [p_{2+i} \, \pi_i]}{4 \, \pi^2}$$

$In[\bullet]:=$ `lhs == rhs`

$Out[\bullet]=$

$\quad$ True

Invariance Under R1r
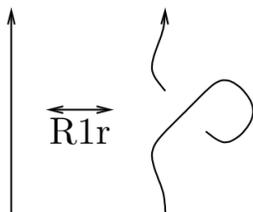
$\textit{In[\circ]:=}$ **Block** $\Big[\big\{ \$\pi = \text{Normal} \big[\# + O[\epsilon]^2 \big] \& \big\},$

   lhs $= \int (\mathbb{E}[\pi_i \, p_i] \, \rho\mathbf{1i}[1, i, i+1] \, \rho\mathbf{1i}[0, i] \, \rho\mathbf{1i}[-1, i+1]) \, \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\} \Big]$

$\textit{Out[\circ]=}$

   $\dfrac{\mathbb{E}[p_{2+i} \, \pi_i]}{4 \, \pi^2}$

$\textit{In[\circ]:=}$ **Block** $\Big[\big\{ \$\pi = \text{Normal} \big[\# + O[\epsilon]^2 \big] \& \big\},$

   rhs $= \int (\mathbb{E}[\pi_i \, p_i] \, \rho\mathbf{1i}[0, i] \, \rho\mathbf{1i}[0, i+1]) \, \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\} \Big]$

$\textit{Out[\circ]=}$

   $\dfrac{\mathbb{E}[p_{2+i} \, \pi_i]}{4 \, \pi^2}$

$\textit{In[\circ]:=}$ **lhs == rhs**

$\textit{Out[\circ]=}$

   True

Invariance Under Sw
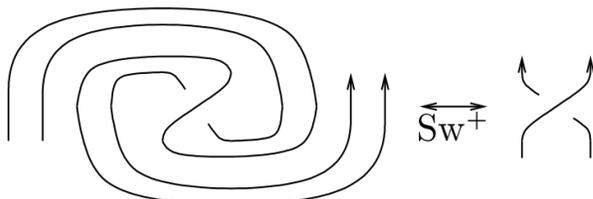
$\textit{In[\circ]:=}$ **Block** $\Big[\big\{ \$\pi = \text{Normal} \big[\# + O[\epsilon]^2 \big] \& \big\},$

   lhs $= \int (\mathbb{E}[\pi_i \, p_i + \pi_j \, p_j] \, \rho\mathbf{1i}[1, i, j] \, \rho\mathbf{1i}[-1, i] \, \rho\mathbf{1i}[1, i+1] \, \rho\mathbf{1i}[-1, j] \, \rho\mathbf{1i}[1, j+1])$

   $\mathbb{d}\{x_i, x_j, p_i, p_j\} \Big]$

$\textit{Out[\circ]=}$

$\dfrac{1}{4 \, \pi^2} \, \sqrt{T} \, \mathbb{E} \Big[\dfrac{3 \, \epsilon}{2} + T \, p_{1+i} \, \pi_i + T \, \epsilon \, p_{1+i} \, \pi_i + \epsilon \, p_{1+j} \, \pi_i +$

$\dfrac{1}{2} \, T \, \epsilon \, p_{1+i} \, p_{1+j} \, \pi_i \, (-\pi_i + T \, \pi_i - 2 \, \pi_j) - \dfrac{1}{2} \, T \, \epsilon \, p_{1+j}^2 \, \pi_i \, (-\pi_i + T \, \pi_i - 2 \, \pi_j) +$

$p_{1+j} \, (\pi_i - T \, \pi_i + \pi_j) - p_{1+i} \, x_{1+i} - \epsilon \, p_{1+i} \, x_{1+i} + p_{2+i} \, x_{1+i} - p_{1+j} \, x_{1+j} - \epsilon \, p_{1+j} \, x_{1+j} + p_{2+j} \, x_{1+j} \Big]$

$In[\circ]:=$ `Block[{$\pi = Normal[# + O[\epsilon]^2] &},`

    `rhs = ∫(𝔼[π_i p_i + π_j p_j] ρ1i[1, i, j] ρ1i[0, i] ρ1i[0, i + 1] ρ1i[0, j] ρ1i[0, j + 1])`

       `𝕕{x_i, x_j, p_i, p_j}]`

$Out[\circ]=$

$$\frac{1}{4\pi^2}\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2} + T\,p_{1+i}\,\pi_i + \frac{1}{2}\,T\,\epsilon\,p_{1+i}\,p_{1+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T\,\epsilon\,p_{1+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$

$$\epsilon\,p_{1+j}\,(T\,\pi_i - \pi_j) + p_{1+j}\,(\pi_i - T\,\pi_i + \pi_j) - p_{1+i}\,x_{1+i} + p_{2+i}\,x_{1+i} - p_{1+j}\,x_{1+j} + p_{2+j}\,x_{1+j}\Big]$$

$In[\circ]:=$ **lhs == rhs**

$Out[\circ]=$

$$\frac{1}{4\pi^2}\sqrt{T}\,\mathbb{E}\Big[\frac{3\,\epsilon}{2} + T\,p_{1+i}\,\pi_i + T\,\epsilon\,p_{1+i}\,\pi_i + \epsilon\,p_{1+j}\,\pi_i +$$

$$\frac{1}{2}\,T\,\epsilon\,p_{1+i}\,p_{1+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T\,\epsilon\,p_{1+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{1+j}\,(\pi_i - T\,\pi_i + \pi_j) -$$

$$p_{1+i}\,x_{1+i} - \epsilon\,p_{1+i}\,x_{1+i} + p_{2+i}\,x_{1+i} - p_{1+j}\,x_{1+j} - \epsilon\,p_{1+j}\,x_{1+j} + p_{2+j}\,x_{1+j}\Big] == \frac{1}{4\pi^2}$$

$$\sqrt{T}\,\mathbb{E}\Big[-\frac{\epsilon}{2} + T\,p_{1+i}\,\pi_i + \frac{1}{2}\,T\,\epsilon\,p_{1+i}\,p_{1+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) - \frac{1}{2}\,T\,\epsilon\,p_{1+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) +$$

$$\epsilon\,p_{1+j}\,(T\,\pi_i - \pi_j) + p_{1+j}\,(\pi_i - T\,\pi_i + \pi_j) - p_{1+i}\,x_{1+i} + p_{2+i}\,x_{1+i} - p_{1+j}\,x_{1+j} + p_{2+j}\,x_{1+j}\Big]$$

$In[\circ]:=$ `Block[{$\pi = Normal[# + O[\epsilon]^2] &},`

    `lhs = ∫(𝔼[π_i p_i + π_j p_j + π_{i+1} p_{i+1} + π_{j+1} p_{j+1} + ξ_{i+1} x_{i+1} + ξ_{j+1} x_{j+1}] ρ1i[1, i, j] ρ1i[-1, i]`

       `ρ1i[1, i + 1] ρ1i[-1, j] ρ1i[1, j + 1]) 𝕕{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}]`

$Out[\circ]=$

$$\frac{1}{16\pi^4}\sqrt{T}\,\mathbb{E}\Big[p_{2+i}\,(T\,\pi_i + \pi_{1+i}) + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) -$$

$$\frac{1}{2}\,T\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j + \pi_{1+j}) + T\,\pi_i\,\xi_{1+i} + \pi_{1+i}\,\xi_{1+i} + \pi_i\,\xi_{1+j} -$$

$$T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j} + \pi_{1+j}\,\xi_{1+j} + \frac{1}{2}\,\epsilon\,p_{2+i}\,\left(-2\,\pi_{1+i} - T\,\pi_i^2\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) +$$

$$\frac{1}{2}\,\epsilon\,p_{2+j}\,\left(2\,T\,\pi_i - 2\,\pi_j - 2\,\pi_{1+j} - T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} -$$

$$2\,T^2\,\pi_i^2\,\xi_{1+j} + 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) + \frac{1}{2}\,\epsilon\,\left(-1 - 2\,\pi_{1+i}\,\xi_{1+i} + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} - 2\,\pi_{1+j}\,\xi_{1+j} -$$

$$T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\right)\Big]$$

$In[\bullet]:=$ `Block` $\Big[\{\$\pi = \text{Normal}[\# + O[\epsilon]^2] \&\},$

     `rhs =`

$$\int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_{i+1}\, p_{i+1} + \pi_{j+1}\, p_{j+1} + \xi_{i+1}\, x_{i+1} + \xi_{j+1}\, x_{j+1}]\, \rho 1i[1, i, j]\, \rho 1i[0, i]\, \rho 1i[0, i+1]$$

$$\rho 1i[0, j]\, \rho 1i[0, j+1])\, \mathbb{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}\Big]$$

$Out[\bullet]=$

$$\frac{1}{16\,\pi^4}\,\sqrt{T}\,\mathbb{E}\Big[p_{2+i}\,(T\,\pi_i + \pi_{1+i}) + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) -$$

$$\frac{1}{2}\,T\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j + \pi_{1+j}) + T\,\pi_i\,\xi_{1+i} + \pi_{1+i}\,\xi_{1+i} +$$

$$\pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j)\,\xi_{1+j} + \pi_j\,\xi_{1+j} + \pi_{1+j}\,\xi_{1+j} + \frac{1}{2}\,\epsilon\,p_{2+j}$$

$$\left(2\,T\,\pi_i - 2\,\pi_j - T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} - 2\,T^2\,\pi_i^2\,\xi_{1+j} + 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) +$$

$$\frac{1}{2}\,\epsilon\,\left(-1 + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} - T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} -\right.$$

$$\left. 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\right)\Big]$$

$In[\bullet]:=$ `lhs == rhs`

$Out[\bullet]=$

$$\frac{1}{16\,\pi^4}\,\sqrt{T}\,\mathbb{E}\Big[p_{2+i}\,(T\,\pi_i + \pi_{1+i}) + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) -$$

$$\frac{1}{2}\,T\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j + \pi_{1+j}) + T\,\pi_i\,\xi_{1+i} + \pi_{1+i}\,\xi_{1+i} + \pi_i\,\xi_{1+j} -$$

$$T\,\pi_i\,\xi_{1+j} + \pi_j\,\xi_{1+j} + \pi_{1+j}\,\xi_{1+j} + \frac{1}{2}\,\epsilon\,p_{2+i}\,\left(-2\,\pi_{1+i} - T\,\pi_i^2\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) +$$

$$\frac{1}{2}\,\epsilon\,p_{2+j}\,\left(2\,T\,\pi_i - 2\,\pi_j - 2\,\pi_{1+j} - T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} - 2\,T^2\,\pi_i^2\,\xi_{1+j} +\right.$$

$$\left. 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) + \frac{1}{2}\,\epsilon\,\left(-1 - 2\,\pi_{1+i}\,\xi_{1+i} + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} - 2\,\pi_{1+j}\,\xi_{1+j} - T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} +\right.$$

$$\left. T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\right)\Big] ==$$

$$\frac{1}{16\,\pi^4}\,\sqrt{T}\,\mathbb{E}\Big[p_{2+i}\,(T\,\pi_i + \pi_{1+i}) + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,p_{2+j}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) -$$

$$\frac{1}{2}\,T\,\epsilon\,p_{2+j}^2\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j) + p_{2+j}\,(\pi_i - T\,\pi_i + \pi_j + \pi_{1+j}) + T\,\pi_i\,\xi_{1+i} + \pi_{1+i}\,\xi_{1+i} +$$

$$\pi_i\,\xi_{1+j} - T\,\pi_i\,\xi_{1+j} + \frac{1}{2}\,T\,\epsilon\,p_{2+i}\,\pi_i\,(-\pi_i + T\,\pi_i - 2\,\pi_j)\,\xi_{1+j} + \pi_j\,\xi_{1+j} + \pi_{1+j}\,\xi_{1+j} + \frac{1}{2}\,\epsilon\,p_{2+j}$$

$$\left(2\,T\,\pi_i - 2\,\pi_j - T\,\pi_i^2\,\xi_{1+i} + T^2\,\pi_i^2\,\xi_{1+i} - 2\,T\,\pi_i\,\pi_j\,\xi_{1+i} + 2\,T\,\pi_i^2\,\xi_{1+j} - 2\,T^2\,\pi_i^2\,\xi_{1+j} + 4\,T\,\pi_i\,\pi_j\,\xi_{1+j}\right) +$$

$$\frac{1}{2}\,\epsilon\,\left(-1 + 2\,T\,\pi_i\,\xi_{1+j} - 2\,\pi_j\,\xi_{1+j} - T\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} + T^2\,\pi_i^2\,\xi_{1+i}\,\xi_{1+j} -\right.$$

$$\left. 2\,T\,\pi_i\,\pi_j\,\xi_{1+i}\,\xi_{1+j} + T\,\pi_i^2\,\xi_{1+j}^2 - T^2\,\pi_i^2\,\xi_{1+j}^2 + 2\,T\,\pi_i\,\pi_j\,\xi_{1+j}^2\right)\Big]$$