

Pensieve header: A first implementation of nilpotent integration.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/icbs24> to compute rotation numbers.

```
In[*]:= CCF[ε_] := ExpandDenominator@ExpandNumerator@Together[ε];
CCF[ε_] := Factor[ε];
CF[ε_List] := CF/@ε;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := Module[{vs = Cases[ε, (x | p)_, ∞] ∪ {x, p}, ps, c},
  Total[CoefficientRules[Expand[ε], vs] /. (ps_ -> c_) -> CCF[c] (Times @@ vs^ps)]];
```

Integration

```
In[*]:= Unprotect[Integrate];
∫ ω_. E[L_] d[vs_List] := Module[{n, Q, G, V, s, t, k, a, b},
  n = Length@vs;
  Q = -Table[(∂_{vs[[a]], vs[[b]]} L) /. Thread[vs -> 0], {a, n}, {b, n}];
  G = Inverse[Q] / 2;
  V = L + vs.Q.vs / 2;
  s = t = V; k = 0;
  While[θ != t,
    s +=
      1 / ((++k)!) (t = CF@Sum[G[[a, b]] ((∂_{vs[[a]], vs[[b]]} t) + (∂_{vs[[a]]} t) (∂_{vs[[b]]} t)), {a, n}, {b, n}]);
  PowerExpand@Factor[ω (Det[Q] (2 π)^n)^{-1/2}] E[CF@s /. Thread[vs -> 0]]
];
Protect[Integrate];
```

```
In[*]:= ∫ E[i λ x_1^2 / 2] d[x_1]
```

```
Out[*]= (-1)^{1/4} E[0] / (√(2 π) √λ)
```

$$\text{In[*]} := \int \mathbb{E} \left[-\frac{i \lambda x_1^2}{2} \right] \mathbb{d} \{x_1\}$$

$$\text{Out[*]} = \frac{(-1)^{3/4} \mathbb{E}[\theta]}{\sqrt{2\pi} \sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[\frac{i}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1, x_2\}$$

$$\text{Out[*]} = \frac{\mathbb{E}[\theta]}{2 \sqrt{b^2 - ac} \pi}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\lambda x_1^2 / 2 \right] \mathbb{d} \{x_1\}$$

$$\text{Out[*]} = \frac{\mathbb{E}[\theta]}{\sqrt{2\pi} \sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-x_1^2 / 2 + \xi x_1 \right] \mathbb{d} \{x_1\}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{\xi^2}{2} \right]}{\sqrt{2\pi}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\frac{1}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1, x_2\}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$\text{In[*]} := \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{x_1, x_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\} \right] \mathbb{d} \{x_1\}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[-\frac{(-b^2 + ac) x_2^2}{2a} + \frac{\xi_1^2}{2a} + \frac{x_2 (-b \xi_1 + a \xi_2)}{a} \right]}{\sqrt{a} \sqrt{2\pi}}$$

$$\text{In[*]} := \int \mathbf{I1} \mathbb{d} \{x_2\}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{c \xi_1^2 - 2b \xi_1 \xi_2 + a \xi_2^2}{2(-b^2 + ac)} \right]}{2 \sqrt{-b^2 + ac} \pi}$$

$$In[*]:= \int \mathbb{E} \left[-\frac{1}{2} \{y_1, y_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{y_1, y_2\} + \{\eta_1, \eta_2\} \cdot \{y_1, y_2\} \right] \mathbb{d}\{y_1, y_2\}$$

Out[*]=

$$\frac{\mathbb{E} \left[\frac{c \eta_1^2 - 2 b \eta_1 \eta_2 + a \eta_2^2}{2 (-b^2 + a c)} \right]}{2 \sqrt{-b^2 + a c} \pi}$$

$$In[*]:= \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{y_1, y_2\} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \{y_1, y_2\} + \{\eta_1, \eta_2\} \cdot \{y_1, y_2\} \right] \mathbb{d}\{y_1\}$$

Out[*]=

$$\frac{\mathbb{E} \left[\frac{b^2 y_2^2 - a c y_2^2 - 2 b y_2 \eta_1 + \eta_1^2 + 2 a y_2 \eta_2}{2 a} \right]}{\sqrt{a} \sqrt{2} \pi}$$

$$In[*]:= \int \mathbf{I1} \mathbb{d}\{y_2\}$$

Out[*]=

$$\frac{\mathbb{E} \left[\frac{a c \eta_1^2 - 2 a b \eta_1 \eta_2 + a^2 \eta_2^2}{2 a (-b^2 + a c)} \right]}{2 \sqrt{-b^2 + a c} \pi}$$

$$In[*]:= \int \mathbb{E} \left[\xi x + \eta y + z (x - y) + x^2 \right] \mathbb{d}\{x, z\}$$

Out[*]=

$$\frac{i \mathbb{E} [y (y + \eta + \xi)]}{2 \pi}$$

The ρ_1 Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

```
In[*]:= q[s_, i_, j_] := x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1});
r1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
L[s_, i_, j_] := -q[s, i, j] + e r1[s, i, j];
gamma1[phi_, k_] := e phi (1 / 2 - x_k p_k);
rho1i[K_] := Module[{Cs, phi, n, s, i, j, k, vs, L},
  {Cs, phi} = Rot[K]; n = Length[Cs];
  L = -x_{2n+1} p_{2n+1};
  Cases[Cs, {s_, i_, j_} -> (L += L[s, i, j])];
  L += e Sum[gamma1[phi[[k]], k], {k, 2 n}];
  CF@ (L + O[e]^2)
];
rho1vs[K_] := Union @@ Table[{x_i, p_i}, {i, 2 Crossings[K] + 1}]
```

In[*]:= `ρ1i[Knot[3, 1]]`

Out[*]=

$$\left(-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T) p_6 x_2}{T} - p_3 x_3 + p_4 x_3 + \frac{(-1+T) p_2 x_4}{T} - p_4 x_4 + \frac{p_5 x_4}{T} - p_5 x_5 + p_6 x_5 + \frac{(-1+T) p_4 x_6}{T} - p_6 x_6 + \frac{p_7 x_6}{T} - p_7 x_7 \right) + \left(1 - p_2 x_2 + p_5 x_2 + \frac{(-1+T) p_2 p_5 x_2^2}{2T} - \frac{(-1+T) p_5^2 x_2^2}{2T} + p_1 x_4 - p_1^2 x_1 x_4 + p_1 p_4 x_1 x_4 - \frac{(-1+T) p_1^2 x_4^2}{2T} + \frac{(-1+T) p_1 p_4 x_4^2}{2T} + p_2 p_5 x_2 x_5 - p_5^2 x_2 x_5 + p_3 x_6 - p_6 x_6 - p_3^2 x_3 x_6 + p_3 p_6 x_3 x_6 - \frac{(-1+T) p_3^2 x_6^2}{2T} + \frac{(-1+T) p_3 p_6 x_6^2}{2T} \right) \epsilon + O[\epsilon]^2$$

In[*]:= `ρ1vs[Knot[3, 1]]`

Out[*]=

{p₁, p₂, p₃, p₄, p₅, p₆, p₇, x₁, x₂, x₃, x₄, x₅, x₆, x₇}

Integration of ε-Series

In[*]:= `Unprotect[Integrate];`

```

∫ ω . E[L_SeriesData] d(vs_List) := Module[{n, L0, Q, Δ, G, V, s, t, k, a, b},
  n = Length@vs; L0 = Normal@L /. ε → 0;
  Q = -Table[(∂vs[[a]], vs[[b]] L0) /. Thread[vs → 0] /. (p | x) → 0, {a, n}, {b, n}];
  If[(Δ = CF@Det[Q]) == 0,
    Return["How dare you ask me to integrate a singular Gaussian!"];
  G = Inverse[Q] / 2;
  V = L + vs.Q.vs / 2;
  s = t = V; k = 0;
  While[0 != Normal@t,
    s +=
      1 / (k + 1) (t = CF@Sum[G[[a, b]] ((∂vs[[a]], vs[[b]] t) + (∂vs[[a]] t) (∂vs[[b]] t)), {a, n}, {b, n}]);
  PowerExpand@Factor[ω (Δ (2 π)^n)^(-1/2)] E[CF@s /. Thread[vs → 0]]
];
Protect[Integrate];

```

In[*]:= $\int \mathbb{E} [x_1 p_1 + \epsilon x_1^7 p_1^7 + O[\epsilon]^2] d\{x_1, p_1\}$

Out[*]=

$$-\frac{i \mathbb{E} [-5040 \epsilon + O[\epsilon]^2]}{2 \pi}$$

$$\text{In[*]:= } \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \epsilon \mathbf{x}_2^7 \mathbf{p}_1^7 + \mathbf{0} [\epsilon]^2] \, d\{\mathbf{x}_1, \mathbf{p}_2\}$$

$$\text{Out[*]= } \frac{i \mathbb{E} [\mathbf{p}_1^7 \mathbf{x}_2^7 \epsilon + \mathbf{0} [\epsilon]^2]}{2 \pi}$$

$$\text{In[*]:= } \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + 3 \mathbf{x}_2 \mathbf{p}_1 + \epsilon \mathbf{p}_2^5 \mathbf{x}_1^5 + \mathbf{0} [\epsilon]^2] \, d\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2\}$$

$$\text{Out[*]= } \frac{\mathbb{E} [-120 \epsilon + \mathbf{0} [\epsilon]^2]}{12 \pi^2}$$

$$\text{In[*]:= } \int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \mathbf{x}_2 \mathbf{p}_3 + \mathbf{x}_3 \mathbf{p}_1 + \epsilon \mathbf{x}_1^5 \mathbf{p}_2^5 + \mathbf{0} [\epsilon]^2] \, d\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$$

$$\text{Out[*]= } \frac{i \mathbb{E} [-120 \epsilon + \mathbf{0} [\epsilon]^2]}{8 \pi^3}$$

$$\text{In[*]:= } \text{MatrixForm@Table} \left[\int \mathbb{E} [\mathbf{x}_1 \mathbf{p}_2 + \mathbf{x}_2 \mathbf{p}_3 + \mathbf{x}_3 \mathbf{p}_1 + \xi_i \mathbf{x}_i + \pi_j \mathbf{p}_j] \, d\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}, \{i, 3\}, \{j, 3\} \right]$$

$$\text{Out[*]//MatrixForm= } \begin{pmatrix} -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} & -\frac{i \mathbb{E} [-\pi_2 \xi_1]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} \\ -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} & -\frac{i \mathbb{E} [-\pi_3 \xi_2]}{8 \pi^3} \\ -\frac{i \mathbb{E} [-\pi_1 \xi_3]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} & -\frac{i \mathbb{E} [\mathbf{0}]}{8 \pi^3} \end{pmatrix}$$

In[*]:= **K = Knot [5, 2];**

{ρ1i@K, ρ1vs@K}

∫ ℰ [ρ1i@K] d (ρ1vs@K)

Out[*]=

$$\left\{ \left(-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T) p_8 x_2}{T} - p_3 x_3 + p_4 x_3 + \frac{(-1+T) p_2 x_4}{T} - p_4 x_4 + \frac{p_5 x_4}{T} - p_5 x_5 + p_6 x_5 - p_6 x_6 + \frac{p_7 x_6}{T} + \frac{(-1+T) p_{10} x_6}{T} - p_7 x_7 + p_8 x_7 + \frac{(-1+T) p_4 x_8}{T} - p_8 x_8 + \frac{p_9 x_8}{T} - p_9 x_9 + p_{10} x_9 + \frac{(-1+T) p_6 x_{10}}{T} - p_{10} x_{10} + \frac{p_{11} x_{10}}{T} - p_{11} x_{11} \right) + \right. \\ \left. \left(2 - p_2 x_2 + p_7 x_2 + \frac{(-1+T) p_2 p_7 x_2^2}{2T} - \frac{(-1+T) p_7^2 x_2^2}{2T} + p_1 x_4 - p_1^2 x_1 x_4 + p_1 p_4 x_1 x_4 - \frac{(-1+T) p_1^2 x_4^2}{2T} + \frac{(-1+T) p_1 p_4 x_4^2}{2T} - p_6 x_6 + p_9 x_6 + \frac{(-1+T) p_6 p_9 x_6^2}{2T} - \frac{(-1+T) p_9^2 x_6^2}{2T} + p_2 p_7 x_2 x_7 - p_7^2 x_2 x_7 + p_3 x_8 - p_8 x_8 - p_3^2 x_3 x_8 + p_3 p_8 x_3 x_8 - \frac{(-1+T) p_3^2 x_8^2}{2T} + \frac{(-1+T) p_3 p_8 x_8^2}{2T} - p_9 x_9 + p_6 p_9 x_6 x_9 - p_9^2 x_6 x_9 + p_5 x_{10} - p_5^2 x_5 x_{10} + p_5 p_{10} x_5 x_{10} - \frac{(-1+T) p_5^2 x_{10}^2}{2T} + \frac{(-1+T) p_5 p_{10} x_{10}^2}{2T} \right) \epsilon + O[\epsilon]^2, \right. \\ \left. \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\} \right\}$$

Out[*]=

$$\frac{i T^4 \mathbb{E} \left[\frac{(-1+T)^2 (5-4T+5T^2) \epsilon}{(2-3T+2T^2)^2} + O[\epsilon]^2 \right]}{2048 \pi^{11} (2-3T+2T^2)}$$

In[*]:= **K = Knot [8, 19];**

{ρ1i@K, ρ1vs@K}

∫ $\mathbb{E} [\rho 1 i @ K] d (\rho 1 v s @ K)$

Out[*]=

$$\left\{ \begin{aligned} & (-p_1 x_1 + T p_2 x_1 + (1 - T) p_5 x_1 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_9 x_3 - \\ & p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - p_6 x_6 + T p_7 x_6 + (1 - T) p_{14} x_6 + (1 - T) p_3 x_7 - p_7 x_7 + \\ & T p_8 x_7 - p_8 x_8 + p_9 x_8 - p_9 x_9 + p_{10} x_9 - p_{10} x_{10} + T p_{11} x_{10} + (1 - T) p_{16} x_{10} - p_{11} x_{11} + \\ & p_{12} x_{11} + (1 - T) p_6 x_{12} - p_{12} x_{12} + T p_{13} x_{12} - p_{13} x_{13} + p_{14} x_{13} + (1 - T) p_{10} x_{14} - \\ & p_{14} x_{14} + T p_{15} x_{14} - p_{15} x_{15} + p_{16} x_{15} + (1 - T) p_{12} x_{16} - p_{16} x_{16} + T p_{17} x_{16} - p_{17} x_{17}) + \\ & \left(-4 + p_1 x_1 - p_4 x_1 + \frac{1}{2} (-1 + T) p_1 p_4 x_1^2 + \frac{1}{2} (1 - T) p_4^2 x_1^2 + p_3 x_3 - p_8 x_3 + \frac{1}{2} (-1 + T) p_3 p_8 x_3^2 + \right. \\ & \frac{1}{2} (1 - T) p_8^2 x_3^2 + p_4 x_4 - p_1 p_4 x_1 x_4 + p_4^2 x_1 x_4 + p_6 x_6 - p_{13} x_6 + \frac{1}{2} (-1 + T) p_6 p_{13} x_6^2 + \\ & \frac{1}{2} (1 - T) p_{13}^2 x_6^2 - p_2 x_7 + p_7 x_7 + p_2^2 x_2 x_7 - p_2 p_7 x_2 x_7 + \frac{1}{2} (1 - T) p_2^2 x_7^2 + \frac{1}{2} (-1 + T) p_2 p_7 x_7^2 - \\ & p_3 p_8 x_3 x_8 + p_8^2 x_3 x_8 + p_{10} x_{10} - p_{15} x_{10} + \frac{1}{2} (-1 + T) p_{10} p_{15} x_{10}^2 + \frac{1}{2} (1 - T) p_{15}^2 x_{10}^2 - \\ & p_5 x_{12} + p_5^2 x_5 x_{12} - p_5 p_{12} x_5 x_{12} + \frac{1}{2} (1 - T) p_5^2 x_{12}^2 + \frac{1}{2} (-1 + T) p_5 p_{12} x_{12}^2 - \\ & p_6 p_{13} x_6 x_{13} + p_{13}^2 x_6 x_{13} - p_9 x_{14} + p_{14} x_{14} + p_9^2 x_9 x_{14} - p_9 p_{14} x_9 x_{14} + \frac{1}{2} (1 - T) p_9^2 x_{14}^2 + \\ & \frac{1}{2} (-1 + T) p_9 p_{14} x_{14}^2 - p_{10} p_{15} x_{10} x_{15} + p_{15}^2 x_{10} x_{15} - p_{11} x_{16} + p_{16} x_{16} + p_{11}^2 x_{11} x_{16} - \\ & \left. p_{11} p_{16} x_{11} x_{16} + \frac{1}{2} (1 - T) p_{11}^2 x_{16}^2 + \frac{1}{2} (-1 + T) p_{11} p_{16} x_{16}^2 \right) \in + O[\epsilon]^2, \\ & \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \\ & x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}\} \} \end{aligned} \right\}$$

Out[*]=

$$\frac{i \mathbb{E} \left[-\frac{(-1+T)^2 (1+T^4) (3+4T^3+3T^6) \epsilon}{(1-T+T^2)^2 (1-T^2+T^4)^2} + O[\epsilon]^2 \right]}{131072 \pi^{17} T (1 - T + T^3 - T^5 + T^6)}$$

Invariance Under Reidemeister 2b

In[*]:= **lhs =**

$$\int \mathbb{E} \left[\pi_i p_i + \pi_j p_j + \mathcal{L}[1, i, j] + \mathcal{L}[-1, i+1, j+1] + O[\epsilon]^2 \right] d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[*]=

$$\frac{\mathbb{E} \left[(p_{2+i} \pi_i + p_{2+j} \pi_j) + O[\epsilon]^2 \right]}{16 \pi^4}$$

$$\text{In[*]:= rhs} = \int \mathbb{E} [\pi_i p_i + \pi_j p_j + x_i (p_{i+1} - p_i) + x_{i+1} (p_{i+2} - p_{i+1}) + x_j (p_{j+1} - p_j) + x_{j+1} (p_{j+2} - p_{j+1}) + \mathbf{O}[\epsilon]^2] \\ \mathfrak{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[*]=

$$\frac{\mathbb{E} [(p_{2+i} \pi_i + p_{2+j} \pi_j) + \mathbf{O}[\epsilon]^2]}{16 \pi^4}$$

$$\text{In[*]:= lhs == rhs}$$

Out[*]=

True

Invariance Under R2c

$$\text{In[*]:= lhs} = \int \mathbb{E} [\pi_i p_i + \pi_j p_j + \mathcal{L}[-1, i, j + 1] + \mathcal{L}[1, i + 1, j] + \gamma_1[-1, j + 1] + \mathbf{O}[\epsilon]^2] \\ \mathfrak{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[*]=

$$\frac{\mathbb{E} [(p_{2+i} \pi_i + p_{2+j} \pi_j) + \frac{\epsilon}{2} + \mathbf{O}[\epsilon]^2]}{16 \pi^4}$$

$$\text{In[*]:= rhs} = \int \mathbb{E} [\pi_i p_i + \pi_j p_j + x_i (p_{i+1} - p_i) + x_{i+1} (p_{i+2} - p_{i+1}) + x_j (p_{j+1} - p_j) + \\ x_{j+1} (p_{j+2} - p_{j+1}) + \gamma_1[-1, j + 1] + \mathbf{O}[\epsilon]^2] \mathfrak{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

Out[*]=

$$\frac{\mathbb{E} [(p_{2+i} \pi_i + p_{2+j} \pi_j) + \frac{\epsilon}{2} + \mathbf{O}[\epsilon]^2]}{16 \pi^4}$$

$$\text{In[*]:= lhs == rhs}$$

Out[*]=

True

Invariance Under Reidemeister 3b

$$\text{In[*]:= lhs} = \int \mathbb{E} [\pi_i p_i + \pi_j p_j + \pi_k p_k + \mathcal{L}[1, i, j] + \mathcal{L}[1, i + 1, k] + \mathcal{L}[1, j + 1, k + 1] + \mathbf{O}[\epsilon]^2] \\ \mathfrak{d}\{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\}$$

Out[*]=

$$\frac{1}{64 \pi^6} \mathbb{E} \left[\left(T^2 p_{2+i} \pi_i - T p_{2+j} (-\pi_i + T \pi_i - \pi_j) + p_{2+k} (\pi_i - T \pi_i + \pi_j - T \pi_j + \pi_k) \right) - \frac{3\epsilon}{2} + \mathbf{O}[\epsilon]^2 \right]$$

$$\text{In[*]:= rhs} = \int \mathbb{E} [\pi_i p_i + \pi_j p_j + \pi_k p_k + \mathcal{L}[1, j, k] + \mathcal{L}[1, i, k + 1] + \mathcal{L}[1, i + 1, j + 1] + \mathbf{O}[\epsilon]^2] \\ \mathfrak{d}\{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\}$$

Out[*]=

$$\frac{\mathbb{E} \left[\left(T^2 p_{2+i} \pi_i - T p_{2+j} (-\pi_i + T \pi_i - \pi_j) + p_{2+k} (\pi_i - T \pi_i + \pi_j - T \pi_j + \pi_k) \right) - \frac{3\epsilon}{2} + \mathbf{O}[\epsilon]^2 \right]}{64 \pi^6}$$


```
In[*]:= lhs == rhs  
Out[*]=  
True
```