

Pensieve header: Proof of invariance of ρ_1 using integration techniques.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
<< NilpotentIntegration.m;
 $\pi = \text{Normal}[\# + 0[\epsilon]^2] \&;$ 
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/icbs24> to compute rotation numbers.

The ρ_1 Integrand

Adopted from pensieve://Projects/APAI/PerturbedGaussianIntegration.nb.

```
In[*]:= q[s_, i_, j_] := x_i ((1 - T^s) p_{i+1} + (T^s - 1) p_{j+1});
r1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
 $\gamma_1[\varphi_, k_] := \epsilon \varphi (1 / 2 - x_k p_k);$ 
 $\rho_{1i}[s_, i_, j_] := T^{s/2} \mathbb{E}[-q[s, i, j] + \epsilon r_1[s, i, j]];$ 
 $\rho_{1i}[\varphi_, k_] := T^{\varphi/2} \mathbb{E}[-x_k (p_k - p_{k+1}) + \gamma_1[\varphi, k]];$ 
 $\rho_{1i}[\text{End}, k_] := \mathbb{E}[-x_k p_k];$ 
 $\rho_{1i}[K_] := \text{Module}[\{Cs, \varphi, n, c, k, \epsilon\},$ 
  {Cs,  $\varphi$ } = Rot[K]; n = Length[Cs];
   $\epsilon = \rho_{1i}[\text{End}, 2 n + 1];$ 
  Do[ $\epsilon *= \rho_{1i}[\varphi[[c], c], \{c, Cs\}];$ 
  Do[ $\epsilon *= \rho_{1i}[\varphi[[k], k], \{k, 2 n\}];$ 
  CF@ $\epsilon$ 
];
 $\rho_{1vs}[K_] := \text{Union}@@\text{Table}[\{x_i, p_i\}, \{i, 2 \text{Crossings}[K] + 1\}]$ 
```

In[*]:= $\rho 1i[\text{Knot}[3, 1]]$

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\frac{1}{T^2} \mathbb{E} \left[\begin{aligned} & \in -p_1 x_1 + p_2 x_1 - p_2 x_2 - \in p_2 x_2 + \frac{p_3 x_2}{T} + \in p_5 x_2 + \frac{(-1+T) p_6 x_2}{T} + \\ & \frac{(-1+T) \in p_2 p_5 x_2^2}{2T} - \frac{(-1+T) \in p_5^2 x_2^2}{2T} - p_3 x_3 + p_4 x_3 + \in p_1 x_4 + \frac{(-1+T) p_2 x_4}{T} - \\ & p_4 x_4 + \frac{p_5 x_4}{T} - \in p_1^2 x_1 x_4 + \in p_1 p_4 x_1 x_4 - \frac{(-1+T) \in p_1^2 x_4^2}{2T} + \frac{(-1+T) \in p_1 p_4 x_4^2}{2T} - \\ & p_5 x_5 + p_6 x_5 + \in p_2 p_5 x_2 x_5 - \in p_5^2 x_2 x_5 + \in p_3 x_6 + \frac{(-1+T) p_4 x_6}{T} - p_6 x_6 - \in p_6 x_6 + \\ & \frac{p_7 x_6}{T} - \in p_3^2 x_3 x_6 + \in p_3 p_6 x_3 x_6 - \frac{(-1+T) \in p_3^2 x_6^2}{2T} + \frac{(-1+T) \in p_3 p_6 x_6^2}{2T} - p_7 x_7 \end{aligned} \right]$$

In[*]:= $\rho 1vs[\text{Knot}[3, 1]]$

Out[*]=

$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

In[*]:= $K = \text{Knot}[5, 2]; \rho 1i[K]$

Out[*]=

$$\frac{1}{T^3} \mathbb{E} \left[\begin{aligned} & 2 \in -p_1 x_1 + p_2 x_1 - p_2 x_2 - \in p_2 x_2 + \frac{p_3 x_2}{T} + \in p_7 x_2 + \frac{(-1+T) p_8 x_2}{T} + \frac{(-1+T) \in p_2 p_7 x_2^2}{2T} - \\ & \frac{(-1+T) \in p_7^2 x_2^2}{2T} - p_3 x_3 + p_4 x_3 + \in p_1 x_4 + \frac{(-1+T) p_2 x_4}{T} - p_4 x_4 + \frac{p_5 x_4}{T} - \in p_1^2 x_1 x_4 + \in p_1 p_4 x_1 x_4 - \\ & \frac{(-1+T) \in p_1^2 x_4^2}{2T} + \frac{(-1+T) \in p_1 p_4 x_4^2}{2T} - p_5 x_5 + p_6 x_5 - p_6 x_6 - \in p_6 x_6 + \frac{p_7 x_6}{T} + \in p_9 x_6 + \\ & \frac{(-1+T) p_{10} x_6}{T} + \frac{(-1+T) \in p_6 p_9 x_6^2}{2T} - \frac{(-1+T) \in p_9^2 x_6^2}{2T} - p_7 x_7 + p_8 x_7 + \in p_2 p_7 x_2 x_7 - \in p_7^2 x_2 x_7 + \\ & \in p_3 x_8 + \frac{(-1+T) p_4 x_8}{T} - p_8 x_8 - \in p_8 x_8 + \frac{p_9 x_8}{T} - \in p_3^2 x_3 x_8 + \in p_3 p_8 x_3 x_8 - \frac{(-1+T) \in p_3^2 x_8^2}{2T} + \\ & \frac{(-1+T) \in p_3 p_8 x_8^2}{2T} - p_9 x_9 - \in p_9 x_9 + p_{10} x_9 + \in p_6 p_9 x_6 x_9 - \in p_9^2 x_6 x_9 + \in p_5 x_{10} + \frac{(-1+T) p_6 x_{10}}{T} - \\ & p_{10} x_{10} + \frac{p_{11} x_{10}}{T} - \in p_5^2 x_5 x_{10} + \in p_5 p_{10} x_5 x_{10} - \frac{(-1+T) \in p_5^2 x_{10}^2}{2T} + \frac{(-1+T) \in p_5 p_{10} x_{10}^2}{2T} - p_{11} x_{11} \end{aligned} \right]$$

In[*]:= $K = \text{Knot}[5, 2]; \int \rho 1i[K] d(\rho 1vs@K)$

Out[*]=

$$\frac{i T \mathbb{E} \left[\frac{(-1+T)^2 (5-4T+5T^2) \in}{(2-3T+2T^2)^2} \right]}{2048 \pi^{11} (2-3T+2T^2)}$$

$$In[*]:= K = \text{Knot}[8, 19]; \int \rho_{1i}[K] \, d(\rho_{1vs}@K)$$

Out[*]=

$$\frac{i T^3 \mathbb{E} \left[- \frac{(-1+T)^2 (1+T^4) (3+4T^3+3T^6) \epsilon}{(1-T+T^2)^2 (1-T^2+T^4)^2} \right]}{131072 \pi^{17} (1 - T + T^3 - T^5 + T^6)}$$

Concatenating edges

$$In[*]:= \text{lhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[\varphi_1, i] \rho_{1i}[\varphi_2, i+1]) \, d\{x_i, p_i, x_{i+1}, p_{i+1}\}$$

$$\text{rhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[\varphi_1 + \varphi_2, i]) \, d\{x_i, p_i\}$$

Out[*]=

$$\frac{T^{\frac{\varphi_1}{2} + \frac{\varphi_2}{2}} \mathbb{E} \left[\frac{1}{2} \epsilon (-\varphi_1 - \varphi_2) + p_{2+i} \pi_i - \epsilon (\varphi_1 + \varphi_2) p_{2+i} \pi_i \right]}{4 \pi^2}$$

Out[*]=

$$\frac{i T^{\frac{\varphi_1}{2} + \frac{\varphi_2}{2}} \mathbb{E} \left[\frac{1}{2} \epsilon (-\varphi_1 - \varphi_2) + p_{1+i} \pi_i - \epsilon (\varphi_1 + \varphi_2) p_{1+i} \pi_i \right]}{2 \pi}$$

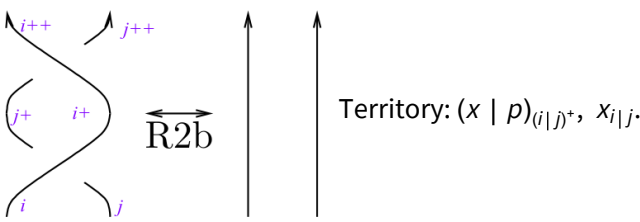
Invariance Under Reidemeister 3b

```
In[*]:= lhs =
  Integrate[(E[\pi_i p_i + \pi_j p_j + \pi_k p_k] \rho1i[1, i, j] \rho1i[1, i + 1, k] \rho1i[1, j + 1, k + 1] \rho1i[0, i] \rho1i[0, j]
    \rho1i[0, k] \rho1i[0, i + 1] \rho1i[0, j + 1] \rho1i[0, k + 1])
    d[\{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\}]
  rhs = Integrate[(E[\pi_i p_i + \pi_j p_j + \pi_k p_k] \rho1i[1, j, k] \rho1i[1, i, k + 1] \rho1i[1, i + 1, j + 1]
    \rho1i[0, i] \rho1i[0, j] \rho1i[0, k] \rho1i[0, i + 1] \rho1i[0, j + 1] \rho1i[0, k + 1])
    d[\{x_i, x_j, x_k, p_i, p_j, p_k, x_{i+1}, x_{j+1}, x_{k+1}, p_{i+1}, p_{j+1}, p_{k+1}\}];
  lhs == rhs
```

```
Out[*]=
  1
  ---
  64 \pi^6
  T^{3/2} E[-\frac{3}{2} \in + T^2 p_{2+i} \pi_i + \frac{1}{2} T^3 \in p_{2+i} p_{2+j} \pi_i (-\pi_i + T \pi_i - 2 \pi_j) - \frac{1}{2} T^3 \in p_{2+j}^2 \pi_i (-\pi_i + T \pi_i - 2 \pi_j) +
  T \in p_{2+j} (T \pi_i - \pi_j) - T p_{2+j} (-\pi_i + T \pi_i - \pi_j) - \frac{1}{2} T \in p_{2+k}^2 (\pi_i + \pi_j) (-\pi_i + T \pi_i - \pi_j + T \pi_j - 2 \pi_k) +
  \frac{1}{2} T^2 \in p_{2+i} p_{2+k} \pi_i (-\pi_i + T \pi_i - 2 \pi_j + 2 T \pi_j - 2 \pi_k) +
  \in p_{2+k} (T \pi_i - \pi_j + 2 T \pi_j - 2 \pi_k) + p_{2+k} (\pi_i - T \pi_i + \pi_j - T \pi_j + \pi_k) - \frac{1}{2} T \in p_{2+j} p_{2+k}
  (\pi_i^2 - 2 T \pi_i^2 + T^2 \pi_i^2 + 2 \pi_i \pi_j - 4 T \pi_i \pi_j + 2 T^2 \pi_i \pi_j + \pi_j^2 - T \pi_j^2 + 2 \pi_i \pi_k - 2 T \pi_i \pi_k + 2 \pi_j \pi_k) ]
```

```
Out[*]=
  True
```

Invariance Under Reidemeister 2b

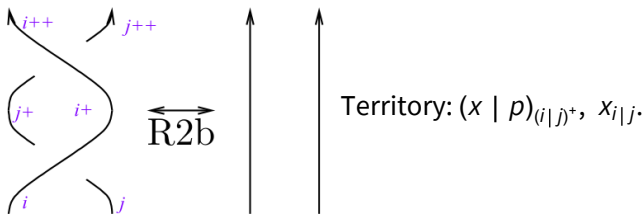


$$\begin{aligned}
 \text{In[*]}:= \text{lhs} &= \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \rho_{1i}[1, i, j] \rho_{1i}[-1, i+1, j+1] \rho_{1i}[0, i] \\
 &\quad \rho_{1i}[0, j] \rho_{1i}[0, i+1] \rho_{1i}[0, j+1]) \mathfrak{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\} \\
 \text{rhs} &= \int (\mathbb{E}[\pi_i p_i + \pi_j p_j] \rho_{1i}[0, i] \rho_{1i}[0, j] \rho_{1i}[0, i+1] \rho_{1i}[0, j+1]) \\
 &\quad \mathfrak{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}; \\
 \text{lhs} &= \text{rhs}
 \end{aligned}$$

$$\text{Out[*]}= \frac{\mathbb{E}[p_{2+i} \pi_i + p_{2+j} \pi_j]}{16 \pi^4}$$

Out[*]= True

Invariance Under Reidemeister 2b (no source terms)



$$\begin{aligned}
 \text{In[*]}:= \text{lhs} &= \int (\rho_{1i}[1, i, j] \rho_{1i}[-1, i+1, j+1] \rho_{1i}[0, i] \rho_{1i}[0, j] \rho_{1i}[0, i+1] \rho_{1i}[0, j+1]) \\
 &\quad \mathfrak{d}\{x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{Out[*]}= &\frac{1}{4 \pi^2} \mathbb{E} \left[-p_i x_i + \epsilon p_i x_i + p_{2+i} x_i - \epsilon p_j x_i - T \in p_{2+j} x_i + \frac{1}{2} (-1 + T) \in p_i p_j x_i^2 + \right. \\
 &\frac{1}{2} (1 - T) \in p_j^2 x_i^2 + \frac{1}{2} (1 - T) \in p_{2+i} p_{2+j} x_i^2 + \frac{1}{2} (-1 + T) \in p_{2+j}^2 x_i^2 - p_j x_j + \\
 &\left. p_{2+j} x_j + \epsilon p_{2+j} x_j - \epsilon p_i p_j x_i x_j + \epsilon p_j^2 x_i x_j + \epsilon p_{2+i} p_{2+j} x_i x_j - \epsilon p_{2+j}^2 x_i x_j \right]
 \end{aligned}$$

$$\text{In[*]}:= \text{CF}[\text{lhs} /. \{p_i \rightarrow (1 + \epsilon) p_{i+2} - \epsilon (1 + T) p_{j+2}, p_j \rightarrow (1 + \epsilon) p_{j+2}\}] /. \epsilon^- \rightarrow 0$$

$$\text{Out[*]}= \frac{\mathbb{E}[0]}{4 \pi^2}$$

$$\text{In[*]}:= \text{rhs} = \int (\rho_{1i}[0, i] \rho_{1i}[0, j] \rho_{1i}[0, i+1] \rho_{1i}[0, j+1]) \mathfrak{d}\{x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

$$\text{Out[*]}= \frac{\mathbb{E}[-p_i x_i + p_{2+i} x_i - p_j x_j + p_{2+j} x_j]}{4 \pi^2}$$

$$\text{In[*]}:= \text{CF}[\text{rhs} /. \{p_i \rightarrow p_{i+2}, p_j \rightarrow p_{j+2}\}]$$

$$\text{Out[*]}= \frac{\mathbb{E}[0]}{4 \pi^2}$$

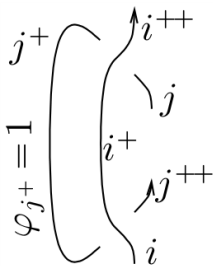
```

In[*]:= Coefficient[(4 π² lhs) [[1]], ε, 0] == Coefficient[(4 π² rhs) [[1]], ε, 0]
Out[*]= True

In[*]:= diff = CF[Coefficient[(4 π² lhs) [[1]], ε, 0] +
    ε (Coefficient[(4 π² lhs) [[1]], ε, 1] - Coefficient[(4 π² rhs) [[1]], ε, 1])]
Out[*]=
    - p_i x_i + ε p_i x_i + p_{2+i} x_i - ε p_j x_i - T ε p_{2+j} x_i + 1/2 (-1 + T) ε p_i p_j x_i² +
    1/2 (1 - T) ε p_j² x_i² + 1/2 (1 - T) ε p_{2+i} p_{2+j} x_i² + 1/2 (-1 + T) ε p_{2+j}² x_i² - p_j x_j +
    p_{2+j} x_j + ε p_{2+j} x_j - ε p_i p_j x_i x_j + ε p_j² x_i x_j + ε p_{2+i} p_{2+j} x_i x_j - ε p_{2+j}² x_i x_j

In[*]:= ∫ E[diff + π_i p_i + π_j p_j] d{x_i, x_j, p_i, p_j}
Out[*]=
    E[p_{2+i} π_i + p_{2+j} π_j]
    -----
    4 π²
    
```

Invariance Under R2c



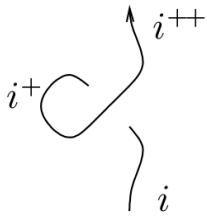
```

In[*]:= lhs = ∫ (E[π_i p_i + π_j p_j] ρ_{1i}[-1, i, j + 1] ρ_{1i}[1, i + 1, j] ρ_{1i}[0, i]
    ρ_{1i}[0, j] ρ_{1i}[0, i + 1] ρ_{1i}[1, j + 1]) d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}
Out[*]=
    √T E[-ε/2 + p_{2+i} π_i + p_{2+j} π_j - ε p_{2+j} π_j]
    -----
    16 π⁴
    
```

```

In[*]:= rhs = ∫ (E[π_i p_i + π_j p_j] ρ_{1i}[0, i] ρ_{1i}[0, j] ρ_{1i}[0, i + 1] ρ_{1i}[1, j + 1])
    d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};
lhs == rhs
Out[*]= True
    
```

Invariance Under R1l



$$\text{In[*]:= lhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[1, i+1, i] \rho_{1i}[0, i] \rho_{1i}[1, i+1]) \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\}$$

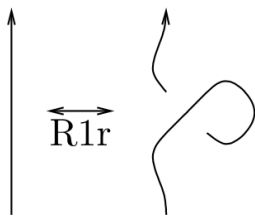
$$\text{Out[*]:=} \frac{\mathbb{E}[p_{2+i} \pi_i]}{4 \pi^2}$$

$$\text{In[*]:= rhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[0, i] \rho_{1i}[0, i+1]) \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\};$$

lhs == rhs

Out[*]:= True

Invariance Under R1r



$$\text{In[*]:= lhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[1, i, i+1] \rho_{1i}[0, i] \rho_{1i}[-1, i+1]) \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\}$$

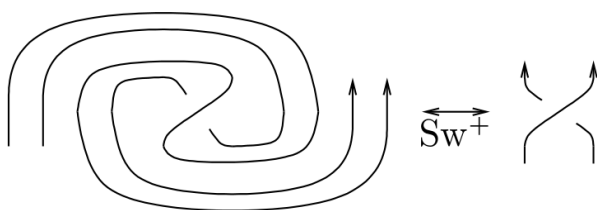
$$\text{Out[*]:=} \frac{\mathbb{E}[p_{2+i} \pi_i]}{4 \pi^2}$$

$$\text{In[*]:= rhs} = \int (\mathbb{E}[\pi_i p_i] \rho_{1i}[0, i] \rho_{1i}[0, i+1]) \mathbb{d}\{x_i, p_i, x_{i+1}, p_{i+1}\};$$

lhs == rhs

Out[*]:= True

Invariance Under Sw



In[*]:= CF /@ {ρ1i[1, j], ρ1i[1, i, j]}

Out[*]=

$$\left\{ \sqrt{T} \mathbb{E} \left[-\frac{\epsilon}{2} - p_j x_j - \epsilon p_j x_j + p_{1+j} x_j \right], \right. \\ \left. \sqrt{T} \mathbb{E} \left[-\frac{\epsilon}{2} + \epsilon p_i x_i + (-1 + T) p_{1+i} x_i - \epsilon p_j x_i + (1 - T) p_{1+j} x_i + \frac{1}{2} (-1 + T) \epsilon p_i p_j x_i^2 + \right. \right. \\ \left. \left. \frac{1}{2} (1 - T) \epsilon p_j^2 x_i^2 - \epsilon p_i p_j x_i x_j + \epsilon p_j^2 x_i x_j \right] \right\}$$

In[*]:= lhs = ∫ (E[πi pi + πj pj + ε πi+1 pi+1 + ε πj+1 pj+1 + ξi+1 xi+1 + ξj+1 xj+1] ρ1i[1, i, j] ρ1i[-1, i] ρ1i[1, i + 1] ρ1i[-1, j] ρ1i[1, j + 1]) d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}}

Out[*]=

$$\frac{1}{16 \pi^4} \sqrt{T} \mathbb{E} \left[T p_{2+i} \pi_i + \frac{1}{2} T \epsilon p_{2+i} p_{2+j} \pi_i (-\pi_i + T \pi_i - 2 \pi_j) - \right. \\ \left. \frac{1}{2} T \epsilon p_{2+j}^2 \pi_i (-\pi_i + T \pi_i - 2 \pi_j) + p_{2+j} (\pi_i - T \pi_i + \pi_j) + T \pi_i \xi_{1+i} + \pi_i \xi_{1+j} - \right. \\ \left. T \pi_i \xi_{1+j} + \pi_j \xi_{1+j} + \frac{1}{2} \epsilon p_{2+i} (2 \pi_{1+i} - T \pi_i^2 \xi_{1+j} + T^2 \pi_i^2 \xi_{1+j} - 2 T \pi_i \pi_j \xi_{1+j}) + \right. \\ \left. \frac{1}{2} \epsilon p_{2+j} (2 T \pi_i - 2 \pi_j + 2 \pi_{1+j} - T \pi_i^2 \xi_{1+i} + T^2 \pi_i^2 \xi_{1+i} - 2 T \pi_i \pi_j \xi_{1+i} + 2 T \pi_i^2 \xi_{1+j} - \right. \\ \left. 2 T^2 \pi_i^2 \xi_{1+j} + 4 T \pi_i \pi_j \xi_{1+j}) + \frac{1}{2} \epsilon (-1 + 2 \pi_{1+i} \xi_{1+i} + 2 T \pi_i \xi_{1+j} - 2 \pi_j \xi_{1+j} + 2 \pi_{1+j} \xi_{1+j} - \right. \\ \left. T \pi_i^2 \xi_{1+i} \xi_{1+j} + T^2 \pi_i^2 \xi_{1+i} \xi_{1+j} - 2 T \pi_i \pi_j \xi_{1+i} \xi_{1+j} + T \pi_i^2 \xi_{1+j}^2 - T^2 \pi_i^2 \xi_{1+j}^2 + 2 T \pi_i \pi_j \xi_{1+j}^2) \right]$$

In[*]:= rhs = ∫ (E[πi pi + πj pj + ε πi+1 pi+1 + ε πj+1 pj+1 + ξi+1 xi+1 + ξj+1 xj+1] ρ1i[1, i, j] ρ1i[0, i] ρ1i[0, i + 1] ρ1i[0, j] ρ1i[0, j + 1]) d{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}};

lhs == rhs

Out[*]=

True