

Pensieve header: Mathematica notebook for the 2-variable perturbed Alexander invariant.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << ../APAI/Rot.m];
TS = T S;
( $\alpha_-^+$ )^+ :=  $\alpha^{++}$ ; (* this is for cosmetic reasons only *)
 $\delta_{i,j}$  := If[i == j, 1, 0];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/APAI> to compute rotation numbers.

Lower Rank 2 Gassner: (Modified from Rank2Gassner.nb)

```
In[*]:= LR2G $_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i \mapsto T e_i + (1 - T) e_j$ ,  $f_i \mapsto S f_i + (1 - S) f_j$ ,
 $g_i \mapsto T S g_i + (1 - T S) g_j$ 
+  $S (T - 1) e_j f_i + (S - 1) T e_i f_j + (S + T - 2 S T) e_j f_j$ }]];
 $\overline{\text{LR2G}}_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i \mapsto T^{-1} e_i + (1 - T^{-1}) e_j$ ,  $f_i \mapsto S^{-1} f_i + (1 - S^{-1}) f_j$ ,
 $g_i \mapsto T^{-1} S^{-1} g_i + (1 - T^{-1} S^{-1}) g_j$ 
-  $S^{-1} (1 - T^{-1}) e_j f_i - T^{-1} (1 - S^{-1}) e_i f_j + (S^{-1} + T^{-1} - 2 S^{-1} T^{-1}) e_j f_j$ }]];
bas =
{ $e_1, e_2, e_3, f_1, f_2, f_3, e_1 f_1, e_1 f_2, e_1 f_3, e_2 f_1, e_2 f_2, e_2 f_3, e_3 f_1, e_3 f_2, e_3 f_3, g_1, g_2, g_3$ };
(bas // LR2G $_{1,2}$  //  $\overline{\text{LR2G}}_{1,2}$ ) == bas
(bas // LR2G $_{3,2}$  //  $\overline{\text{LR2G}}_{3,2}$ ) == bas
(1hs = bas // LR2G $_{1,2}$  // LR2G $_{1,3}$  // LR2G $_{2,3}$ ) == (bas // LR2G $_{2,3}$  // LR2G $_{1,3}$  // LR2G $_{1,2}$ )
```

Out[*]=
True

Out[*]=
True

Out[*]=
True

```

In[*]:= R1[s_, i_, j_] := TBD; (* a sign(s)-dependent quadratic polynomial in  $g_{1\alpha\beta}$ ,  $g_{2\alpha\beta}$ ,  $g_{3\alpha\beta}$ ,
and  $y_{\alpha\beta\gamma}$ , where  $\deg y_{\alpha\beta\gamma}=2$  and  $\alpha, \beta, \gamma \in \{i, j\}$ , with coefficients in  $\mathbb{Z}[T^{\pm 1}, S^{\pm 1}]$ . *)
CF[E_] := Factor@Together[E];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];
  Δ = T(-Total[φ]-Total[Cs[[All,1]])/2 Det[A];
  G = Inverse[A];
  gEval[E_] := CF[E /. α_+ => α + 1 /.
    {g1, α, β_ => G[[α, β]], g2, α, β_ => (G[[α, β]] /. T -> S), g3, α, β_ => (G[[α, β]] /. T -> TS)}];
  Y[α_, β_, γ_] := Sum[{s, i, j} = c;
    g3, α, i (Ss (Ts - 1) g1, j+, β g2, i+, γ + (Ss - 1) Ts g1, i+, β g2, j+, γ + (Ss + Ts - 2 TSs) g1, j+, β g2, j+, γ),
    {c, Cs}];
  yEval[E_] := CF[E /. y_α, β, γ_ => Y[α, β, γ]];
  λ1 = Sumk=1n R1@@Cs[[k]] - Sumk=12n φ[[k]] (g1, k, k + g2, k, k + g3, k, k);
  {Δ, Δ (Δ /. T -> S) (Δ /. T -> TS) λ1} // yEval // gEval
];

```

Step-by-step Run-Through

```
In[*]:= CF[E_] := Factor@Together[E];
```

```
In[*]:= K = {
  PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
  Knot[6, 2]
}[[1]]
```

```
Out[*]= PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]
```

```
In[*]:= {Cs, φ} = Rot[K]
```

```
Out[*]= {{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

```
In[*]:= n = Length[Cs];
```

```
A = IdentityMatrix[2 n + 1];
```

```
Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];
```

In[*]:= **A // MatrixForm**

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[*]:= $\Delta = T^{(-\text{Total}[\varphi] - \text{Total}[\text{Cs}[\text{All}, 1]])/2} \text{Det}[A]$

Out[*]=

$$\frac{1 - T + T^2}{T}$$

In[*]:= **G = Inverse[A];**

G // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[*]:= **gEval[\mathcal{E}_-] := CF[$\mathcal{E} / . \alpha_- \rightarrow \alpha + 1 / .$**

{ $g_{1,\alpha,-,\beta_-} \rightarrow G[\alpha, \beta]$, $g_{2,\alpha,-,\beta_-} \rightarrow (G[\alpha, \beta] / . T \rightarrow S)$, $g_{3,\alpha,-,\beta_-} \rightarrow (G[\alpha, \beta] / . T \rightarrow TS)$ }];

In[*]:= **Clear[Y]**

Y[α_- , β_- , γ_-] := Y[α , β , γ] = Sum[{ s , i , j] = c ;

$g_{3,\alpha,i} (S^5 (T^5 - 1) g_{1,j^*,\beta} g_{2,i^*,\gamma} + (S^5 - 1) T^5 g_{1,i^*,\beta} g_{2,j^*,\gamma} + (S^5 + T^5 - 2 TS^5) g_{1,j^*,\beta} g_{2,j^*,\gamma})$,
{ c , Cs }];

Column[MatrixForm/@Table[Y[α , β , γ], { α , 3}, { β , 3}, { γ , 3}]]

Out[*]=

$$\begin{pmatrix} (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,3,1} \end{pmatrix}$$

```
In[*]:= Column[MatrixForm/@Table[Y[α, β, γ] // gEval, {α, 3}, {β, 3}, {γ, 3}]]
```

```
Out[*]=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(-1+S)^2 T}{1-S+S^2} \\ 0 & -\frac{S(-1+T)^2}{1-T+T^2} & -\frac{-S+S^2-T+2ST-2S^2T+T^2-2ST^2+2S^2T^2}{(1-S+S^2)(1-T+T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

```
In[*]:= yEval[ε_] := CF[ε /. y_{α,β,γ} -> Y[α, β, γ]]
```

The g-Rules

```
In[*]:= gRules_{s_,i_,j_} := {
  g_{1,i,β} -> δ_{i,β} + T^S g_{1,i^+,β} + (1 - T^S) g_{1,j^+,β}, g_{1,j,β} -> δ_{j,β} + g_{1,j^+,β},
  g_{1,α,i} -> T^{-S} (g_{1,α,i^+} - δ_{α,i^+}), g_{1,α,j} -> g_{1,α,j^+} - (1 - T^S) g_{1,α,i} - δ_{α,j^+},
  g_{2,i,β} -> δ_{i,β} + S^S g_{2,i^+,β} + (1 - S^S) g_{2,j^+,β}, g_{2,j,β} -> δ_{j,β} + g_{2,j^+,β},
  g_{2,α,i} -> S^{-S} (g_{2,α,i^+} - δ_{α,i^+}), g_{2,α,j} -> g_{2,α,j^+} - (1 - S^S) g_{2,α,i} - δ_{α,j^+},
  g_{3,i,β} -> δ_{i,β} + TS^S g_{3,i^+,β} + (1 - TS^S) g_{3,j^+,β}, g_{3,j,β} -> δ_{j,β} + g_{3,j^+,β},
  g_{3,α,i} -> TS^{-S} (g_{3,α,i^+} - δ_{α,i^+}), g_{3,α,j} -> g_{3,α,j^+} - (1 - TS^S) g_{3,α,i} - δ_{α,j^+}
}
```

```
In[*]:= (gs = Table[{g_{1,α,β}, g_{2,α,β}, g_{3,α,β}}, {α, 2n+1}, {β, 2n+1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{{{g_{1,1,1}, g_{2,1,1}}, {g_{1,1,2}, g_{2,1,2}}, {{g_{1,2,1}, g_{2,2,1}}, {g_{1,2,2}, g_{2,2,2}}}}
```

```
In[*]:= Cs
```

```
Out[*]=
```

```
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

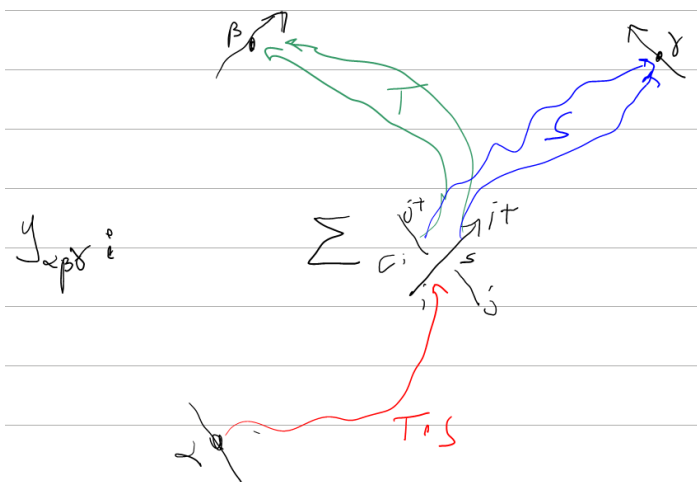
```
In[*]:= (gs /. gRules_{Sequence@@Cs[[1]]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{
  {{(1 + T) g_{1,1^+,1} + (1 - T) g_{1,4^+,1}, (1 + S) g_{2,1^+,1} + (1 - S) g_{2,4^+,1}},
  {T g_{1,1^+,2} + (1 - T) g_{1,4^+,2}, S g_{2,1^+,2} + (1 - S) g_{2,4^+,2}},
  {{-If[2 == 1^+, 1, 0] + g_{1,2,1^+}, -If[2 == 1^+, 1, 0] + g_{2,2,1^+}}, {g_{1,2,2}, g_{2,2,2}}}
}
```

```
In[ ]:= gEval[gs - (gs /. Flatten@Table[gRulesSequence@@c, {c, Cs}]]]
Out[ ]:=
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

The y-Rules



```
In[ ]:= yRuless, i, j := {
  yi, beta, gamma -> TSs yi+, beta, gamma + (1 - TSs) yj+, beta, gamma +
    (Ss (Ts - 1) g1, j+, beta g2, i+, gamma + (Ss - 1) Ts g1, i+, beta g2, j+, gamma + (Ss + Ts - 2 TSs) g1, j+, beta g2, j+, gamma),
  yj, beta, gamma -> yj+, beta, gamma,
  yalpha, i, gamma -> T-s yalpha, i+, gamma + (1 - Ss) g3, alpha, i g2, j+, gamma,
  yalpha, j, gamma -> yalpha, j+, gamma - (1 - Ts) yalpha, i, gamma - g3, alpha, i (Ss (Ts - 1) g2, i+, gamma + (Ss + Ts - 2 TSs) g2, j+, gamma),
  yalpha, beta, i -> S-s yalpha, beta, i+ + (1 - Ts) g3, alpha, i g1, j+, beta,
  yalpha, beta, j -> yalpha, beta, j+ - (1 - Ss) yalpha, beta, i - g3, alpha, i (Ts (Ss - 1) g1, i+, beta + (Ss + Ts - 2 TSs) g1, beta, j+)
}
```

```
In[ ]:= (ys = Table[yalpha, beta, gamma, {alpha, 2 n + 1}, {beta, 2 n + 1}, {gamma, 2 n + 1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[ ]:=
{{{y1,1,1, y1,1,2}, {y1,2,1, y1,2,2}, {{y2,1,1, y2,1,2}, {y2,2,1, y2,2,2}}}
```

```
In[ ]:= Cs
```

```
Out[ ]:=
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

In[*]:= (ys /. yRules_{Sequence@@Cs[[1]]}) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]

Out[*]=

$$\left\{ \left\{ \left\{ S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,1^+} + (1-ST)y_{4^+,1^+}, S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,1^+} + (1-ST)y_{4^+,1^+}, \right. \right. \\ \left. \left\{ S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,2^+} + (1-ST)y_{4^+,2^+}, \right. \right. \\ \left. \left\{ S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,2^+} + (1-ST)y_{4^+,2^+}, S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,2^+} + (1-ST)y_{4^+,2^+} \right\} \right\}, \\ \left\{ \left\{ (1-S)g_{2,4^+}g_{3,2^+} + \frac{y_{2,1^+,1}}{T}, (1-S)g_{2,4^+}g_{3,2^+} + \frac{y_{2,1^+,2}}{T} \right\}, \right. \\ \left. \left\{ (1-T)g_{1,4^+}g_{3,2^+} + \frac{y_{2,2^+,1}}{S}, y_{2,2^+,2} \right\} \right\}$$

In[*]:= yEval[ys - (ys /. yRules_{Sequence@@Cs[[1]]})] [[1, 1, 1]]

Out[*]=

$$Sg_{1,4^+}g_{2,1^+} - STg_{1,4^+}g_{2,1^+} + Tg_{1,1^+}g_{2,4^+} - STg_{1,1^+}g_{2,4^+} - Sg_{1,4^+}g_{2,4^+} - Tg_{1,4^+}g_{2,4^+} + 2STg_{1,4^+}g_{2,4^+} - Sg_{1,4^+}g_{2,1^+}g_{3,1^+} + STg_{1,4^+}g_{2,1^+}g_{3,1^+} - Tg_{1,1^+}g_{2,4^+}g_{3,1^+} + STg_{1,1^+}g_{2,4^+}g_{3,1^+} + Sg_{1,4^+}g_{2,4^+}g_{3,1^+} + Tg_{1,4^+}g_{2,4^+}g_{3,1^+} - 2STg_{1,4^+}g_{2,4^+}g_{3,1^+} - Sg_{1,6^+}g_{2,3^+}g_{3,1^+} + STg_{1,6^+}g_{2,3^+}g_{3,1^+} - Tg_{1,3^+}g_{2,6^+}g_{3,1^+} + STg_{1,3^+}g_{2,6^+}g_{3,1^+} + Sg_{1,6^+}g_{2,6^+}g_{3,1^+} + Tg_{1,6^+}g_{2,6^+}g_{3,1^+} - 2STg_{1,6^+}g_{2,6^+}g_{3,1^+} + Sg_{1,2^+}g_{2,2^+}g_{3,1^+} + Tg_{1,2^+}g_{2,2^+}g_{3,1^+} - 2STg_{1,2^+}g_{2,2^+}g_{3,1^+} - Tg_{1,5^+}g_{2,2^+}g_{3,1^+} + STg_{1,5^+}g_{2,2^+}g_{3,1^+} - Sg_{1,2^+}g_{2,5^+}g_{3,1^+} + STg_{1,2^+}g_{2,5^+}g_{3,1^+} + S^2Tg_{1,4^+}g_{2,1^+}g_{3,1^+} - S^2T^2g_{1,4^+}g_{2,1^+}g_{3,1^+} + S^2T^2g_{1,1^+}g_{2,4^+}g_{3,1^+} - S^2T^2g_{1,1^+}g_{2,4^+}g_{3,1^+} + 2S^2T^2g_{1,4^+}g_{2,4^+}g_{3,1^+} - S^2T^2g_{1,4^+}g_{2,4^+}g_{3,1^+} + 2S^2T^2g_{1,4^+}g_{2,4^+}g_{3,1^+} + S^2T^2g_{1,6^+}g_{2,3^+}g_{3,1^+} - S^2T^2g_{1,6^+}g_{2,3^+}g_{3,1^+} + S^2T^2g_{1,3^+}g_{2,6^+}g_{3,1^+} - S^2T^2g_{1,3^+}g_{2,6^+}g_{3,1^+} + S^2T^2g_{1,3^+}g_{2,6^+}g_{3,1^+} - S^2T^2g_{1,6^+}g_{2,6^+}g_{3,1^+} + 2S^2T^2g_{1,6^+}g_{2,6^+}g_{3,1^+} - S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+} + S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+} - S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+} + 2S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+} + S^2T^2g_{1,5^+}g_{2,2^+}g_{3,1^+} - S^2T^2g_{1,5^+}g_{2,2^+}g_{3,1^+} + S^2T^2g_{1,2^+}g_{2,5^+}g_{3,1^+} - S^2T^2g_{1,2^+}g_{2,5^+}g_{3,1^+} + Sg_{1,4^+}g_{2,1^+}g_{3,4^+} - STg_{1,4^+}g_{2,1^+}g_{3,4^+} - S^2T^2g_{1,4^+}g_{2,1^+}g_{3,4^+} + S^2T^2g_{1,4^+}g_{2,1^+}g_{3,4^+} + S^2T^2g_{1,4^+}g_{2,1^+}g_{3,4^+} + Tg_{1,1^+}g_{2,4^+}g_{3,4^+} - STg_{1,1^+}g_{2,4^+}g_{3,4^+} - S^2T^2g_{1,1^+}g_{2,4^+}g_{3,4^+} + S^2T^2g_{1,1^+}g_{2,4^+}g_{3,4^+} + S^2T^2g_{1,1^+}g_{2,4^+}g_{3,4^+} - Sg_{1,4^+}g_{2,4^+}g_{3,4^+} - Tg_{1,4^+}g_{2,4^+}g_{3,4^+} + 2STg_{1,4^+}g_{2,4^+}g_{3,4^+} + S^2T^2g_{1,4^+}g_{2,4^+}g_{3,4^+} + Sg_{1,6^+}g_{2,3^+}g_{3,4^+} - STg_{1,6^+}g_{2,3^+}g_{3,4^+} - S^2T^2g_{1,6^+}g_{2,3^+}g_{3,4^+} + S^2T^2g_{1,6^+}g_{2,3^+}g_{3,4^+} + Tg_{1,3^+}g_{2,6^+}g_{3,4^+} - STg_{1,3^+}g_{2,6^+}g_{3,4^+} - S^2T^2g_{1,3^+}g_{2,6^+}g_{3,4^+} + S^2T^2g_{1,3^+}g_{2,6^+}g_{3,4^+} - Sg_{1,6^+}g_{2,6^+}g_{3,4^+} - Tg_{1,6^+}g_{2,6^+}g_{3,4^+} + 2STg_{1,6^+}g_{2,6^+}g_{3,4^+} + S^2T^2g_{1,6^+}g_{2,6^+}g_{3,4^+} - 2S^2T^2g_{1,6^+}g_{2,6^+}g_{3,4^+} - Sg_{1,2^+}g_{2,2^+}g_{3,4^+} - Tg_{1,2^+}g_{2,2^+}g_{3,4^+} + 2STg_{1,2^+}g_{2,2^+}g_{3,4^+} + S^2T^2g_{1,2^+}g_{2,2^+}g_{3,4^+} - S^2T^2g_{1,2^+}g_{2,2^+}g_{3,4^+} + Tg_{1,5^+}g_{2,2^+}g_{3,4^+} - STg_{1,5^+}g_{2,2^+}g_{3,4^+} - S^2T^2g_{1,5^+}g_{2,2^+}g_{3,4^+} + S^2T^2g_{1,5^+}g_{2,2^+}g_{3,4^+} + Sg_{1,2^+}g_{2,5^+}g_{3,4^+} - STg_{1,2^+}g_{2,5^+}g_{3,4^+} - S^2T^2g_{1,2^+}g_{2,5^+}g_{3,4^+} + S^2T^2g_{1,2^+}g_{2,5^+}g_{3,4^+}$$

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

```

In[*]:= Clear[i, j, k];
Cs = {{1, i, j}, {1, i+, k}, {1, j+, k+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
lhs = Simplify[Z //. gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+ /. _If → 0]

Out[*]=
{{1, i, j}, {1, i+, k}, {1, j+, k+}}

Out[*]=
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i +
(S (-1 + T) g1,k+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+ +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+

Out[*]=
1
S^2 T^2 ((S (-1 + T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,i+,γ - (-1 + S) g2,k+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ) +
(S + T - 2 S T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ)) g3,α,i+ +
S T ((-1 + S) T g1,i+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,i+ +
S T ((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,j+)

In[*]:= Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k+}, {1, i+, j+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
rhs = Simplify[Z //. gRules1,j,k ∪ gRules1,i,k+ ∪ gRules1,i+,j+ /. _If → 0]

Out[*]=
{{1, j, k}, {1, i, k+}, {1, i+, j+}}

Out[*]=
((S + T - 2 S T) g1,k+,β g2,k+,γ + (-1 + S) T g1,i+,β g2,k+,γ + S (-1 + T) g1,k+,β g2,i+,γ) g3,α,i +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j +
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i+

Out[*]=
1
S^2 T^2 (S T ((-1 + S) T g1,i+,β g2,j+,γ + g1,j+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,j+,γ)) g3,α,i+ +
(S (-1 + T) g1,k+,β (S g2,i+,γ - (-1 + S) g2,j+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,j+,β) g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+ +
((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ))
((-1 + S T) g3,α,i+ + S T g3,α,j+))

```



```
In[*]:= Simplify[lhs == rhs]
```

```
Out[*]=
True
```

Setting up R_1

```
In[*]:= gs = Flatten@Table[{gp,i,i, gp,i,j, gp,j,i, gp,j,j}, {p, 3}];
bas = {1} ∪ gs ∪ Flatten@Table[gs[[p] × gs[[q]], {p, Length[gs]}, {q, p, Length[gs]}] ∪
  {yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
len = Length[bas];
vars = Table[cp = 0; cp = .; cp, {p, 2 len}]
R1[1, i_, j_] := Evaluate[Sum[cp bas[[p]], {p, len}]];
R1[-1, i_, j_] := Evaluate[Sum[c1len+p bas[[p]], {p, Length[bas]}]];
```

```
Out[*]=
{1, g1,i,i, g1,i,i2, g1,i,j, g1,i,i g1,i,j, g1,i,j2, g1,j,i, g1,i,i g1,j,i, g1,i,j g1,j,i, g1,j,i2, g1,j,j,
g1,i,i g1,j,j, g1,i,j g1,j,j, g1,j,i g1,j,j, g1,j,j2, g2,i,i, g1,i,i g2,i,i, g1,i,j g2,i,i, g1,j,i g2,i,i,
g1,j,j g2,i,i, g2,i,i2, g2,i,j, g1,i,i g2,i,j, g1,i,j g2,i,j, g1,j,i g2,i,j, g1,j,j g2,i,j, g2,i,i g2,i,j,
g2,i,j2, g2,j,i, g1,i,i g2,j,i, g1,i,j g2,j,i, g1,j,i g2,j,i, g1,j,j g2,j,i, g2,i,i g2,j,i, g2,i,j g2,j,i,
g2,j,i2, g2,j,j, g1,i,i g2,j,j, g1,i,j g2,j,j, g1,j,i g2,j,j, g1,j,j g2,j,j, g2,i,i g2,j,j, g2,i,j g2,j,j,
g2,j,i g2,j,j, g2,j,j2, g3,i,i, g1,i,i g3,i,i, g1,i,j g3,i,i, g1,j,i g3,i,i, g1,j,j g3,i,i, g2,i,i g3,i,i,
g2,i,j g3,i,i, g2,j,i g3,i,i, g2,j,j g3,i,i, g3,i,i2, g3,i,j, g1,i,i g3,i,j, g1,i,j g3,i,j, g1,j,i g3,i,j,
g1,j,j g3,i,j, g2,i,i g3,i,j, g2,i,j g3,i,j, g2,j,i g3,i,j, g2,j,j g3,i,j, g3,i,i g3,i,j, g3,i,j2,
g3,j,i, g1,i,i g3,j,i, g1,i,j g3,j,i, g1,j,i g3,j,i, g1,j,j g3,j,i, g2,i,i g3,j,i, g2,i,j g3,j,i,
g2,j,i g3,j,i, g2,j,j g3,j,i, g3,i,i g3,j,i, g3,i,j g3,j,i, g3,j,i2, g3,j,j, g1,i,i g3,j,j, g1,i,j g3,j,j,
g1,j,i g3,j,j, g1,j,j g3,j,j, g2,i,i g3,j,j, g2,i,j g3,j,j, g2,j,i g3,j,j, g2,j,j g3,j,j, g3,i,i g3,j,j,
g3,i,j g3,j,j, g3,j,i g3,j,j, g3,j,j2, yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
```

```
Out[*]=
{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21, c22,
c23, c24, c25, c26, c27, c28, c29, c30, c31, c32, c33, c34, c35, c36, c37, c38, c39, c40, c41,
c42, c43, c44, c45, c46, c47, c48, c49, c50, c51, c52, c53, c54, c55, c56, c57, c58, c59, c60,
c61, c62, c63, c64, c65, c66, c67, c68, c69, c70, c71, c72, c73, c74, c75, c76, c77, c78, c79,
c80, c81, c82, c83, c84, c85, c86, c87, c88, c89, c90, c91, c92, c93, c94, c95, c96, c97, c98,
c99, c100, c101, c102, c103, c104, c105, c106, c107, c108, c109, c110, c111, c112, c113, c114, c115,
c116, c117, c118, c119, c120, c121, c122, c123, c124, c125, c126, c127, c128, c129, c130, c131, c132,
c133, c134, c135, c136, c137, c138, c139, c140, c141, c142, c143, c144, c145, c146, c147, c148, c149,
c150, c151, c152, c153, c154, c155, c156, c157, c158, c159, c160, c161, c162, c163, c164, c165, c166,
c167, c168, c169, c170, c171, c172, c173, c174, c175, c176, c177, c178, c179, c180, c181, c182,
c183, c184, c185, c186, c187, c188, c189, c190, c191, c192, c193, c194, c195, c196, c197, c198}
```

In[*]:= **R₁**[-1, i⁺, 7]

Out[*]=

$$\begin{aligned}
 & C_{100} + C_{110} g_{1,7,7} + C_{114} g_{1,7,7}^2 + C_{106} g_{1,7,i^+} + C_{113} g_{1,7,7} g_{1,7,i^+} + C_{109} g_{1,7,i^+}^2 + C_{103} g_{1,i^+,7} + C_{112} g_{1,7,7} g_{1,i^+,7} + \\
 & C_{108} g_{1,7,i^+} g_{1,i^+,7} + C_{105} g_{1,i^+,7}^2 + C_{101} g_{1,i^+,i^+} + C_{111} g_{1,7,7} g_{1,i^+,i^+} + C_{107} g_{1,7,i^+} g_{1,i^+,i^+} + \\
 & C_{104} g_{1,i^+,7} g_{1,i^+,i^+} + C_{102} g_{1,i^+,i^+}^2 + C_{136} g_{2,7,7} + C_{140} g_{1,7,7} g_{2,7,7} + C_{139} g_{1,7,i^+} g_{2,7,7} + C_{138} g_{1,i^+,7} g_{2,7,7} + \\
 & C_{137} g_{1,i^+,i^+} g_{2,7,7} + C_{144} g_{2,7,7}^2 + C_{128} g_{2,7,i^+} + C_{132} g_{1,7,7} g_{2,7,i^+} + C_{131} g_{1,7,i^+} g_{2,7,i^+} + C_{130} g_{1,i^+,7} g_{2,7,i^+} + \\
 & C_{129} g_{1,i^+,i^+} g_{2,7,i^+} + C_{143} g_{2,7,7} g_{2,7,i^+} + C_{135} g_{2,7,i^+}^2 + C_{121} g_{2,i^+,7} + C_{125} g_{1,7,7} g_{2,i^+,7} + \\
 & C_{124} g_{1,7,i^+} g_{2,i^+,7} + C_{123} g_{1,i^+,7} g_{2,i^+,7} + C_{122} g_{1,i^+,i^+} g_{2,i^+,7} + C_{142} g_{2,7,7} g_{2,i^+,7} + C_{134} g_{2,7,i^+} g_{2,i^+,7} + \\
 & C_{127} g_{2,i^+,7}^2 + C_{115} g_{2,i^+,i^+} + C_{119} g_{1,7,7} g_{2,i^+,i^+} + C_{118} g_{1,7,i^+} g_{2,i^+,i^+} + C_{117} g_{1,i^+,7} g_{2,i^+,i^+} + \\
 & C_{116} g_{1,i^+,i^+} g_{2,i^+,i^+} + C_{141} g_{2,7,7} g_{2,i^+,i^+} + C_{133} g_{2,7,i^+} g_{2,i^+,i^+} + C_{126} g_{2,i^+,7} g_{2,i^+,i^+} + C_{120} g_{2,i^+,i^+}^2 + \\
 & C_{178} g_{3,7,7} + C_{182} g_{1,7,7} g_{3,7,7} + C_{181} g_{1,7,i^+} g_{3,7,7} + C_{180} g_{1,i^+,7} g_{3,7,7} + C_{179} g_{1,i^+,i^+} g_{3,7,7} + \\
 & C_{186} g_{2,7,7} g_{3,7,7} + C_{185} g_{2,7,i^+} g_{3,7,7} + C_{184} g_{2,i^+,7} g_{3,7,7} + C_{183} g_{2,i^+,i^+} g_{3,7,7} + C_{190} g_{3,7,7}^2 + C_{166} g_{3,7,i^+} + \\
 & C_{170} g_{1,7,7} g_{3,7,i^+} + C_{169} g_{1,7,i^+} g_{3,7,i^+} + C_{168} g_{1,i^+,7} g_{3,7,i^+} + C_{167} g_{1,i^+,i^+} g_{3,7,i^+} + C_{174} g_{2,7,7} g_{3,7,i^+} + \\
 & C_{173} g_{2,7,i^+} g_{3,7,i^+} + C_{172} g_{2,i^+,7} g_{3,7,i^+} + C_{171} g_{2,i^+,i^+} g_{3,7,i^+} + C_{189} g_{3,7,7} g_{3,7,i^+} + C_{177} g_{3,7,i^+}^2 + \\
 & C_{155} g_{3,i^+,7} + C_{159} g_{1,7,7} g_{3,i^+,7} + C_{158} g_{1,7,i^+} g_{3,i^+,7} + C_{157} g_{1,i^+,7} g_{3,i^+,7} + C_{156} g_{1,i^+,i^+} g_{3,i^+,7} + \\
 & C_{163} g_{2,7,7} g_{3,i^+,7} + C_{162} g_{2,7,i^+} g_{3,i^+,7} + C_{161} g_{2,i^+,7} g_{3,i^+,7} + C_{160} g_{2,i^+,i^+} g_{3,i^+,7} + C_{188} g_{3,7,7} g_{3,i^+,7} + \\
 & C_{176} g_{3,7,i^+} g_{3,i^+,7} + C_{165} g_{3,i^+,7}^2 + C_{145} g_{3,i^+,i^+} + C_{149} g_{1,7,7} g_{3,i^+,i^+} + C_{148} g_{1,7,i^+} g_{3,i^+,i^+} + \\
 & C_{147} g_{1,i^+,7} g_{3,i^+,i^+} + C_{146} g_{1,i^+,i^+} g_{3,i^+,i^+} + C_{153} g_{2,7,7} g_{3,i^+,i^+} + C_{152} g_{2,7,i^+} g_{3,i^+,i^+} + C_{151} g_{2,i^+,7} g_{3,i^+,i^+} + \\
 & C_{150} g_{2,i^+,i^+} g_{3,i^+,i^+} + C_{187} g_{3,7,7} g_{3,i^+,i^+} + C_{175} g_{3,7,i^+} g_{3,i^+,i^+} + C_{164} g_{3,i^+,7} g_{3,i^+,i^+} + C_{154} g_{3,i^+,i^+}^2 + \\
 & C_{198} y_{7,7,7} + C_{197} y_{7,7,i^+} + C_{196} y_{7,i^+,7} + C_{195} y_{7,i^+,i^+} + C_{194} y_{i^+,7,7} + C_{193} y_{i^+,7,i^+} + C_{192} y_{i^+,i^+,7} + C_{191} y_{i^+,i^+,i^+}
 \end{aligned}$$

Solving R2b

In[*]:= **Clear**[i, j];

eqn = **Expand**[**R₁**[1, i, j] + **R₁**[-1, i⁺, j⁺]] //.

gRules_{1,i,j} **U** **gRules**_{-1,i⁺,j⁺} **U** **yRules**_{1,i,j} **U** **yRules**_{-1,i⁺,j⁺} /. **_If** → 0]

Out[*]=

$$\begin{aligned}
 & C_1 + C_{100} + C_2 g_{1,i^{**},i^{**}} - C_{93} g_{1,i^{**},i^{**}} + 2 S C_{93} g_{1,i^{**},i^{**}} + \frac{S C_{93} g_{1,i^{**},i^{**}}}{T^2} + \frac{C_{93} g_{1,i^{**},i^{**}}}{T} - \frac{3 S C_{93} g_{1,i^{**},i^{**}}}{T} + \\
 & \dots 2412 \dots + C_{194} y_{j^{**},j^{**},i^{**}} - S C_{194} y_{j^{**},j^{**},i^{**}} + \frac{C_{194} y_{j^{**},j^{**},i^{**}}}{T} - \frac{C_{194} y_{j^{**},j^{**},i^{**}}}{S T} + S C_{197} y_{j^{**},j^{**},i^{**}} + \\
 & C_{198} y_{j^{**},j^{**},i^{**}} - S C_{198} y_{j^{**},j^{**},i^{**}} + C_{99} y_{j^{**},j^{**},j^{**}} + C_{194} y_{j^{**},j^{**},j^{**}} - \frac{C_{194} y_{j^{**},j^{**},j^{**}}}{S T} + C_{198} y_{j^{**},j^{**},j^{**}}
 \end{aligned}$$

Full expression not available (original memory size: 1.2 MB) ⚙️

```
In[*]:= covars = Cases[eqn, g_., ∞] ∪ Cases[eqn, c_. * (g1 : g_.,_) (g2 : g_.,_) ⇒ g1 g2, ∞] ∪
Cases[eqn, g_., ∞] ∪ Cases[eqn, y_., ∞]
```

Out[*]=

$$\left\{ \begin{aligned} &g_{1,i^{++},i^{++}}, g_{1,i^{++},i^{++}}^2, g_{1,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,i^{++},j^{++}}, g_{1,i^{++},j^{++}}^2, g_{1,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},i^{++}}, \\ &g_{1,i^{++},j^{++}} g_{1,j^{++},i^{++}}, g_{1,j^{++},i^{++}}^2, g_{1,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{1,j^{++},j^{++}}, \\ &g_{1,j^{++},i^{++}} g_{1,j^{++},j^{++}}, g_{1,j^{++},j^{++}}^2, g_{2,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},i^{++}}, \\ &g_{1,j^{++},j^{++}} g_{2,i^{++},i^{++}}, g_{2,i^{++},i^{++}}^2, g_{2,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},j^{++}}, \\ &g_{1,j^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g_{2,i^{++},j^{++}}^2, g_{2,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},i^{++}}, \\ &g_{1,j^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{2,j^{++},i^{++}}, g_{2,j^{++},i^{++}}^2, g_{2,j^{++},j^{++}}, \\ &g_{1,i^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},j^{++}}, \\ &g_{2,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{2,j^{++},j^{++}}^2, g_{3,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},i^{++}}, \\ &g_{1,j^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},i^{++}}, \\ &g_{2,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{3,i^{++},i^{++}}^2, g_{3,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{3,i^{++},j^{++}}, \\ &g_{1,j^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},j^{++}} g_{3,i^{++},j^{++}}, \\ &g_{3,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{3,i^{++},j^{++}}^2, g_{3,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{3,j^{++},i^{++}}, \\ &g_{1,j^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},j^{++}} g_{3,j^{++},i^{++}}, \\ &g_{3,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{3,j^{++},i^{++}}^2, g_{3,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},j^{++}}, \\ &g_{1,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},j^{++}}, \\ &g_{2,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{3,j^{++},j^{++}}^2, \\ &y_{i^{++},i^{++},i^{++}}, y_{i^{++},i^{++},j^{++}}, y_{i^{++},j^{++},i^{++}}, y_{i^{++},j^{++},j^{++}}, y_{j^{++},i^{++},i^{++}}, y_{j^{++},i^{++},j^{++}}, y_{j^{++},j^{++},i^{++}}, y_{j^{++},j^{++},j^{++}} \end{aligned} \right\}$$

```
In[*]:= Short[eqns = {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y) _.,_ → 0}, 5]
```

Out[*]//Short=

$$\left\{ \begin{aligned} &c_6 + \frac{c_{105}}{T^2} = 0, c_3 + c_{102} - c_{104} + \frac{c_{104}}{T} + c_{105} + \frac{c_{105}}{T^2} - \frac{2 c_{105}}{T} = 0, \langle\langle 95 \rangle\rangle, \\ &c_{67} + 2 c_{99} - S c_{99} - T c_{99} - c_{145} + S T c_{145} + 2 c_{155} - \frac{c_{155}}{S T} - S T c_{155} + S T c_{166} + c_{178} - S T c_{178} + c_{192} - \frac{c_{192}}{S} + \\ &T c_{192} - S T c_{192} + c_{193} + S c_{193} - \frac{c_{193}}{T} - S T c_{193} + \frac{2 c_{194}}{S} - 2 S c_{194} + \frac{2 c_{194}}{T} - \frac{2 c_{194}}{S T} - 2 T c_{194} + \\ &2 S T c_{194} + T c_{196} - S T c_{196} + S c_{197} - S T c_{197} + 2 c_{198} - 2 S c_{198} - 2 T c_{198} + 2 S T c_{198} = 0, c_1 + c_{100} = 0 \end{aligned} \right\}$$

```
In[*]:= Short[{sol} = Solve[eqns, vars], 5]
```

Solve: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ c_2 \rightarrow -c_{101} - \left(-1 + \frac{1}{T}\right) c_{103} - \frac{(-S - T + 3ST + T^2 - 2ST^2) c_{192}}{ST^2}, \right. \right.$$

$$c_3 \rightarrow -c_{102} - \left(-1 + \frac{1}{T}\right) c_{104} - \frac{(1 - 2T + T^2) c_{105}}{T^2}, c_4 \rightarrow -\frac{c_{103}}{T} - \frac{(-S - T + 2ST) c_{192}}{ST^2},$$

$$c_5 \rightarrow -\frac{c_{104}}{T} + \frac{2(-1 + T) c_{105}}{T^2}, c_6 \rightarrow -\frac{c_{105}}{T^2}, \langle\langle 90 \rangle\rangle,$$

$$c_{97} \rightarrow -\frac{(-1 + ST) c_{192}}{S} - \frac{(-1 + T + ST - ST^2) c_{194}}{ST} - T c_{196} - (1 - T) c_{198},$$

$$c_{98} \rightarrow -\frac{(-1 + ST) c_{193}}{T} - \frac{(-1 + S + ST - S^2 T) c_{194}}{ST} - S c_{197} - (1 - S) c_{198},$$

$$c_{99} \rightarrow -\frac{(-1 + ST) c_{194}}{ST} - c_{198}, c_{100} \rightarrow -c_1 \left. \right\}$$

```
In[*]:= Short[sol /. (v_ -> val_) :> (v = CF[val]), 5]
```

Out[*]//Short=

$$\left\{ -\frac{ST^2 c_{101} + \langle\langle 8 \rangle\rangle + T^2 \langle\langle 1 \rangle\rangle - 2ST^2 c_{192}}{ST^2}, \right.$$

$$\left. -\frac{T^2 c_{102} + \langle\langle 6 \rangle\rangle + T^2 c_{\langle\langle 3 \rangle\rangle}}{T^2}, -\frac{\langle\langle 1 \rangle\rangle}{S \langle\langle 1 \rangle\rangle}, \langle\langle 93 \rangle\rangle, -\frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle}, -\frac{\langle\langle 1 \rangle\rangle}{ST}, -c_1 \right\}$$

```
In[*]:= Short[R1[1, i, j], 5]
```

Out[*]//Short=

$\langle\langle 1 \rangle\rangle$

Solving R3

```
In[*]:= Clear[i, j, k];
```


```
lhs = Expand[R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+]] /. gRules1,i,j U
      gRules1,i+,k U gRules1,j+,k+ U yRules1,i,j U yRules1,i+,k U yRules1,j+,k+ /. _If -> 0];
rhs = Expand[R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+]] /. gRules1,j,k U
      gRules1,i,k+ U gRules1,i+,j+ U yRules1,j,k U yRules1,i,k+ U yRules1,i+,j+ /. _If -> 0];
eqn = lhs - rhs
```

Out[*]=

$$-2 c_{194} + \frac{c_{194}}{S} + \frac{c_{194}}{T} - c_{103} g_{1,i+,j++} + \frac{c_{103} g_{1,i+,j++}}{T} - \frac{c_{192} g_{1,i+,j++}}{T^2} + \frac{2 c_{192} g_{1,i+,j++}}{T} -$$

$$\frac{c_{192} g_{1,i+,j++}}{ST} - c_{104} g_{1,i+,i++} g_{1,i+,j++} + \dots 3208 \dots + \frac{c_{198} y_{k+,j++,j++}}{S} + \frac{c_{198} y_{k+,j++,j++}}{T} - \frac{2 c_{198} y_{k+,j++,j++}}{ST} -$$

$$c_{198} y_{k+,j++,k++} + \frac{c_{198} y_{k+,j++,k++}}{T} - \frac{c_{198} y_{k+,k++,i++}}{S^2} + \frac{c_{198} y_{k+,k++,i++}}{S} - c_{198} y_{k+,k++,j++} + \frac{c_{198} y_{k+,k++,j++}}{S}$$

Full expression not available (original memory size: 1.6 MB) 

In[*]:= vars = Union@Cases[eqn, c_, ∞]

Out[*]=

{C101, C102, C103, C104, C105, C106, C107, C108, C109, C110, C111, C112, C113, C114, C115, C116, C117, C118, C119, C120, C121, C122, C123, C124, C125, C126, C127, C128, C129, C130, C131, C132, C133, C134, C135, C136, C137, C138, C139, C140, C141, C142, C143, C144, C145, C146, C147, C148, C149, C150, C151, C152, C153, C154, C155, C156, C157, C158, C159, C160, C161, C162, C163, C164, C165, C166, C167, C168, C169, C170, C171, C172, C173, C174, C175, C176, C177, C178, C179, C180, C181, C182, C183, C184, C185, C186, C187, C188, C189, C190, C191, C192, C193, C194, C195, C196, C197, C198}

In[*]:= Short[covars = Cases[eqn, c_. * (g1 : g_.,_) (g2 : g_.,_) => g1 g2, ∞] ∪ Cases[eqn, g^2, ∞] ∪ Cases[eqn, y_., ∞], 5]

Out[*]//Short=


{g1,i+,i+ g1,i+,j+, g1^2,i+,j+, g1,i+,i+ g1,i+,k+, g1,i+,j+ g1,i+,k+, g1^2,i+,k+, g1,i+,i+ g1,j+,i+, g1,i+,k+ g1,j+,i+, <<298>>, yk+,i+,j+, yk+,i+,k+, yk+,j+,i+, yk+,j+,j+, yk+,j+,k+, yk+,k+,i+, yk+,k+,j+}

In[*]:= Short[eqns = {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y) _.,_ → 0}, 5]

Out[*]//Short=

{2 C102 - $\frac{2 C102}{T} = 0$, $-2 C102 + \frac{2 C102}{T} = 0$, $C104 - \frac{C104}{T} = 0$, $-C104 + \frac{C104}{T} = 0$, <<293>>, $-C182 + \frac{C182}{S T} + 2 C196 - 6 T C196 + \frac{2 T C196}{S} + 4 T^2 C196 - \frac{2 T^2 C196}{S} - 5 C198 + \frac{C198}{S} + \frac{C198}{T} + \frac{C198}{S T} + 5 T C198 - \frac{2 T C198}{S} - 2 T^2 C198 + \frac{T^2 C198}{S} = 0$, $-C191 + \frac{C191}{S} + \frac{C192}{S^2} - \frac{C192}{S} - 2 C196 + \frac{C196}{S} + \frac{C196}{T} - T C196 + 2 T^2 C196 - \frac{T^2 C196}{S} - C198 + 3 T C198 - \frac{T C198}{S} - 2 T^2 C198 + \frac{T^2 C198}{S} = 0$, $-2 C194 + \frac{C194}{S} + \frac{C194}{T} = 0$ }

In[*]:= Short[{sol} = Solve[eqns, vars], 5]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ \begin{aligned} &C_{102} \rightarrow 0, C_{104} \rightarrow 0, C_{105} \rightarrow 0, C_{108} \rightarrow -\frac{T C_{107}}{1+T} - \frac{T C_{113}}{1+T}, \\ &C_{109} \rightarrow -\frac{(-1+T) C_{107}}{1+T} - \frac{(1-T) C_{113}}{T(1+T)}, C_{111} \rightarrow -\frac{T C_{107}}{1+T} - \frac{T C_{113}}{1+T}, C_{112} \rightarrow 0, C_{114} \rightarrow 0, C_{116} \rightarrow 0, \\ &C_{117} \rightarrow 0, C_{119} \rightarrow \frac{T C_{118}}{-1+T} - \frac{S C_{129}}{-1+S}, C_{120} \rightarrow 0, C_{122} \rightarrow 0, C_{123} \rightarrow 0, C_{124} \rightarrow 0, \langle\langle 45 \rangle\rangle, \\ &C_{184} \rightarrow 0, C_{185} \rightarrow -\frac{(-1+S) T C_{171}}{-1+ST}, C_{186} \rightarrow 0, C_{187} \rightarrow -\frac{ST C_{175}}{1+ST} - \frac{ST C_{189}}{1+ST}, C_{188} \rightarrow 0, \\ &C_{190} \rightarrow 0, C_{191} \rightarrow 0, C_{192} \rightarrow 0, C_{193} \rightarrow 0, C_{194} \rightarrow 0, C_{195} \rightarrow 0, C_{196} \rightarrow 0, C_{197} \rightarrow 0, C_{198} \rightarrow 0 \end{aligned} \right\} \right\}$$

In[*]:= Short[sol /. (v_ -> val_) :-> (v = CF[val])]

Out[*]//Short=

$$\left\{ 0, 0, 0, -\frac{T(C_{107} + C_{113})}{1+T}, \langle\langle 66 \rangle\rangle, 0, 0, 0, 0 \right\}$$

In[*]:= Short[R1[1, i, j], 5]

Out[*]//Short=

$$\begin{aligned} &C_1 - \frac{(S T^2 C_{101} + S T C_{103} - S T^2 C_{103}) g_{1,i,i} - C_{103} g_{1,i,j} - (-S T^2 C_{101} + \langle\langle 10 \rangle\rangle) g_{1,j,i}}{S T^2} - \\ &\frac{(\langle\langle 1 \rangle\rangle) \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{T^2} + \langle\langle 73 \rangle\rangle + \frac{S(-C_{152} + \langle\langle 4 \rangle\rangle) g_{\langle\langle 1 \rangle\rangle} g_{3,j,j}}{(-1+S)(-1+ST)} - \\ &\frac{\left(-\frac{(-1+S) S^3 T^2 C_{171}}{-1+ST} + \frac{S^3 T(\langle\langle 1 \rangle\rangle)}{(-1+S)(\langle\langle 1 \rangle\rangle)} - \frac{S^4 T(\langle\langle 1 \rangle\rangle)}{(-1+S)(-1+ST)}\right) g_{\langle\langle 1 \rangle\rangle} g_{\langle\langle 1 \rangle\rangle}}{S^2 T} + \\ &\frac{S T(C_{175} + C_{189}) g_{3,i,i} g_{3,j,j}}{1+ST} - \frac{\left(S^3 T^3 C_{189} + \frac{2 S^3 T^3 (C_{175} + C_{189})}{1+ST} - \frac{2 S^4 T^4 (C_{175} + C_{189})}{1+ST}\right) g_{3,j,i} g_{3,j,j}}{S^2 T^2} \end{aligned}$$

In[*]:= Simplify[R₁[1, i, j] /. {c₁₅₅ → 0, c₁₈₉ → 0, c₁₇₅ → 0, c₁₇₈ → 0, c₁₆₆ → 0, c₁₄₅ → 0}]

Out[*]=

$$\begin{aligned}
& c_1 - \frac{(T c_{101} - (-1 + T) c_{103}) g_{1,i,i} - c_{103} g_{1,i,j}}{T} - \\
& \frac{((-1 + T) T c_{101} - (-1 + T)^2 c_{103} + T (T c_{106} - (-1 + T) c_{110})) g_{1,j,i}}{T} - \\
& \frac{T ((-1 + 3 T) c_{107} + 2 (-1 + T) c_{113}) g_{1,i,i} g_{1,j,i}}{1 + T} + \frac{T (c_{107} + c_{113}) g_{1,i,j} g_{1,j,i}}{1 + T} - \\
& \frac{(-1 + T) T ((-1 + 2 T) c_{107} + (-2 + T) c_{113}) g_{1,j,i}^2}{1 + T} - \frac{((-1 + T) c_{103} + T c_{110}) g_{1,j,j}}{T} + \\
& \frac{T (c_{107} + c_{113}) g_{1,i,i} g_{1,j,j}}{1 + T} + \frac{T (2 (-1 + T) c_{107} + (-3 + T) c_{113}) g_{1,j,i} g_{1,j,j}}{1 + T} - \\
& \frac{(S c_{115} - (-1 + S) c_{121}) g_{2,i,i} - S (-1 + T) c_{129} g_{1,j,i} g_{2,i,i}}{S} + \left(-\frac{T c_{118}}{-1 + T} + \frac{S c_{129}}{-1 + S} \right) g_{1,j,j} g_{2,i,i} - \\
& \frac{c_{121} g_{2,i,j}}{S} - \frac{((-1 + S) S c_{115} - (-1 + S)^2 c_{121} + S (S c_{128} - (-1 + S) c_{136})) g_{2,j,i}}{S} - \\
& \frac{(-1 + S) T c_{118} g_{1,i,i} g_{2,j,i}}{-1 + T} - ((-1 + S) T c_{118} + S (-1 + T) c_{129}) g_{1,j,i} g_{2,j,i} + S c_{129} g_{1,j,j} g_{2,j,i} - \\
& \frac{S ((-1 + 3 S) c_{133} + 2 (-1 + S) c_{143}) g_{2,i,i} g_{2,j,i}}{1 + S} + \frac{S (c_{133} + c_{143}) g_{2,i,j} g_{2,j,i}}{1 + S} - \\
& \frac{(-1 + S) S ((-1 + 2 S) c_{133} + (-2 + S) c_{143}) g_{2,j,i}^2}{1 + S} - \frac{((-1 + S) c_{121} + S c_{136}) g_{2,j,j}}{S} + \\
& \frac{((-1 + S) T c_{118} - S (-1 + T) c_{129}) g_{1,i,i} g_{2,j,j}}{(-1 + S) (-1 + T)} + T c_{118} g_{1,j,i} g_{2,j,j} + \frac{S (c_{133} + c_{143}) g_{2,i,i} g_{2,j,j}}{1 + S} + \\
& \frac{S (2 (-1 + S) c_{133} + (-3 + S) c_{143}) g_{2,j,i} g_{2,j,j}}{1 + S} - \frac{S (-1 + T) T c_{167} g_{1,j,i} g_{3,i,i}}{-1 + S T} + \\
& T \left(-\frac{c_{148}}{-1 + T} + \frac{S c_{167}}{-1 + S T} \right) g_{1,j,j} g_{3,i,i} - \frac{(-1 + S) S T c_{171} g_{2,j,i} g_{3,i,i}}{-1 + S T} + \\
& S \left(-\frac{c_{152}}{-1 + S} + \frac{T c_{171}}{-1 + S T} \right) g_{2,j,j} g_{3,i,i} - \frac{T (-1 + S T) c_{148} g_{1,i,i} g_{3,j,i}}{-1 + T} - \\
& T ((-1 + S T) c_{148} + S (-1 + T) c_{167}) g_{1,j,i} g_{3,j,i} + S T c_{167} g_{1,j,j} g_{3,j,i} - \\
& \frac{S (-1 + S T) c_{152} g_{2,i,i} g_{3,j,i}}{-1 + S} - S ((-1 + S T) c_{152} + (-1 + S) T c_{171}) g_{2,j,i} g_{3,j,i} + \\
& S T c_{171} g_{2,j,j} g_{3,j,i} + \frac{T ((-1 + S T) c_{148} - S (-1 + T) c_{167}) g_{1,i,i} g_{3,j,j}}{(-1 + T) (-1 + S T)} + \\
& T c_{148} g_{1,j,i} g_{3,j,j} + \frac{S ((-1 + S T) c_{152} - (-1 + S) T c_{171}) g_{2,i,i} g_{3,j,j}}{(-1 + S) (-1 + S T)} + S c_{152} g_{2,j,i} g_{3,j,j}
\end{aligned}$$

The most general no-mixed-terms solution:

In[*]:= Simplify[{R₁[-1, i, j], R₁[1, i, j]} /. {c₁₁₈|c₁₂₉|c₁₄₈|c₁₅₂|c₁₆₇|c₁₇₁ → 0}]

Out[*]=

$$\left\{ -c_1 + c_{101} g_{1,i,i} + c_{103} g_{1,i,j} + c_{106} g_{1,j,i} + c_{107} g_{1,i,i} g_{1,j,i} - \frac{T (c_{107} + c_{113}) g_{1,i,j} g_{1,j,i}}{1 + T} - \right.$$

$$\begin{aligned}
 & \frac{(-1+T)(Tc_{107}-c_{113})g_{1,j,i}^2}{T(1+T)} + c_{110}g_{1,j,j} - \frac{T(c_{107}+c_{113})g_{1,i,i}g_{1,j,j}}{1+T} + c_{113}g_{1,j,i}g_{1,j,j} + \\
 & c_{115}g_{2,i,i} + c_{121}g_{2,i,j} + c_{128}g_{2,j,i} + c_{133}g_{2,i,i}g_{2,j,i} - \frac{S(c_{133}+c_{143})g_{2,i,j}g_{2,j,i}}{1+S} - \\
 & \frac{(-1+S)(Sc_{133}-c_{143})g_{2,j,i}^2}{S(1+S)} + c_{136}g_{2,j,j} - \frac{S(c_{133}+c_{143})g_{2,i,i}g_{2,j,j}}{1+S} + c_{143}g_{2,j,i}g_{2,j,j} + \\
 & c_{145}g_{3,i,i} + c_{155}g_{3,i,j} + c_{166}g_{3,j,i} + c_{175}g_{3,i,i}g_{3,j,i} - \frac{ST(c_{175}+c_{189})g_{3,i,j}g_{3,j,i}}{1+ST} - \\
 & \frac{(-1+ST)(STc_{175}-c_{189})g_{3,j,i}^2}{ST(1+ST)} + c_{178}g_{3,j,j} - \frac{ST(c_{175}+c_{189})g_{3,i,i}g_{3,j,j}}{1+ST} + c_{189}g_{3,j,i}g_{3,j,j}, \\
 c_1 - & \frac{(Tc_{101}-(-1+T)c_{103})g_{1,i,i}}{T} - \frac{c_{103}g_{1,i,j}}{T} - \\
 & \frac{((-1+T)Tc_{101}-(-1+T)^2c_{103}+T(Tc_{106}-(-1+T)c_{110}))g_{1,j,i}}{T} - \\
 & \frac{T((-1+3T)c_{107}+2(-1+T)c_{113})g_{1,i,i}g_{1,j,i}}{1+T} + \frac{T(c_{107}+c_{113})g_{1,i,j}g_{1,j,i}}{1+T} - \\
 & \frac{(-1+T)T((-1+2T)c_{107}+(-2+T)c_{113})g_{1,j,i}^2}{1+T} - \\
 & \frac{((-1+T)c_{103}+Tc_{110})g_{1,j,j}}{T} + \frac{T(c_{107}+c_{113})g_{1,i,i}g_{1,j,j}}{1+T} + \\
 & \frac{T(2(-1+T)c_{107}+(-3+T)c_{113})g_{1,j,i}g_{1,j,j}}{1+T} - \frac{(Sc_{115}-(-1+S)c_{121})g_{2,i,i}}{S} - \\
 & \frac{c_{121}g_{2,i,j}}{S} - \frac{((-1+S)Sc_{115}-(-1+S)^2c_{121}+S(Sc_{128}-(-1+S)c_{136}))g_{2,j,i}}{S} - \\
 & \frac{S((-1+3S)c_{133}+2(-1+S)c_{143})g_{2,i,i}g_{2,j,i}}{1+S} + \frac{S(c_{133}+c_{143})g_{2,i,j}g_{2,j,i}}{1+S} - \\
 & \frac{(-1+S)S((-1+2S)c_{133}+(-2+S)c_{143})g_{2,j,i}^2}{1+S} - \frac{((-1+S)c_{121}+Sc_{136})g_{2,j,j}}{S} + \\
 & \frac{S(c_{133}+c_{143})g_{2,i,i}g_{2,j,j}}{1+S} + \frac{S(2(-1+S)c_{133}+(-3+S)c_{143})g_{2,j,i}g_{2,j,j}}{1+S} - \\
 & \frac{(STc_{145}+(1-ST)c_{155})g_{3,i,i}}{ST} - \frac{c_{155}g_{3,i,j}}{ST} - \\
 & \frac{(ST(-1+ST)c_{145}-(-1+ST)^2c_{155}+ST(STc_{166}+(1-ST)c_{178}))g_{3,j,i}}{ST} - \\
 & \frac{ST((-1+3ST)c_{175}+2(-1+ST)c_{189})g_{3,i,i}g_{3,j,i}}{1+ST} + \frac{ST(c_{175}+c_{189})g_{3,i,j}g_{3,j,i}}{1+ST} - \\
 & \frac{ST(-1+ST)((-1+2ST)c_{175}+(-2+ST)c_{189})g_{3,j,i}^2}{1+ST} - \frac{((-1+ST)c_{155}+STc_{178})g_{3,j,j}}{ST} + \\
 & \left. \frac{ST(c_{175}+c_{189})g_{3,i,i}g_{3,j,j}}{1+ST} + \frac{ST(2(-1+ST)c_{175}+(-3+ST)c_{189})g_{3,j,i}g_{3,j,j}}{1+ST} \right\}
 \end{aligned}$$

The most general no-mixed-terms quadratic-only solution:

In[*]:= **Simplify**[{**R**₁[-1, i, j], **R**₁[1, i, j]} /. {**C**_{1|101|103|106|110|115|118|121|128|129|136|145|148|152|155|166|167|171|178} → 0}]

Out[*]=

$$\left\{ \begin{aligned} & C_{107} g_{1,i,i} g_{1,j,i} - \frac{T (C_{107} + C_{113}) g_{1,i,j} g_{1,j,i}}{1 + T} - \\ & \frac{(-1 + T) (T C_{107} - C_{113}) g_{1,j,i}^2}{T (1 + T)} - \frac{T (C_{107} + C_{113}) g_{1,i,i} g_{1,j,j}}{1 + T} + C_{113} g_{1,j,i} g_{1,j,j} + \\ & C_{133} g_{2,i,i} g_{2,j,i} - \frac{S (C_{133} + C_{143}) g_{2,i,j} g_{2,j,i}}{1 + S} - \frac{(-1 + S) (S C_{133} - C_{143}) g_{2,j,i}^2}{S (1 + S)} - \\ & \frac{S (C_{133} + C_{143}) g_{2,i,i} g_{2,j,j}}{1 + S} + C_{143} g_{2,j,i} g_{2,j,j} + C_{175} g_{3,i,i} g_{3,j,i} - \frac{S T (C_{175} + C_{189}) g_{3,i,j} g_{3,j,i}}{1 + S T} - \\ & \frac{(-1 + S T) (S T C_{175} - C_{189}) g_{3,j,i}^2}{S T (1 + S T)} - \frac{S T (C_{175} + C_{189}) g_{3,i,i} g_{3,j,j}}{1 + S T} + C_{189} g_{3,j,i} g_{3,j,j}, \\ & - \frac{T ((-1 + 3 T) C_{107} + 2 (-1 + T) C_{113}) g_{1,i,i} g_{1,j,i}}{1 + T} + \frac{T (C_{107} + C_{113}) g_{1,i,j} g_{1,j,i}}{1 + T} - \\ & \frac{(-1 + T) T ((-1 + 2 T) C_{107} + (-2 + T) C_{113}) g_{1,j,i}^2}{1 + T} + \frac{T (C_{107} + C_{113}) g_{1,i,i} g_{1,j,j}}{1 + T} + \\ & \frac{T (2 (-1 + T) C_{107} + (-3 + T) C_{113}) g_{1,j,i} g_{1,j,j}}{1 + T} - \frac{S ((-1 + 3 S) C_{133} + 2 (-1 + S) C_{143}) g_{2,i,i} g_{2,j,i}}{1 + S} + \\ & \frac{S (C_{133} + C_{143}) g_{2,i,j} g_{2,j,i}}{1 + S} - \frac{(-1 + S) S ((-1 + 2 S) C_{133} + (-2 + S) C_{143}) g_{2,j,i}^2}{1 + S} + \\ & \frac{S (C_{133} + C_{143}) g_{2,i,i} g_{2,j,j}}{1 + S} + \frac{S (2 (-1 + S) C_{133} + (-3 + S) C_{143}) g_{2,j,i} g_{2,j,j}}{1 + S} - \\ & \frac{S T ((-1 + 3 S T) C_{175} + 2 (-1 + S T) C_{189}) g_{3,i,i} g_{3,j,i}}{1 + S T} + \frac{S T (C_{175} + C_{189}) g_{3,i,j} g_{3,j,i}}{1 + S T} - \\ & \frac{S T (-1 + S T) ((-1 + 2 S T) C_{175} + (-2 + S T) C_{189}) g_{3,j,i}^2}{1 + S T} + \\ & \left. \frac{S T (C_{175} + C_{189}) g_{3,i,i} g_{3,j,j}}{1 + S T} + \frac{S T (2 (-1 + S T) C_{175} + (-3 + S T) C_{189}) g_{3,j,i} g_{3,j,j}}{1 + S T} \right\} \end{aligned}$$

The most general quadratic-only solution: (has mixed terms!)

In[*]:= **Simplify**[{**R**₁[-1, i, j], **R**₁[1, i, j]} /. {**C**_{1|101|103|106|110|115|121|128|136|145|155|166|178} → 0}]

Out[*]=

$$\left\{ \begin{aligned} & C_{107} g_{1,i,i} g_{1,j,i} - \frac{T (C_{107} + C_{113}) g_{1,i,j} g_{1,j,i}}{1 + T} - \\ & \frac{(-1 + T) (T C_{107} - C_{113}) g_{1,j,i}^2}{T (1 + T)} - \frac{T (C_{107} + C_{113}) g_{1,i,i} g_{1,j,j}}{1 + T} + C_{113} g_{1,j,i} g_{1,j,j} + \\ & C_{118} g_{1,j,i} g_{2,i,i} + \frac{((-1 + S) T C_{118} - S (-1 + T) C_{129}) g_{1,j,j} g_{2,i,i}}{(-1 + S) (-1 + T)} + C_{129} g_{1,i,i} g_{2,j,i} - \end{aligned} \right.$$

$$\begin{aligned}
 & \frac{((-1+S) T c_{118} + S (-1+T) c_{129}) g_{1,j,i} g_{2,j,i}}{S T} - \frac{(-1+S) T c_{118} g_{1,j,j} g_{2,j,i}}{S (-1+T)} + \\
 & c_{133} g_{2,i,i} g_{2,j,i} - \frac{S (c_{133} + c_{143}) g_{2,i,j} g_{2,j,i}}{1+S} - \frac{(-1+S) (S c_{133} - c_{143}) g_{2,j,i}^2}{S (1+S)} + \\
 & \left(-\frac{T c_{118}}{-1+T} + \frac{S c_{129}}{-1+S} \right) g_{1,i,i} g_{2,j,j} - \frac{S (-1+T) c_{129} g_{1,j,i} g_{2,j,j}}{(-1+S) T} - \frac{S (c_{133} + c_{143}) g_{2,i,i} g_{2,j,j}}{1+S} + \\
 & c_{143} g_{2,j,i} g_{2,j,j} + c_{148} g_{1,j,i} g_{3,i,i} + \frac{T ((-1+ST) c_{148} - S (-1+T) c_{167}) g_{1,j,j} g_{3,i,i}}{(-1+T) (-1+ST)} + \\
 & c_{152} g_{2,j,i} g_{3,i,i} + \frac{S ((-1+ST) c_{152} - (-1+S) T c_{171}) g_{2,j,j} g_{3,i,i}}{(-1+S) (-1+ST)} + c_{167} g_{1,i,i} g_{3,j,i} - \\
 & \frac{((-1+ST) c_{148} + S (-1+T) c_{167}) g_{1,j,i} g_{3,j,i}}{S T} - \frac{(-1+ST) c_{148} g_{1,j,j} g_{3,j,i}}{S (-1+T)} + \\
 & c_{171} g_{2,i,i} g_{3,j,i} - \frac{((-1+ST) c_{152} + (-1+S) T c_{171}) g_{2,j,i} g_{3,j,i}}{S T} - \\
 & \frac{(-1+ST) c_{152} g_{2,j,j} g_{3,j,i}}{(-1+S) T} + c_{175} g_{3,i,i} g_{3,j,i} - \frac{S T (c_{175} + c_{189}) g_{3,i,j} g_{3,j,i}}{1+ST} - \\
 & \frac{(-1+ST) (S T c_{175} - c_{189}) g_{3,j,i}^2}{S T (1+ST)} + T \left(-\frac{c_{148}}{-1+T} + \frac{S c_{167}}{-1+ST} \right) g_{1,i,i} g_{3,j,j} - \\
 & \frac{S (-1+T) c_{167} g_{1,j,i} g_{3,j,j}}{-1+ST} + S \left(-\frac{c_{152}}{-1+S} + \frac{T c_{171}}{-1+ST} \right) g_{2,i,i} g_{3,j,j} - \\
 & \frac{(-1+S) T c_{171} g_{2,j,i} g_{3,j,j}}{-1+ST} - \frac{S T (c_{175} + c_{189}) g_{3,i,i} g_{3,j,j}}{1+ST} + c_{189} g_{3,j,i} g_{3,j,j}, \\
 & - \frac{T ((-1+3T) c_{107} + 2 (-1+T) c_{113}) g_{1,i,i} g_{1,j,i}}{1+T} + \frac{T (c_{107} + c_{113}) g_{1,i,j} g_{1,j,i}}{1+T} - \\
 & \frac{(-1+T) T ((-1+2T) c_{107} + (-2+T) c_{113}) g_{1,j,i}^2}{1+T} + \frac{T (c_{107} + c_{113}) g_{1,i,i} g_{1,j,j}}{1+T} + \\
 & T \frac{(2 (-1+T) c_{107} + (-3+T) c_{113}) g_{1,j,i} g_{1,j,j}}{1+T} - \frac{S (-1+T) c_{129} g_{1,j,i} g_{2,i,i}}{-1+S} + \\
 & \left(-\frac{T c_{118}}{-1+T} + \frac{S c_{129}}{-1+S} \right) g_{1,j,j} g_{2,i,i} - \frac{(-1+S) T c_{118} g_{1,i,i} g_{2,j,i}}{-1+T} - \\
 & ((-1+S) T c_{118} + S (-1+T) c_{129}) g_{1,j,i} g_{2,j,i} + S c_{129} g_{1,j,j} g_{2,j,i} - \\
 & \frac{S ((-1+3S) c_{133} + 2 (-1+S) c_{143}) g_{2,i,i} g_{2,j,i}}{1+S} + \frac{S (c_{133} + c_{143}) g_{2,i,j} g_{2,j,i}}{1+S} - \\
 & \frac{(-1+S) S ((-1+2S) c_{133} + (-2+S) c_{143}) g_{2,j,i}^2}{1+S} + \frac{((-1+S) T c_{118} - S (-1+T) c_{129}) g_{1,i,i} g_{2,j,j}}{(-1+S) (-1+T)} + \\
 & T c_{118} g_{1,j,i} g_{2,j,j} + \frac{S (c_{133} + c_{143}) g_{2,i,i} g_{2,j,j}}{1+S} + \frac{S (2 (-1+S) c_{133} + (-3+S) c_{143}) g_{2,j,i} g_{2,j,j}}{1+S} - \\
 & \frac{S (-1+T) T c_{167} g_{1,j,i} g_{3,i,i}}{-1+ST} + T \left(-\frac{c_{148}}{-1+T} + \frac{S c_{167}}{-1+ST} \right) g_{1,j,j} g_{3,i,i} - \\
 & \frac{(-1+S) S T c_{171} g_{2,j,i} g_{3,i,i}}{-1+ST} + S \left(-\frac{c_{152}}{-1+S} + \frac{T c_{171}}{-1+ST} \right) g_{2,j,j} g_{3,i,i} - \frac{T (-1+ST) c_{148} g_{1,i,i} g_{3,j,i}}{-1+T} -
 \end{aligned}$$

$$\begin{aligned}
 & T ((-1 + ST) c_{148} + S(-1 + T) c_{167}) g_{1,j,i} g_{3,j,i} + ST c_{167} g_{1,j,j} g_{3,j,i} - \\
 & \frac{S(-1 + ST) c_{152} g_{2,i,i} g_{3,j,i}}{-1 + S} - S((-1 + ST) c_{152} + (-1 + S) T c_{171}) g_{2,j,i} g_{3,j,i} + \\
 & ST c_{171} g_{2,j,j} g_{3,j,i} - \frac{ST((-1 + 3ST) c_{175} + 2(-1 + ST) c_{189}) g_{3,i,i} g_{3,j,i}}{1 + ST} + \\
 & \frac{ST(c_{175} + c_{189}) g_{3,i,j} g_{3,j,i}}{1 + ST} - \frac{ST(-1 + ST)((-1 + 2ST) c_{175} + (-2 + ST) c_{189}) g_{3,j,i}^2}{1 + ST} + \\
 & \frac{T((-1 + ST) c_{148} - S(-1 + T) c_{167}) g_{1,i,i} g_{3,j,j}}{(-1 + T)(-1 + ST)} + T c_{148} g_{1,j,i} g_{3,j,j} + \\
 & \frac{S((-1 + ST) c_{152} - (-1 + S) T c_{171}) g_{2,i,i} g_{3,j,j}}{(-1 + S)(-1 + ST)} + S c_{152} g_{2,j,i} g_{3,j,j} + \\
 & \left. \frac{ST(c_{175} + c_{189}) g_{3,i,i} g_{3,j,j}}{1 + ST} + \frac{ST(2(-1 + ST) c_{175} + (-3 + ST) c_{189}) g_{3,j,i} g_{3,j,j}}{1 + ST} \right\}
 \end{aligned}$$