Pensieve header: Full testing of the sl_2 portfolio. Continues pensieve://Projects/SL2Portfolio2/PortfolioTesting.nb. Time: 136.744.

Startup

(Alt) In[∘]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine.m
<< Objects.m
<< KT.m
```

(Alt) Out[∘]=
```
{2021, 11, 29, 10, 54, 5.2722470}
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

(Alt) In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

(Alt) In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

(Alt) In[∘]:=
```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, cm → cm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
   aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
   dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, K̄ink → K̄ink_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

$$\text{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\ 0\right]$$

$$\text{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,(\beta_i+\beta_j)+y_k\,(\eta_i+\eta_j),\ -y_k\,\beta_i\,\eta_j\right]$$

$$\text{cm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+y_k\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)+b_k\,(\beta_i+\beta_j+\eta_j\,\xi_i)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\right.$$
$$\left.a_k\,\eta_j\,\xi_i-\frac{1}{2}\,b_k\,\eta_j^2\,\xi_i^2-\frac{y_k\,\eta_j\,(\beta_i+\eta_j\,\xi_i)}{\mathcal{A}_i}-\frac{x_k\,\xi_i\,(\beta_j+\eta_j\,\xi_i)}{\mathcal{A}_j}\right]$$

$$\text{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i+\alpha_j)+b_k\,\beta_i+b_k\,\beta_j+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}-\frac{(-1+B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$$
$$\left.-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}-\frac{(-1+3\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}-\frac{(-1+3\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{(-1+B_k)\times(-1+3\,B_k)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right]$$

$$\text{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\hbar\,x_j\,y_i,\ -\frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right]$$

$$\overline{\text{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i-\frac{\hbar\,x_j\,y_i}{B_i},\ -\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right]$$

$$\text{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar}+\frac{\eta_i\,\xi_j}{\hbar},\ \frac{\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$$

$$\text{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\overline{\text{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ \hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$$

$$\text{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ -\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

(Alt) Out[∘]=

$$\overline{\text{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ \frac{\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$$

$$\text{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-b_i\,\beta_i-\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i-\frac{(-1+B_i)\,\mathcal{A}_i\,\eta_i\,\xi_i}{\hbar\,B_i},\right.$$
$$\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}+\frac{(-1+B_i)\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}-$$
$$\left.\frac{(-1+B_i)\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}-\frac{(-3+B_i)\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i}{2\,B_i^2}-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2-\frac{(-3+B_i)\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2}{2\,B_i}-\frac{(-3+B_i)\times(-1+B_i)\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\right]$$

$$\text{a}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+x_j\,\xi_i+x_k\,\xi_i,\ -\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$$

$$\text{b}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[(b_j+b_k)\,\beta_i+B_k\,y_j\,\eta_i+y_k\,\eta_i,\ \frac{1}{2}\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\right]$$

$$\text{d}\triangle \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+(b_j+b_k)\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$$
$$\left.\frac{1}{2}\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$$

$$\text{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2},\ -\frac{\hbar\,a_i}{2}\right]$$

$$\overline{\text{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2},\ \frac{\hbar\,a_i}{2}\right]$$

$$\text{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2}+\hbar\,a_i\,b_i+\hbar\,x_i\,y_i,\ \frac{\hbar\,a_i}{2}-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2\right]$$

$$\overline{\text{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2}-\hbar\,a_i\,b_i-\frac{\hbar\,x_i\,y_i}{B_i},\ -\frac{\hbar\,a_i}{2}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right]$$

$$\text{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-t_i\,\beta_i+y_i\,\eta_i+x_i\,\xi_i,\ a_i\,\beta_i\right]$$

$$\text{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+y_i\,\eta_i+x_i\,\xi_i-b_i\,\tau_i,\ a_i\,\tau_i\right]$$

Check that on the generators this agrees with our conventions in the handout:

```
In[•]:= 𝔼2Λ[ℰ_] := Module[{k}, Sum[ℰ[k] ϵ^(k-1), {k, 0, ℰ[$]}]];
Timing@Block[{$k = 2}, {
   {
    "[a,x]" → 𝔼2Λ[𝔼_{}→{1,2}[0, a₂ x₁] // am_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, a₁ x₂] // am_{1,2→1}],
    "[b,y]" → 𝔼2Λ[𝔼_{}→{1,2}[0, y₂ b₁, 0] // bm_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, y₁ b₂, 0] // bm_{1,2→1}]
   } /. z_1 → z,
   {
   "Δ[y]" → Last[𝔼_{}→{1}[0, y₁] // bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼_{}→{1}[0, b₁] // bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{}→{1}[0, a₁] // aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{}→{1}[0, x₁] // aΔ_{1→1,2}]},
   {
    "S(a)" → ((𝔼_{}→{1}[0, a₁] // aS₁)[1]),
    "S(x)" → ((𝔼_{}→{1}[0, x₁] // aS₁)[1]),
    "S(b)" → ((𝔼_{}→{1}[0, b₁] // bS₁)[1]),
    "S(y)" → ((𝔼_{}→{1}[0, y₁] // bS₁)[1])
   } /. z_1 → z
  }]
```

Out[•]= $\{3.57813,$
$\{\{[a,x] \to -x,\ [b,y] \to -y\,\epsilon\},\ \{\Delta[y] \to B_2\, y_1 + y_2,\ \Delta[b] \to b_1 + b_2,\ \Delta[a] \to a_1 + a_2,\ \Delta[x] \to x_1 + x_2\},$
$\{S(a) \to -a,\ S(x) \to -x,\ S(b) \to -b,\ S(y) \to -\frac{y}{B}\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
In[•]:= Timing@Block[{$k = 3},
   HL /@ {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
  ]
```

Out[•]= {0.28125, {True, True}}

R and *P* are inverses:

```
In[•]:= Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ aσ_{k→j}]}]
```

Out[•]= $\{0.453125, \{\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, b_i + \hbar\, x_j\, y_i,\ -\frac{1}{4}\hbar^3 x_j^2 y_i^2,\ \frac{1}{9}\hbar^5 x_j^3 y_i^3,\ \frac{1}{48}\left(\hbar^5 x_j^2 y_i^2 - 3\hbar^7 x_j^4 y_i^4\right)\right],$
$\mathbb{E}_{\{i,k\}\to\{\}}\left[\frac{\alpha_k \beta_i}{\hbar} + \frac{\eta_i \xi_k}{\hbar},\ \frac{\eta_i^2 \xi_k^2}{4\hbar},\ \frac{1}{8}\eta_i^2 \xi_k^2 + \frac{5\eta_i^3 \xi_k^3}{36\hbar},\ \frac{1}{24}\hbar\,\eta_i^2 \xi_k^2 + \frac{1}{6}\eta_i^3 \xi_k^3 + \frac{5\eta_i^4 \xi_k^4}{48\hbar}\right],\ \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
In[•]:= Timing[HL /@ {(aS̄₁ // aS₁) ≡ aσ_{1→1}, (bS̄₁ // bS₁) ≡ bσ_{1→1}}]
```

Out[•]= {0.640625, {True, True}}

(co)-associativity on both sides

```
In[∘]:= Timing[
   HL /@ {(aΔ1→1,2 // aΔ2→2,3) ≡ (aΔ1→1,3 // aΔ1→1,2), (bΔ1→1,2 // bΔ2→2,3) ≡ (bΔ1→1,3 // bΔ1→1,2),
     (am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}]
Out[∘]= {0.546875, {True, True, True, True}}
```

Δ is an algebra morphism

```
In[∘]:= Timing[HL /@ {(am1,2→1 // aΔ1→1,2) ≡ ((aΔ1→1,3 aΔ2→2,4) // (am3,4→2 am1,2→1)),
     (bm1,2→1 // bΔ1→1,2) ≡ ((bΔ1→1,3 bΔ2→2,4) // (bm3,4→2 bm1,2→1))}]
Out[∘]= {1.5, {True, True}}
```

An explicit formula for aS_i

```
In[∘]:= Timing@Block[{$k = 4}, HL[
     aSi ≡ (Δ2𝔼{i}→{i,j}[
        -αi aj - ξi xi +
         Log@Sum[Expand[(𝕖^(ξi xi) (-ℏ ϵ)^k)/(2^k k!) Nest[Expand[xi^2 ∂{xi,2} #] &, 𝕖^(-ξi 𝕖^(ℏ ϵ ai) xi), k]], {k, 0, $k}]
        ] // ami,j→i)
    ]]
Out[∘]= {6.21875, True}
```

S is convolution inverse of id

```
In[∘]:= Timing[HL[# ≡ sϵ1 sη1] & /@ {
     (aΔ1→1,2 // aS1) // am1,2→1, (aΔ1→1,2 // aS2) // am1,2→1,
     (bΔ1→1,2 // bS1) // bm1,2→1, (bΔ1→1,2 // bS2) // bm1,2→1}]
Out[∘]= {1., {True, True, True, True}}
```

But not with the opposite product:

In[∘]:=
```
Timing[Short[# ≡ sϵ₁ sη₁] & /@ {
   (aΔ₁→₁,₂ ~ B₁ ~ aS₁) ~ B₁,₂ ~ am₂,₁→₁, (aΔ₁→₁,₂ ~ B₂ ~ aS₂) ~ B₁,₂ ~ am₂,₁→₁,
   (bΔ₁→₁,₂ ~ B₁ ~ bS₁) ~ B₁,₂ ~ bm₂,₁→₁, (bΔ₁→₁,₂ ~ B₂ ~ bS₂) ~ B₁,₂ ~ bm₂,₁→₁}]
```

Out[∘]=

$$\{0.015625,\ \{B_{1,2}[B_1[\mathbb{E}_{\{1\}\to\{1,2\}}[\ll 1\gg],\ \mathbb{E}_{\{1\}\to\{1\}}[-a_1\alpha_1 - x_1\mathcal{A}_1\xi_1,\ -\hbar a_1 x_1 \mathcal{A}_1\xi_1 - \frac{1}{2}\ll 3\gg \ll 1\gg,\ \ll 1\gg]],$$
$$\ll 1\gg] \equiv \ll 1\gg,\ B_{1,2}[B_2[\mathbb{E}_{\{1\}\to\{1,2\}}[\ll 1\gg],$$
$$\mathbb{E}_{\{2\}\to\{2\}}[-a_2\alpha_2 - x_2\mathcal{A}_2\xi_2,\ -\hbar a_2 x_2\mathcal{A}_2\xi_2 - \frac{1}{2}\ll 3\gg \ll 1\gg,\ \ll 1\gg]],\ \ll 1\gg] \equiv \ll 1\gg,$$
$$B_{1,2}[\ll 1\gg,\ \mathbb{E}_{\{2,1\}\to\{1\}}[b_1(\beta_1+\beta_2) + y_1(\eta_1+\eta_2),\ -y_1\beta_2\eta_1,\ \frac{1}{2}y_1\beta_2^2\eta_1]] \equiv \mathbb{E}_{\{1\}\to\ll 1\gg}[0,0,0],$$
$$B_{1,2}[\ll 1\gg,\ \mathbb{E}_{\{2,1\}\to\{1\}}[b_1(\beta_1+\beta_2) + y_1(\eta_1+\eta_2),\ -y_1\beta_2\eta_1,\ \frac{1}{2}y_1\beta_2^2\eta_1]] \equiv \mathbb{E}_{\{1\}\to\ll 1\gg}[0,0,0]\}\}$$

S is an algebra anti-(co)morphism

In[∘]:=
```
Timing[HL /@ {(am₁,₂→₁ // aS₁) ≡ ((aS₁ aS₂) // am₂,₁→₁), (bm₁,₂→₁ // bS₁) ≡ ((bS₁ bS₂) // bm₂,₁→₁),
   (aS₁ // aΔ₁→₁,₂) ≡ (aΔ₁→₂,₁ // (aS₁ aS₂)), (bS₁ // bΔ₁→₁,₂) ≡ (bΔ₁→₂,₁ // (bS₁ bS₂))}]
```

Out[∘]= {1.03125, {True, True, True, True}}

Pairing axioms

In[∘]:=
```
Timing[HL /@ {((bm₁,₂→₁ sϒ₃→₀,₀,₃,₃ // sϵ₀) // P₁,₃) ≡
    (( (sϒ₁→₁,₁,₀,₀ // sϵ₀) (sϒ₂→₂,₂,₀,₀ // sϵ₀) aΔ₃→₄,₅) // P₁,₄ // P₂,₅),
   ((bΔ₁→₁,₂ (sϒ₃→₀,₀,₃,₃ // sϵ₀) (sϒ₄→₀,₀,₄,₄ // sϵ₀)) // P₁,₃ // P₂,₄) ≡
    (((sϒ₁→₁,₁,₀,₀ // sϵ₀) am₃,₄→₃) // P₁,₃ )}]
```

Out[∘]= {1.375, {True, True}}

In[∘]:=
```
Timing[HL /@ {((bS₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ aS₂) // P₁,₂),
   ((b̅S̅₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ a̅S̅₂) // P₁,₂)}]
```

Out[∘]= {0.796875, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

In[∘]:=
```
(*Timing@{{
    "[a,y]"→ ((𝔼{}→{1,2}[0,0,y2a1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,y1a2]~B1,2~dm1,2→1)[3]),
    "[b,x]"   →
     ((𝔼{}→{1,2}[0,0,x2b1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,x1b2]~B1,2~dm1,2→1)[3]),"xy-qyx"→
     ((𝔼{}→{1,2}[0,0,x1y2]~B1,2~dm1,2→1)[3]-(1+ϵ)(𝔼{}→{1,2}[0,0,y1x2]~B1,2~dm1,2→1)[3])
   }/.{z_1→z}//Expand//Factor,
  {
   "Δ(a)"→((𝔼{}→{1}[0,0,a1]~B1~dΔ1→1,2)[3]),
   "Δ(x)"→((𝔼{}→{1}[0,0,x1]~B1~dΔ1→1,2)[3]),
   "Δ(b)"→((𝔼{}→{1}[0,0,b1]~B1~dΔ1→1,2)[3]),
   "Δ(y)"→((𝔼{}→{1}[0,0,y1]~B1~dΔ1→1,2)[3])
  }//Simplify,
  {
    "S(a)"→((𝔼{}→{1}[0,0,a1]~B1~dS1)[3]),
    "S(x)"→((𝔼{}→{1}[0,0,x1]~B1~dS1)[3]),
    "S(b)"→((𝔼{}→{1}[0,0,b1]~B1~dS1)[3]),
    "S(y)"→((𝔼{}→{1}[0,0,y1]~B1~dS1)[3])
   }/.{z_1→z}//Simplify
 }*)
```

In[∘]:=
```
{HL[((sΥ1→0,0,1,1 // sϵ0) (sΥ2→0,0,2,2 // sϵ0) // dm1,2→1) ≡ am1,2→1],
 HL[((sΥ1→1,1,0,0 // sϵ0) (sΥ2→2,2,0,0 // sϵ0) // dm1,2→1) ≡ bm1,2→1]}
```

Out[∘]= {True, True}

(co)-associativity

In[∘]:=
```
Timing[Block[{$k = 1},
  HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
]
```

Out[∘]= {0.421875, {True, True}}

Δ is an algebra morphism

In[∘]:=
```
Timing@HL[(dm1,2→1 // dΔ1→1,2) ≡ ((dΔ1→1,3 dΔ2→2,4) // (dm3,4→2 dm1,2→1))]
```

Out[∘]= {1.96875, True}

dS and $\overline{\mathrm{dS}}$ are inverses:

In[∘]:=
```
Timing@HL[(dS̄1 // dS1) ≡ dσ1→1]
```

Out[∘]= {1.625, True}

S_2 inverts R, but not S_1:

In[∘]:=
```
Timing@{(R_{1,2} // dS_1) ≡ R̄_{1,2}, HL[(R_{1,2} // dS_2) ≡ R̄_{1,2}]}
```

Out[∘]=

$$\left\{0.3125, \left\{\frac{\hbar^2 x_2 y_1}{B_1} - \frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} == -\frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} \,\&\&\, -\frac{\hbar^3 x_2 y_1}{2 B_1} + \frac{\hbar^3 a_2 x_2 y_1}{B_1} - \frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{2\hbar^4 x_2^2 y_1^2}{B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3} == -\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{\hbar^4 x_2^2 y_1^2}{2 B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3}, \text{True}\right\}\right\}$$

dS is convolution inverse of id

In[∘]:=
```
Timing[HL[# ≡ dϵ_1 dη_1] & /@ {(dΔ_{1→1,2} // dS_1) // dm_{1,2→1}, (dΔ_{1→1,2} // dS_2) // dm_{1,2→1}}]
```

Out[∘]= {2.10938, {True, True}}

dS is a (co)-algebra anti-morphism

In[∘]:=
```
Timing[HL /@
  Expand /@ {(dm_{1,2→1} // dS_1) ≡ ((dS_1 dS_2) // dm_{2,1→1}), (dS_1 // dΔ_{1→1,2}) ≡ (dΔ_{1→2,1} // (dS_1 dS_2))}]
```

Out[∘]= {3.42188, {True, True}}

Quasi-triangular axiom 1:

In[∘]:=
```
Timing[
 HL /@ {(R_{1,3} // dΔ_{1→1,2}) ≡ ((R_{1,4} R_{2,3}) // dm_{3,4→3}), (R_{1,2} // dΔ_{2→2,3}) ≡ ((R_{1,2} R_{4,3}) // dm_{1,4→1})}]
```

Out[∘]= {0.546875, {True, True}}

Quasi-triangular axiom 2:

In[∘]:=
```
Timing@HL[((dΔ_{1→1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2})) ≡ ((R_{1,2} dΔ_{1→3,4}) // (dm_{1,4→1} dm_{2,3→2}))]
```

Out[∘]= {1.82813, True}

The Drinfel'd element inverse property, $(u_1\, \bar{u}_2)$ // $dm_{1,2\to 1} \equiv d\epsilon_i$:

In[∘]:=
```
Timing@HL[(((R_{1,2} // dS_1 // dm_{2,1→i}) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j})) // dm_{i,j→i}) ≡ dη_i]
```

Out[∘]= {2.20313, True}

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner $C=uv^{-1}$. It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy.

In[∘]:=
```
Timing@
 Block[{$k = 2}, (((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j})) // dm_{i,j→i}]
```

Out[∘]=

$$\left\{3.85938, \mathbb{E}_{\{\}\to\{i\}}\left[\text{Log}\left[\frac{1}{B_i}\right], \hbar\, a_i, 0\right]\right\}$$

In[◦]:= $\texttt{Timing@Block}\left[\{\$k = 2\},\ \texttt{HL}\ /@ \left\{\left(\left(C_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv \left(\left(\left(R_{1,2} // dS_1 // dm_{2,1\to i}\right) // dS_i\right)\left(R_{1,2} // dS_2 // dS_2 // dm_{2,1\to j}\right) // dm_{i,j\to i}\right)\right\}\right]$

Out[◦]= $\left\{3.35938,\ \left\{\text{True},\ \hbar\, b_i == \text{Log}\left[\frac{1}{B_i}\right]\right\}\right\}$

In[◦]:= $\texttt{Timing@Block}\left[\{\$k = 2\},\ \texttt{HL}\ /@ \left\{\left(\left(C_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv \left(\left(\left(\left(R_{1,2} // dS_1 // dm_{2,1\to i}\right) // dS_i\right)\left(R_{1,2} // dS_2 // dS_2 // dm_{2,1\to j}\right)\right) // dm_{i,j\to i}\right)\right\}\right]$

Out[◦]= $\left\{3.54688,\ \left\{\text{True},\ \hbar\, b_i == \text{Log}\left[\frac{1}{B_i}\right]\right\}\right\}$

Reidemeister 2:

In[◦]:= $\texttt{Timing}\left[\texttt{HL}[\# \equiv d\eta_1\, d\eta_2]\ \&\ /@ \left\{\left(\overline{R}_{1,2}\, R_{3,4}\right) // \left(dm_{1,3\to 1}\, dm_{2,4\to 2}\right),\ \left(R_{1,2}\, \overline{R}_{3,4}\right) // \left(dm_{1,3\to 1}\, dm_{2,4\to 2}\right)\right\}\right]$

Out[◦]= {1.73438, {True, True}}

Cyclic Reidemeister 2:

In[◦]:= $\texttt{Timing@HL}\left[\left(\left(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3\right) // dm_{2,4\to 2} // dm_{1,3\to 1} // dm_{1,5\to 1}\right) \equiv \overline{C}_1\, d\eta_2\right]$

Out[◦]= {0.640625, True}

Reidemeister 3:

In[◦]:= $\texttt{Timing@HL}\left[\left(R_{1,2}\, R_{6,3}\, R_{4,5} // dm_{1,6\to 1}\, dm_{2,4\to 2}\, dm_{3,5\to 3}\right) \equiv \left(R_{2,3}\, R_{1,4}\, R_{5,6} // dm_{1,5\to 1}\, dm_{2,6\to 2}\, dm_{3,4\to 3}\right)\right]$

Out[◦]= {3.25, True}

Relations between the four kinks:

In[◦]:= $\texttt{Timing}\left[\texttt{HL}\ /@ \left\{\text{Kink}_i \equiv \left(\left(R_{3,1}\, C_2\right) // dm_{1,2\to 1} // dm_{1,3\to i}\right),\ \overline{\text{Kink}}_j \equiv \left(\left(\overline{R}_{3,1}\,\overline{C}_2\right) // dm_{1,2\to 1} // dm_{1,3\to j}\right),\ \left(\left(\text{Kink}_i\,\overline{\text{Kink}}_j\right) // dm_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$

Out[◦]= $\left\{2.96875,\ \left\{\frac{\hbar\, b_i}{2} + \hbar\, a_i\, b_i + \hbar\, x_i\, y_i == \hbar\, a_i\, b_i + \frac{1}{2}\left(\text{Log}\left[\frac{1}{B_i^2}\right] - \hbar\, b_i\right) + \hbar\, x_i\, y_i,\ -\frac{\hbar\, b_j}{2} - \hbar\, a_j\, b_j - \frac{\hbar\, x_j\, y_j}{B_j} == -\hbar\, a_j\, b_j + \frac{1}{2}\left(\text{Log}\left[B_j^2\right] + \hbar\, b_j\right) - \frac{\hbar\, x_j\, y_j}{B_j},\ \text{True}\right\}\right\}$

The Trefoil

In[◦]:=
```
Timing@Block[{$k = 1},
  Z31 = R1,5 R6,2 R3,7 C̄4 K̄ink8 K̄ink9 K̄ink10;
  Do[Z31 = Z31 // dm1,r→1, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31 // b2t1 /. T1 → T)}]
```

Out[◦]=

$$\left\{2.28125, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{2}\left(\text{Log}\left[\frac{1}{\left(1-B_1+B_1^2\right)^2}\right]-2\hbar b_1\right),\right.\right.\right.$$
$$\left.-\frac{\hbar\left(B_1-2B_1^2-2B_1^4-a_1\left(-1+B_1-B_1^3+B_1^4\right)+2\hbar x_1 y_1+B_1^3\left(3+2\hbar x_1 y_1\right)\right)}{\left(1-B_1+B_1^2\right)^2}\right],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[\frac{1}{2}\text{Log}\left[\frac{1}{\left(1-T_1+T_1^2\right)^2}\right]+\hbar t_1,\right.$$
$$\left.\left.\left.-\frac{\hbar\left(T_1-2T_1^2-2T_1^4-2a_1\left(-1+T_1-T_1^3+T_1^4\right)+2\hbar x_1 y_1+T_1^3\left(3+2\hbar x_1 y_1\right)\right)}{\left(1-T_1+T_1^2\right)^2}\right]\right\}\right\}$$

b2t, t2b, knot tensors.

In[◦]:= `HL[(b2ti // t2bi) ≡ dσi→i]`

Out[◦]= True

In[◦]:= `t2bi // b2ti`

Out[◦]= $\mathbb{E}_{\{i\}\to\{i\}}[a_i\,\alpha_i + y_i\,\eta_i + x_i\,\xi_i + t_i\,\tau_i,\ 0,\ 0]$

Reidemeister 2:

In[◦]:=
```
Timing[HL[# ≡ dη1 dη2] & /@ {(k̄R1,2 kR3,4) // (km1,3→1 km2,4→2), (kR1,2 k̄R3,4) // (km1,3→1 km2,4→2)}]
```

Out[◦]= {2.64063, {True, True}}

Cyclic Reidemeister 2:

In[◦]:=
```
Timing@HL[((kR1,4 k̄R5,2 k̄C3) // km2,4→2 // km1,3→1 // km1,5→1) ≡ k̄C1 dη2]
```

Out[◦]= {0.671875, True}

Reidemeister 3:

In[◦]:=
```
Timing@HL[(kR1,2 kR4,3 kR5,6 // km1,4→1 // km2,5→2 // km3,6→3 ) ≡
   (kR1,6 kR2,3 kR4,5 // km1,4→1 // km2,5→2 // km3,6→3)]
```

Out[◦]= {1.26563, True}

Relations between the four kinks:

In[∘]:=
```
Timing[HL /@ {kKink_i ≡ ((kR_{3,1} kC_2) // km_{1,2→1} // km_{1,3→i}),
   \overline{kKink}_j ≡ ((\overline{kR}_{3,1} \overline{kC}_2) // km_{1,2→1} // km_{1,3→j}), ((kKink_i \overline{kKink}_j) // km_{i,j→1}) ≡ dη_1}]
```

Out[∘]=

$$\left\{2.25, \left\{-\frac{t\hbar}{2} - t\hbar a_i + \hbar x_i y_i == \frac{1}{2}\left(t\hbar + \mathrm{Log}\left[\frac{1}{T^2}\right]\right) - t\hbar a_i + \hbar x_i y_i,\ \frac{t\hbar}{2} + t\hbar a_j - \frac{\hbar x_j y_j}{T} == \frac{1}{2}\left(-t\hbar + \mathrm{Log}\left[T^2\right]\right) + t\hbar a_j - \frac{\hbar x_j y_j}{T},\ \mathrm{True}\right\}\right\}$$

The Trefoil

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} \overline{kC}_4 \overline{kKink}_8 \overline{kKink}_9 \overline{kKink}_{10};
  Do[Z31 = Z31 // km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[∘]=

$$\left\{3.17188, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar + \mathrm{Log}\left[\frac{1}{1-T+T^2}\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\times\left(-1+T-T^3+T^4\right)a_1 + 2\times\left(1+T^3\right)\hbar x_1 y_1\right)}{\left(1-T+T^2\right)^2}\right]\right\}$$

In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} \overline{kC}_4 \overline{kKink}_8 \overline{kKink}_9 \overline{kKink}_{10};
  Do[Z31 = Z31 // km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

Out[∘]=

$$\left\{1.92188, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar + \mathrm{Log}\left[\frac{1}{1-T+T^2}\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\times\left(-1+T-T^3+T^4\right)a_1 + 2\times\left(1+T^3\right)\hbar x_1 y_1\right)}{\left(1-T+T^2\right)^2}\right]\right\}$$

In[∘]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

» 6

Out[∘]=

$$\left\{16.9375, \mathbb{E}_{\{\}\to\{1\}}\left[-t\hbar + \mathrm{Log}\left[-\frac{\sqrt{\frac{1}{T^2}}\,T^5}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}\right], \frac{(-1+T)\times(1+T)\times\left(1-T+T^2\right)\times\left(3-5T+3T^2\right)\hbar}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6} + \frac{2a(-1+T)\times(1+T)\times\left(1-T+T^2\right)\times\left(3-5T+3T^2\right)\hbar}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6} - \frac{2\times(1+T)\times\left(1-T+T^2\right)\times\left(3-5T+3T^2\right)x\,y\,\hbar^2}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}\right]\right\}$$

CU

Associativity of CU:

```
In[∘]:= Timing@Block[{$k = 3}, HL[(cm1,2→1 // cm1,3→1) ≡ (cm2,3→2 // cm1,2→1)]]
Out[∘]= {0.890625, True}
```

Associativity, co-associativity, and Δ is an algebra morphism:

```
In[∘]:= Timing@Block[{$k = 3}, HL /@ {(cm1,2→1 // cm1,3→1) ≡ (cm2,3→2 // cm1,2→1),
          (cΔ1→1,2 // cΔ2→2,3) ≡ (cΔ1→1,3 // cΔ1→1,2),
          (cm1,2→1 // cΔ1→1,2) ≡ ((cΔ1→1,3 cΔ2→2,4) // (cm3,4→2 cm1,2→1))}]
Out[∘]= {1.57813, {True, True, True}}
```

S is convolution inverse of id:

```
In[∘]:= Timing@Block[{$k = 3}, HL[# ≡ cϵ1 cη1] & /@ {
          (cΔ1→1,2 // cS1) // cm1,2→1, (cΔ1→1,2 // cS2) // cm1,2→1}]
Out[∘]= {2.10938, {True, True}}
```

S is an algebra anti-(co)morphism

```
In[∘]:= Timing@Block[{$k = 3},
        HL /@ {(cm1,2→1 // cS1) ≡ ((cS1 cS2) // cm2,1→1), (cS1 // cΔ1→1,2) ≡ (cΔ1→2,1 // (cS1 cS2))}]
Out[∘]= {1.78125, {True, True}}
```

Classical is the $\hbar \to 0$ limit of quantum:

```
In[∘]:= ClassicalLimit[f_] := Normal@Series[Normal[f] // U2l, {ℏ, 0, 0}] // l2U;
        Timing[HL /@ Simplify /@
          {cm1,2→3 ≡ ClassicalLimit /@ dm1,2→3,
           (cΔ1→2,3 /. τ1 → 0) ≡ ClassicalLimit /@ dΔ1→2,3, cS1 ≡ ClassicalLimit /@ dS1}]
Out[∘]= {1.15625, {True, True, True}}
```

```
In[∘]:= PrintProfile[]
Out[∘]= ProfileRoot is root. Profiled time: 107.181
          (      1)   0.188/ 25.546 above Z
          (     59)   0.897/ 25.264 above Boot
          (   1329)   2.140/  3.538 above CF
          (    198)   1.389/ 24.716 above EZip3
          (      1)       0/      0 above RVK
          (    198)   2.308/  3.340 above Zip1
          (    198)   2.417/  8.425 above Zip2
          (    198)   6.485/ 16.352 above Zip3
        CF: called 113174 times, time in 44.256/68.449
          (    146)   0.813/  1.408 under Z
          (    407)   0.450/  0.621 under Boot
          (   1218)   6.843/ 16.210 under EZip3
```

```
  (  1329)   2.140/  3.538 under ProfileRoot
  (   648)   0.582/  1.896 under Zip1
  ( 25752)   8.390/ 11.045 under Zip2
  ( 83674)  25.038/ 33.731 under Zip3
  ( 86749)  24.193/ 24.193 above CCF
Zip3: called 648 times, time in 26.037/59.768
  (    40)   1.215/  4.983 under Z
  (    86)   3.063/  7.490 under Boot
  (   324)  15.274/ 30.943 under EZip3
  (   198)   6.485/ 16.352 under ProfileRoot
  ( 83674)  25.038/ 33.731 above CF
CCF: called 86749 times, time in 24.193/24.193
  ( 86749)  24.193/ 24.193 under CF
Zip1: called 324 times, time in 4.783/6.679
  (    40)   0.481/  0.921 under Z
  (    86)   1.994/  2.418 under Boot
  (   198)   2.308/  3.340 under ProfileRoot
  (   648)   0.582/  1.896 above CF
Zip2: called 324 times, time in 4.195/15.24
  (    40)   0.534/  2.672 under Z
  (    86)   1.244/  4.143 under Boot
  (   198)   2.417/  8.425 under ProfileRoot
  ( 25752)   8.390/ 11.045 above CF
EZip3: called 324 times, time in 2.461/49.614
  (    40)   0.793/ 15.140 under Z
  (    86)   0.279/  9.758 under Boot
  (   198)   1.389/ 24.716 under ProfileRoot
  (  1218)   6.843/ 16.210 above CF
  (   324)  15.274/ 30.943 above Zip3
Boot: called 86 times, time in 1.068/37.871
  (     3)   0.015/  0.234 under Z
  (    24)   0.156/ 12.373 under Boot
  (    59)   0.897/ 25.264 under ProfileRoot
  (    24)   0.156/ 12.373 above Boot
  (   407)   0.450/  0.621 above CF
  (    86)   0.279/  9.758 above EZip3
  (    86)   1.994/  2.418 above Zip1
  (    86)   1.244/  4.143 above Zip2
  (    86)   3.063/  7.490 above Zip3
Z: called 1 times, time in 0.188/25.546
  (     1)   0.188/ 25.546 under ProfileRoot
  (     3)   0.015/  0.234 above Boot
  (   146)   0.813/  1.408 above CF
  (    40)   0.793/ 15.140 above EZip3
  (    40)   0.481/  0.921 above Zip1
  (    40)   0.534/  2.672 above Zip2
  (    40)   1.215/  4.983 above Zip3
```

```
RVK: called 1 times, time in 0./0.
  (     1)      0/     0 under ProfileRoot
```