Pensieve header: Exponentiation and logarithms in ybax algebras.

Startup

In[•]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 2;
<< Engine.m
<< Objects.m
<< KT.m
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, , ]];
```

Out[•]= {2021, 8, 22, 10, 25, 16.7085381}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

Exponentials

Task. Define $\mathrm{Exp}_m[U : \mathbb{U}_{\{\_\_\}\to\{i\_\}}[\_\_]]$ to compute $e^{\mathfrak{D}(U)}$ to order $\epsilon^{\mathrm{Length@}\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where U is an ϵ-dependent sub-balanced near-docile element, giving the answer in $\mathbb{E}$-form.
Example: $\mathrm{Exp}_{\mathrm{dm},1}[\mathbb{U}_{\{\}\to\{2\}}[b_2\, a_2 + y_2\, x_2,\, 0]]$ is the exponential of the arrow on strand 2, computed to degree 1.

```
In[•]:= Exp_m_[U : 𝕌_is_→{i1_}[__]] :=
 Module[{λ, μ, k, n, F, f, i, j, lhs, rhs, U1, MI(*multi-index*), mis, mi, yax},
  MI /: Coefficient[ℰ_, MI[p_, n_, q_]] :=
   Coefficient[Coefficient[Coefficient[ℰ, y_i, p], a_i, n], x_i, q];
  yax /: yax^MI[p_,n_,q_] := y_i^p a_i^n x_i^q;
  U1 = U /. (v : (y | b | t | a | x | B | T | 𝔄))_i1 → v_i;
  F = 𝔼_is→{i}[];
  Do[AppendTo[F, 0]; Do[
    mis = Flatten@
      Table[MI[p, n, q], {p, 0, Min[k + 1, 2 k + 2 - 2 n]}, {q, 0, Min[k + 1, 2 k + 2 - 2 n - p]}];
    F[[-1]] += Sum[f_mi[λ] yax^mi, {mi, mis}];
    lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // m_i,j→i]) /. μ → 0 /. f_[0] → 0 /.
      Table[f_mi'[0] → Coefficient[U1[[k + 1]], mi], {mi, mis}];
    rhs = ∂_λ U2l@Last[F];
    F = l2U[F /. First@DSolve[Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_mi[0] == 0, {mi, mis}],
        Table[f_mi, {mi, mis}], λ]],
   {n, k + 1, 0, -1}], {k, 0, Length[U1] - 1}];
  CF@l2U[F /. {λ → 1, i → i1}] ]
```

```
In[•]:= Exp_dm[𝕌_{}→{i}[ℏ x_i y_i, 0]] // l2U
```

$$\text{Out[•]}= \mathbb{E}_{\{\}\to\{i\}}\left[\frac{e^{-B_i}\left(-e+e^{B_i}\right)\hbar\, x_i\, y_i}{-1+B_i},\right.$$
$$\left.\frac{e^{-B_i}\hbar^2 a_i B_i\left(e^{B_i}-e\,B_i\right)x_i\,y_i}{(-1+B_i)^2}+\frac{e^{-2B_i}\hbar^3\left(-e^2-e^{2B_i}+5\,e^2\,B_i-5\,e^{2B_i}\,B_i-6\,e^2\,B_i^2+8\,e^{1+B_i}\,B_i^2\right)x_i^2\,y_i^2}{4\,(-1+B_i)^3}\right]$$

```
In[•]:= Exp_dm[𝕌_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, 0]]
```

$$\text{Out[•]}= \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i+\frac{e^{-B_i}\left(-e+e^{B_i}\right)\hbar\, B_i\, x_i\, y_i}{-1+B_i},\right.$$
$$\frac{e^{-B_i}\left(-e+e^{B_i}\right)\hbar^2 B_i\, x_i\, y_i}{-1+B_i}+\frac{e^{-B_i}\hbar^2 a_i B_i\left(-e+e^{B_i}+e\,B_i-e\,B_i^2\right)x_i\,y_i}{(-1+B_i)^2}+\frac{1}{4\,(-1+B_i)^3}$$
$$\left.e^{-2B_i}\hbar^3 B_i^2\left(-5\,e^2-5\,e^{2B_i}+8\,e^{1+B_i}+9\,e^2\,B_i-e^{2B_i}\,B_i-8\,e^{1+B_i}\,B_i-6\,e^2\,B_i^2+8\,e^{1+B_i}\,B_i^2\right)x_i^2\,y_i^2\right]$$

```
In[•]:= Exp_dm[𝕌_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, c (x_i + y_i)]]
```

$$\text{Out[•]}= \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i+\frac{e^{-B_i}\left(-e+e^{B_i}\right)\hbar\, B_i\, x_i\, y_i}{-1+B_i},\ \frac{c\,e^{-B_i}\left(e^{B_i}-e\,B_i\right)x_i}{-1+\hbar\, b_i+B_i}+\frac{c\,e^{-B_i}\left(e^{B_i}-e\,B_i\right)y_i}{-1+\hbar\, b_i+B_i}+\right.$$
$$\frac{e^{-B_i}\left(-e+e^{B_i}\right)\hbar^2 B_i\, x_i\, y_i}{-1+B_i}+\frac{e^{-B_i}\hbar^2 a_i B_i\left(-e+e^{B_i}+e\,B_i-e\,B_i^2\right)x_i\,y_i}{(-1+B_i)^2}+\frac{1}{4\,(-1+B_i)^3}$$
$$\left.e^{-2B_i}\hbar^3 B_i^2\left(-5\,e^2-5\,e^{2B_i}+8\,e^{1+B_i}+9\,e^2\,B_i-e^{2B_i}\,B_i-8\,e^{1+B_i}\,B_i-6\,e^2\,B_i^2+8\,e^{1+B_i}\,B_i^2\right)x_i^2\,y_i^2\right]$$

In[•]:=
```
Exp_cm[𝕎_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, c (x_i + y_i)]]
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i - \frac{(-1+B_i)\, x_i\, y_i}{b_i},\right.$$
$$\left. c\, x_i + c\, y_i - \frac{\hbar\,(-1+B_i)\, x_i\, y_i}{b_i} + \frac{a_i\,(-1+B_i+\hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1-4\,B_i+3\,B_i^2+2\,\hbar\, b_i\, B_i^2\right) x_i^2\, y_i^2}{2\, b_i^3}\right]$$

In[•]:=
```
PrintProfile[]
```

Out[•]=
```
ProfileRoot is root. Profiled time: 5.593
  (   1)  0.265/ 0.265 above Boot
  (  18)  0.079/ 0.079 above CF
  (   5)  0.078/ 2.593 above EZip3
  (   5)  0.125/ 0.188 above Zip1
  (   5)  0.108/ 0.468 above Zip2
  (   5)  0.689/ 2.000 above Zip3
CF: called 1562 times, time in 2.391/4.047
  (  13)  1.172/ 2.030 under EZip3
  (  18)  0.079/ 0.079 under ProfileRoot
  (  10)  0.047/ 0.063 under Zip1
  ( 373)  0.203/ 0.360 under Zip2
  ( 1148)  0.890/ 1.515 under Zip3
  ( 791)  1.656/ 1.656 above CCF
CCF: called 791 times, time in 1.656/1.656
  ( 791)  1.656/ 1.656 under CF
Zip3: called 10 times, time in 0.97/2.485
  (   5)  0.281/ 0.485 under EZip3
  (   5)  0.689/ 2.000 under ProfileRoot
  ( 1148)  0.890/ 1.515 above CF
Boot: called 1 times, time in 0.265/0.265
  (   1)  0.265/ 0.265 under ProfileRoot
Zip1: called 5 times, time in 0.125/0.188
  (   5)  0.125/ 0.188 under ProfileRoot
  (  10)  0.047/ 0.063 above CF
Zip2: called 5 times, time in 0.108/0.468
  (   5)  0.108/ 0.468 under ProfileRoot
  ( 373)  0.203/ 0.360 above CF
EZip3: called 5 times, time in 0.078/2.593
  (   5)  0.078/ 2.593 under ProfileRoot
  (  13)  1.172/ 2.030 above CF
  (   5)  0.281/ 0.485 above Zip3
```

In[•]:=
```
Exp_cm[𝕎_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, c_1 (x_i + y_i), 0]]
```

» 0.422

» 1.531

» 5.125

» 8.547

Out[•]= $$\mathbb{E}_{\{\}\to\{i\}}\Big[\hbar\, a_i\, b_i + \frac{(1-B_i)\, x_i\, y_i}{b_i},$$

$$c_1\, x_i + c_1\, y_i + \frac{(\hbar-\hbar\, B_i)\, x_i\, y_i}{b_i} + \frac{a_i\,(-1+B_i+\hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1-4\,B_i+3\,B_i^2+2\,\hbar\, b_i\, B_i^2\right) x_i^2\, y_i^2}{2\, b_i^3},$$

$$\frac{1}{2}\, b_i\, c_1^2 + \frac{1}{2}\,\hbar\, c_1\, x_i + \frac{1}{2}\,\hbar\, c_1\, y_i + \frac{\left(\hbar^2-\hbar^2\, B_i\right) x_i\, y_i}{2\, b_i} + \frac{a_i\,\left(-\hbar+\hbar\, B_i+\hbar^2\, b_i\, B_i\right) x_i\, y_i}{b_i^2} +$$

$$\frac{a_i^2\,\left(2-2\,B_i-2\,\hbar\, b_i\, B_i-\hbar^2\, b_i^2\, B_i\right) x_i\, y_i}{2\, b_i^3} + \frac{a_i\,\left(-3+12\,B_i+4\,\hbar\, b_i\, B_i-9\,B_i^2-10\,\hbar\, b_i\, B_i^2-4\,\hbar^2\, b_i^2\, B_i^2\right) x_i^2\, y_i^2}{2\, b_i^4} +$$

$$\frac{\left(1+2\,\hbar\, b_i-8\,B_i-8\,\hbar\, b_i\, B_i+7\,B_i^2+12\,\hbar\, b_i\, B_i^2+6\,\hbar^2\, b_i^2\, B_i^2\right) x_i^2\, y_i^2}{2\, b_i^4} +$$

$$\frac{\left(2-15\,B_i+30\,B_i^2+12\,\hbar\, b_i\, B_i^2-17\,B_i^3-18\,\hbar\, b_i\, B_i^3-6\,\hbar^2\, b_i^2\, B_i^3\right) x_i^3\, y_i^3}{3\, b_i^5}\Big]$$

```
In[•]:= PrintProfile[]
```

```
Out[•]= ProfileRoot is root. Profiled time: 228.938
          (   1)   0.265/  0.265 above Boot
          (  61)   0.252/  0.346 above CF
          (  14)   0.326/ 147.160 above EZip3
          (  14)   0.249/  0.438 above Zip1
          (  14)   0.311/  1.280 above Zip2
          (  14)  19.171/ 79.452 above Zip3
        CF: called 4587 times, time in 123.579/207.345
          (  42)  88.470/ 143.910 under EZip3
          (  61)   0.252/  0.346 under ProfileRoot
          (  28)   0.126/  0.189 under Zip1
          ( 1050)   0.576/  0.969 under Zip2
          ( 3406)  34.155/ 61.936 under Zip3
          ( 4249)  83.766/ 83.766 above CCF
        CCF: called 4249 times, time in 83.766/83.766
          ( 4249)  83.766/ 83.766 under CF
        Zip3: called 28 times, time in 20.442/82.378
          (  14)   1.271/  2.926 under EZip3
          (  14)  19.171/ 79.452 under ProfileRoot
          ( 3406)  34.155/ 61.936 above CF
        EZip3: called 14 times, time in 0.326/147.157
          (  14)   0.326/ 147.160 under ProfileRoot
          (  42)  88.470/ 143.910 above CF
          (  14)   1.271/  2.926 above Zip3
        Zip2: called 14 times, time in 0.311/1.28
          (  14)   0.311/  1.280 under ProfileRoot
          ( 1050)   0.576/  0.969 above CF
        Boot: called 1 times, time in 0.265/0.265
          (   1)   0.265/  0.265 under ProfileRoot
        Zip1: called 14 times, time in 0.249/0.438
          (  14)   0.249/  0.438 under ProfileRoot
          (  28)   0.126/  0.189 above CF
```

Testing Multiplicativity

In[•]:=
```
lhs = Exp_dm[𝕎_{}→{i}[λ ħ a_i b_i + λ ħ x_i y_i, 0]] Exp_dm[𝕎_{}→{j}[μ ħ a_j b_j + μ ħ x_j y_j, 0]] // dm_{i,j→i}
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\Bigg[(\lambda+\mu)\,\hbar\,a_i\,b_i+\frac{e^{-((\lambda+\mu)\,(-1+B_i))}\left(-1+e^{(\lambda+\mu)\,(-1+B_i)}\right)\hbar\,B_i^{\lambda+\mu}\,x_i\,y_i}{-1+B_i},$$
$$\frac{e^{-((\lambda+\mu)\,(-1+B_i))}\left(-1+e^{(\lambda+\mu)\,(-1+B_i)}\right)(\lambda+\mu)\,\hbar^2\,B_i^{\lambda+\mu}\,x_i\,y_i}{-1+B_i}-$$
$$\frac{1}{(-1+B_i)^2}e^{-((\lambda+\mu)\,(-1+B_i))}\hbar^2\,a_i\,B_i^{\lambda+\mu}\left(\lambda-e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda+\mu-e^{(\lambda+\mu)\,(-1+B_i)}\,\mu+B_i-\right.$$
$$\left.e^{(\lambda+\mu)\,(-1+B_i)}\,B_i-2\,\lambda\,B_i+e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i-2\,\mu\,B_i+e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i+\lambda\,B_i^2+\mu\,B_i^2\right)x_i\,y_i+$$
$$\frac{1}{4\,(-1+B_i)^3}e^{-2\,(\lambda+\mu)\,(-1+B_i)}\hbar^3\,B_i^{2\,\lambda+2\,\mu}\left(1-e^{2\,(\lambda+\mu)\,(-1+B_i)}-6\,\lambda+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda-4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\lambda-\right.$$
$$6\,\mu+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu-4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\mu-3\,B_i+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,B_i-5\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,B_i+$$
$$12\,\lambda\,B_i-16\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i+4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i+12\,\mu\,B_i-16\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i+$$
$$\left.4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i-6\,\lambda\,B_i^2+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i^2-6\,\mu\,B_i^2+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i^2\right)x_i^2\,y_i^2\Bigg]$$

In[•]:=
```
rhs = Exp_dm[𝕎_{}→{i}[(λ + μ) ħ a_i b_i + (λ + μ) ħ x_i y_i, 0]]
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\Bigg[(\lambda+\mu)\,\hbar\,a_i\,b_i+\frac{e^{-((\lambda+\mu)\,(-1+B_i))}\left(-1+e^{(\lambda+\mu)\,(-1+B_i)}\right)\hbar\,B_i^{\lambda+\mu}\,x_i\,y_i}{-1+B_i},$$
$$\frac{e^{-((\lambda+\mu)\,(-1+B_i))}\left(-1+e^{(\lambda+\mu)\,(-1+B_i)}\right)(\lambda+\mu)\,\hbar^2\,B_i^{\lambda+\mu}\,x_i\,y_i}{-1+B_i}-$$
$$\frac{1}{(-1+B_i)^2}e^{-((\lambda+\mu)\,(-1+B_i))}\hbar^2\,a_i\,B_i^{\lambda+\mu}\left(\lambda-e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda+\mu-e^{(\lambda+\mu)\,(-1+B_i)}\,\mu+B_i-\right.$$
$$\left.e^{(\lambda+\mu)\,(-1+B_i)}\,B_i-2\,\lambda\,B_i+e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i-2\,\mu\,B_i+e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i+\lambda\,B_i^2+\mu\,B_i^2\right)x_i\,y_i+$$
$$\frac{1}{4\,(-1+B_i)^3}e^{-2\,(\lambda+\mu)\,(-1+B_i)}\hbar^3\,B_i^{2\,\lambda+2\,\mu}\left(1-e^{2\,(\lambda+\mu)\,(-1+B_i)}-6\,\lambda+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda-4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\lambda-\right.$$
$$6\,\mu+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu-4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\mu-3\,B_i+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,B_i-5\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,B_i+$$
$$12\,\lambda\,B_i-16\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i+4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i+12\,\mu\,B_i-16\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i+$$
$$\left.4\,e^{2\,(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i-6\,\lambda\,B_i^2+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\lambda\,B_i^2-6\,\mu\,B_i^2+8\,e^{(\lambda+\mu)\,(-1+B_i)}\,\mu\,B_i^2\right)x_i^2\,y_i^2\Bigg]$$

In[•]:=
```
lhs ≡ rhs
```

Out[•]=
```
True
```

Testing dS

In[•]:=
```
η_1 ((∂_{η_1} #)) /. {(η | β | α | ξ)_1 → 0, 𝒜_1 → 1}) & /@ dS_1 /. 𝔼 → 𝕎
```

Out[•]=
$$\mathbb{W}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1},-\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}\right]$$

In[•]:= $\mathbf{Exp_{dm}}\left[\mathbb{W}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1}\right]\right]$

Out[•]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1}\right]$

In[•]:= $\mathbf{Exp_{dm}}\left[\mathbb{W}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1}\right]\right]$

Out[•]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2}\right]$

In[•]:= $\mathbf{Exp_{dm}}\left[\mathbb{W}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1},-\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}\right]\right]$

Out[•]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2},-\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{5\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\hbar^2\,y_1^3\,\eta_1^3}{2\,B_1^3}\right]$

In[•]:= $\mathbb{E}_{\{1\}\to\{1\}}[\eta_1\,y_1,0,0]\;//\;dS_1$

Out[•]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2},-\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{5\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\hbar^2\,y_1^3\,\eta_1^3}{2\,B_1^3}\right]$

```
In[•]:= PrintProfile[]
```

```
Out[•]= ProfileRoot is root. Profiled time: 37.161
          (    2)   0.046/  6.563 above Boot
          (  123)   0.314/  0.424 above CF
          (   32)   0.250/ 15.923 above EZip3
          (   32)   0.297/  1.128 above Zip1
          (   32)   0.451/  2.466 above Zip2
          (   32)   2.441/ 10.657 above Zip3
        CCF: called 9719 times, time in 19.285/19.285
          ( 9719)  19.285/ 19.285 under CF
        CF: called 23221 times, time in 11.013/30.298
          (  124)   0.122/  0.154 under Boot
          (  184)   3.780/ 14.592 under EZip3
          (  123)   0.314/  0.424 under ProfileRoot
          (  114)   0.175/  0.988 under Zip1
          ( 5530)   1.249/  2.736 under Zip2
          ( 17146)   5.373/ 11.404 under Zip3
          ( 9719)  19.285/ 19.285 above CCF
        Zip3: called 114 times, time in 4.678/16.082
          (   25)   0.692/  2.176 under Boot
          (   57)   1.545/  3.249 under EZip3
          (   32)   2.441/ 10.657 under ProfileRoot
          ( 17146)   5.373/ 11.404 above CF
        Zip1: called 57 times, time in 0.892/1.88
          (   25)   0.595/  0.752 under Boot
          (   32)   0.297/  1.128 under ProfileRoot
          (  114)   0.175/  0.988 above CF
        Zip2: called 57 times, time in 0.887/3.623
          (   25)   0.436/  1.157 under Boot
          (   32)   0.451/  2.466 under ProfileRoot
          ( 5530)   1.249/  2.736 above CF
        EZip3: called 57 times, time in 0.282/18.123
          (   25)   0.032/  2.200 under Boot
          (   32)   0.250/ 15.923 under ProfileRoot
          (  184)   3.780/ 14.592 above CF
          (   57)   1.545/  3.249 above Zip3
        Boot: called 20 times, time in 0.124/21.175
          (   18)   0.078/ 14.612 under Boot
          (    2)   0.046/  6.563 under ProfileRoot
          (   18)   0.078/ 14.612 above Boot
          (  124)   0.122/  0.154 above CF
          (   25)   0.032/  2.200 above EZip3
          (   25)   0.595/  0.752 above Zip1
          (   25)   0.436/  1.157 above Zip2
          (   25)   0.692/  2.176 above Zip3
```

Exp step by step #1

Exp step by step #2

In[•]:=
```
{m = cm, is = {}, i1 = i, U = 𝔼{}→{i}[
    gMI[0,1,0] ai + gMI[0,0,0] + gMI[0,0,1] xi + gMI[0,0,2] xi^2 + gMI[1,0,0] yi + gMI[1,0,1] xi yi + gMI[2,0,0] yi^2]}
```

Out[•]=
```
{cm, {}, i, 𝔼{}→{i}[
  gMI[0,0,0] + ai gMI[0,1,0] + gMI[0,0,1] xi + gMI[0,0,2] xi^2 + gMI[1,0,0] yi + gMI[1,0,1] xi yi + gMI[2,0,0] yi^2]}
```

Generic exponentiation

In[•]:=
```
CF @@ Expcm[𝕌{}→{i}[c1 ai bi + c2 xi yi + c3 xi + c4 yi]]
```

» 0.016

Out[•]=
$$a_i b_i c_1 + \frac{B_i^{-\frac{c_2}{\hbar}}\left(B_i^{\frac{c_1}{\hbar}} - B_i^{\frac{c_2}{\hbar}} + b_i B_i^{\frac{c_2}{\hbar}} c_1 - b_i B_i^{\frac{c_2}{\hbar}} c_2\right) c_3 c_4}{b_i (c_1 - c_2)^2} + \frac{B_i^{-\frac{c_2}{\hbar}}\left(-B_i^{\frac{c_1}{\hbar}} + B_i^{\frac{c_2}{\hbar}}\right) c_3 x_i}{b_i (c_1 - c_2)} + \frac{B_i^{-\frac{c_2}{\hbar}}\left(-B_i^{\frac{c_1}{\hbar}} + B_i^{\frac{c_2}{\hbar}}\right) c_4 y_i}{b_i (c_1 - c_2)} - \frac{B_i^{\frac{c_1}{\hbar}-\frac{c_2}{\hbar}}\left(-1 + B_i^{\frac{c_2}{\hbar}}\right) x_i y_i}{b_i}$$

In[•]:=
```
Block[{CCF = FullSimplify}, Expcm[𝕌{}→{i}[c0 + c1 ai bi + c2 xi yi + c3 xi + c4 yi]]]
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\left[c_0 + a_i b_i c_1 + \frac{\left(-1 + B_i^{\frac{c_1-c_2}{\hbar}} + b_i (c_1 - c_2)\right) c_3 c_4}{b_i (c_1 - c_2)^2} + \frac{\left(1 - B_i^{\frac{c_1-c_2}{\hbar}}\right) c_3 x_i}{b_i (c_1 - c_2)} + \frac{\left(1 - B_i^{\frac{c_1-c_2}{\hbar}}\right) c_4 y_i}{b_i (c_1 - c_2)} + \frac{B_i^{\frac{c_1}{\hbar}}\left(-1 + B_i^{-\frac{c_2}{\hbar}}\right) x_i y_i}{b_i}\right]$$

Logarithms

Task. Define $\text{Log}_m[\mathcal{E} : \mathbb{E}_{\{\_\}\to\{i\_\}}[\_]]$ to compute $\text{Log}@\mathbb{O}[e^{\mathcal{E}}]$ to order $\epsilon^{\text{Length@}\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where $\mathcal{E}$ is an ϵ-dependent sub-balanced docile element, giving the answer in $\mathbb{U}$-form.

```
In[•]:= Log_m_[ℰ : 𝔼is_→{i_}[__]] :=
 Module[{e, k, n, G, c, g, eqn, Sanify, MI(*multi-index*), mis, mi, yax, p, q},
  G = 𝕌is→{i}[c1 ai + c2 xi yi]; eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]];
  {eqn, G} = CF /@ ({eqn, G} /. First@Solve[Coefficient[eqn, ai] == 0, c1]);
  Sanify[{{v_ → s_}}] := v → PowerExpand[Normal[s] /. c_ → 0];
  G = CF[G /. Sanify@Solve[Coefficient[eqn, xi yi] == 0, c2]];
  G[[1]] += c0 + c1 xi + c2 yi; eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]];
  {eqn, G} = CF /@ ({eqn, G} /.
      First@Solve[Coefficient[eqn, xi] == 0 ∧ Coefficient[eqn, yi] == 0, {c1, c2}]);
  G = G /. First@Solve[eqn == 0, c0];
  MI /: Coefficient[e_, MI[p_, n_, q_]] :=
   Coefficient[Coefficient[Coefficient[e, yi, p], ai, n], xi, q];
  yax /: yax^MI[p_,n_,q_] := yi^p ai^n xi^q;
  Do[
   mis = Flatten@Table[MI[p, n, q], {n, 0, k + 1},
      {p, 0, Min[k + 1, 2 k + 2 - 2 n]}, {q, 0, Min[k + 1, 2 k + 2 - 2 n - p]}];
   AppendTo[G, Sum[gmi yax^mi, {mi, mis}]];
   eqn = U2l[Last[Exp_m[G]] - ℰ[[k + 1]]];
   G = CF[
     G /. First@Solve[Table[Coefficient[eqn, mi] == 0, {mi, mis}], Table[gmi, {mi, mis}]]],
   {k, Length[ℰ] - 1}];
  CF[l2U@G]
 ]
```

```
In[•]:= Log_cm[𝔼{}→{i}[ħ ai bi + (1 - Bi) xi yi / bi ,
   c1 xi + c1 yi + ((ħ - ħ Bi) xi yi)/bi + (ai (-1 + Bi + ħ bi Bi) xi yi)/bi^2 + ((1 - 4 Bi + 3 Bi^2 + 2 ħ bi Bi^2) xi^2 yi^2)/(2 bi^3)]]
```

```
Out[•]= 𝕌{}→{i}[ħ ai bi + ħ xi yi, c1 xi + c1 yi]
```

```
In[•]:= Log_cm[𝔼_{}→{i}[ħ a_i b_i + ((1 - B_i) x_i y_i)/b_i,
    c_1 x_i + c_1 y_i + ((ħ - ħ B_i) x_i y_i)/b_i + (a_i (-1 + B_i + ħ b_i B_i) x_i y_i)/b_i^2 + ((1 - 4 B_i + 3 B_i^2 + 2 ħ b_i B_i^2) x_i^2 y_i^2)/(2 b_i^3),
    (1/2) b_i c_1^2 + (1/2) ħ c_1 x_i + (1/2) ħ c_1 y_i + ((ħ^2 - ħ^2 B_i) x_i y_i)/(2 b_i) +
     (a_i (-ħ + ħ B_i + ħ^2 b_i B_i) x_i y_i)/b_i^2 + (a_i^2 (2 - 2 B_i - 2 ħ b_i B_i - ħ^2 b_i^2 B_i) x_i y_i)/(2 b_i^3) +
     (a_i (-3 + 12 B_i + 4 ħ b_i B_i - 9 B_i^2 - 10 ħ b_i B_i^2 - 4 ħ^2 b_i^2 B_i^2) x_i^2 y_i^2)/(2 b_i^4) +
     ((1 + 2 ħ b_i - 8 B_i - 8 ħ b_i B_i + 7 B_i^2 + 12 ħ b_i B_i^2 + 6 ħ^2 b_i^2 B_i^2) x_i^2 y_i^2)/(2 b_i^4) +
     ((2 - 15 B_i + 30 B_i^2 + 12 ħ b_i B_i^2 - 17 B_i^3 - 18 ħ b_i B_i^3 - 6 ħ^2 b_i^2 B_i^3) x_i^3 y_i^3)/(3 b_i^5)]]

Out[•]= 𝕌_{}→{i}[ħ a_i b_i + ħ x_i y_i, c_1 x_i + c_1 y_i, 0]
```

Logarithms step by step, #1

```
In[•]:= {m = cm, is = {};
    ℰ = 𝔼_{}→{i}[ħ a_i b_i + ((1 - B_i) x_i y_i)/b_i,
      c_1 x_i + c_1 y_i + ((ħ - ħ B_i) x_i y_i)/b_i + (a_i (-1 + B_i + ħ b_i B_i) x_i y_i)/b_i^2 + ((1 - 4 B_i + 3 B_i^2 + 2 ħ b_i B_i^2) x_i^2 y_i^2)/(2 b_i^3)]}

Out[•]= {cm, 𝔼_{}→{i}[ħ a_i b_i + ((1 - B_i) x_i y_i)/b_i,
    c_1 x_i + c_1 y_i + ((ħ - ħ B_i) x_i y_i)/b_i + (a_i (-1 + B_i + ħ b_i B_i) x_i y_i)/b_i^2 + ((1 - 4 B_i + 3 B_i^2 + 2 ħ b_i B_i^2) x_i^2 y_i^2)/(2 b_i^3)]}

In[•]:= G = 𝕌_is→{i}[c_1 a_i + c_2 x_i y_i]; eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]]

Out[•]= -ħ a_i b_i + a_i c_1 - ((1 - ℯ^(-ħ b_i)) x_i y_i)/b_i - (ℯ^(-c_1+b_i c_2) (-1 + ℯ^(-b_i c_2)) x_i y_i)/b_i

In[•]:= {eqn, G} = CF /@ ({eqn, G} /. First@Solve[Coefficient[eqn, a_i] == 0, c_1])

Out[•]= {-(ℯ^(-b_i (ħ-c_2)) (-1 + ℯ^(b_i (ħ-c_2))) x_i y_i)/b_i, 𝕌_{}→{i}[ħ a_i b_i + c_2 x_i y_i]}
```

```
In[•]:= Solve[Coefficient[eqn, x_i y_i] == 0, c_2]
```

$$\text{Out[•]= } \left\{\left\{c_2 \to \frac{-2\,\mathbb{i}\,\pi\,\mathbb{c}_1+\hbar\,b_i}{b_i} \text{ if } \mathbb{c}_1 \in \mathbb{Z}\right\}\right\}$$

```
In[•]:= Sanify[{{v_ → s_}}] := v → PowerExpand[Normal[s] /. ℂ_ → 0]

In[•]:= Sanify@Solve[Coefficient[eqn, x_i y_i] == 0, c_2]
```

$$\text{Out[•]= } c_2 \to \hbar$$

```
In[•]:= G = CF[G /. Sanify@Solve[Coefficient[eqn, x_i y_i] == 0, c_2]]
```

$$\text{Out[•]= } \mathbb{U}_{\{\}\to\{i\}}[\hbar\,a_i\,b_i+\hbar\,x_i\,y_i]$$

```
In[•]:= G[[1]] += c_0 + c_1 x_i + c_2 y_i; eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]]
```

$$\text{Out[•]= } \frac{1}{2}\times(2\,c_0+b_i\,c_1\,c_2)+c_1\,x_i+c_2\,y_i-\frac{\left(1-\mathbb{e}^{-\hbar\,b_i}\right)\,x_i\,y_i}{b_i}-\frac{\left(-1+\mathbb{e}^{-\hbar\,b_i}\right)\,x_i\,y_i}{b_i}$$

Logarithms step by step, #2

```
In[•]:= {m = cm, is = {}, ℰ = 𝔼_{}→{i}[-t ħ/2 - t ħ a_i + ħ x_i y_i, ħ a_i + ħ a_i^2 - 1/4 ħ^3 x_i^2 y_i^2, 1/9 ħ^5 x_i^3 y_i^3]}
```

$$\text{Out[•]= } \left\{cm,\ \{\},\ \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{t\,\hbar}{2}-t\,\hbar\,a_i+\hbar\,x_i\,y_i,\ \hbar\,a_i+\hbar\,a_i^2-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2,\ \frac{1}{9}\,\hbar^5\,x_i^3\,y_i^3\right]\right\}$$

```
In[•]:= G = 𝕌_{is→{i}}[c_1 a_i + c_2 x_i y_i]; eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]]
```

$$\text{Out[•]= } \frac{t\,\hbar}{2}+t\,\hbar\,a_i+a_i\,c_1-\hbar\,x_i\,y_i-\frac{\mathbb{e}^{-c_1+b_i\,c_2}\left(-1+\mathbb{e}^{-b_i\,c_2}\right)\,x_i\,y_i}{b_i}$$

```
In[•]:= {eqn, G} = CF /@ ({eqn, G} /. First@Solve[Coefficient[eqn, a_i] == 0, c_1])
```

$$\text{Out[•]= } \left\{\frac{t\,\hbar}{2}-\frac{\left(\mathbb{e}^{t\,\hbar}-\mathbb{e}^{t\,\hbar+b_i\,c_2}+\hbar\,b_i\right)\,x_i\,y_i}{b_i},\ \mathbb{U}_{\{\}\to\{i\}}[-t\,\hbar\,a_i+c_2\,x_i\,y_i]\right\}$$

```
In[•]:= Coefficient[eqn, x_i y_i] == 0
```

$$\text{Out[•]= } -\frac{\mathbb{e}^{t\,\hbar}-\mathbb{e}^{t\,\hbar+b_i\,c_2}+\hbar\,b_i}{b_i} == 0$$

```
In[•]:= Solve[-(ℯ^(t ħ) - ℯ^(t ħ + b_i c_2) + ħ b_i)/b_i == 0, c_2]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\text{Out[•]= } \left\{\left\{c_2 \to \frac{-t\,\hbar+\text{Log}\left[\mathbb{e}^{t\,\hbar}+\hbar\,b_i\right]}{b_i}\right\}\right\}$$

```
In[•]:= G = CF[G /. First@Solve[Coefficient[eqn, x_i y_i] == 0, c_2, Reals]]
```

Solve: This system cannot be solved with the methods available to Solve.

ReplaceAll: $\left\{-\frac{e^{t\hbar} - e^{\text{Times[≪2≫]+Times[≪2≫]}} + \hbar b_i}{b_i} == 0\right\}$ is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

```
Out[•]= 𝕌{}→{i}[-t ℏ a_i + c_2 x_i y_i] /. -(e^(t ℏ) - e^(t ℏ+b_i c_2) + ℏ b_i)/b_i == 0
```

Log[Kink]

```
In[•]:= Kink_i
```

$$\text{Out[•]= } \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar b_i}{2} + \hbar a_i b_i + \hbar x_i y_i,\ \frac{\hbar a_i}{2} - \frac{1}{4}\hbar^3 x_i^2 y_i^2,\ \frac{1}{9}\hbar^5 x_i^3 y_i^3\right]$$

```
In[•]:= Block[{$k = 0}, Log_dm[Kink_i]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\text{Out[•]= } \mathbb{U}_{\{\}\to\{i\}}\left[\frac{\hbar b_i}{2} + \hbar a_i b_i - \frac{\hbar^2 b_i x_i y_i}{-1 + B_i}\right]$$

Now step by step:

```
In[•]:= {m = dm, is = {}, ℰ = 𝔼{}→{i}[ℏ b_i/2 + ℏ a_i b_i + ℏ x_i y_i]}
```

$$\text{Out[•]= } \left\{\text{dm}, \{\}, \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar b_i}{2} + \hbar a_i b_i + \hbar x_i y_i\right]\right\}$$

```
In[•]:= G = 𝕌is→{i}[c_1 a_i + c_2 x_i y_i]
Out[•]= 𝕌{}→{i}[a_i c_1 + c_2 x_i y_i]

In[•]:= Last[Exp_m[G]]
```

$$\text{Out[•]= } a_i c_1 + \frac{e^{-c_1 - \frac{B_i c_2}{\hbar}}\left(-e^{\frac{c_2}{\hbar}} + e^{\frac{B_i c_2}{\hbar}}\right)\hbar x_i y_i}{-1 + B_i}$$

```
In[•]:= eqn = U2l[Last[Exp_m[G]] - ℰ[[1]]]
```

$$\text{Out[•]= } -\frac{\hbar b_i}{2} - \hbar a_i b_i + a_i c_1 - \hbar x_i y_i + \frac{e^{-c_1 - \frac{e^{-\hbar b_i} c_2}{\hbar}}\left(-e^{\frac{c_2}{\hbar}} + e^{\frac{e^{-\hbar b_i} c_2}{\hbar}}\right)\hbar x_i y_i}{-1 + e^{-\hbar b_i}}$$

```
In[•]:= First@Solve[Coefficient[eqn, a_i] == 0, c_1]
Out[•]= {c_1 → ℏ b_i}
```

In[•]:=
```
{eqn, G} = CF /@ ({eqn, G} /. First@Solve[Coefficient[eqn, a_i] == 0, c_1])
```

Out[•]=
$$\left\{-\frac{\hbar\, b_i}{2}-\frac{e^{-\frac{e^{-\hbar b_i} c_2}{\hbar}}\left(-e^{\frac{c_2}{\hbar}}+e^{\hbar b_i+\frac{e^{-\hbar b_i} c_2}{\hbar}}\right)\hbar\, x_i\, y_i}{-1+e^{\hbar b_i}},\ \mathbb{U}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i+c_2\, x_i\, y_i\right]\right\}$$

In[•]:=
```
l2U[eqn]
```

Out[•]=
$$-\frac{\hbar\, b_i}{2}-\frac{e^{-\frac{B_i c_2}{\hbar}}\,\hbar\left(-e^{\frac{c_2}{\hbar}}+\frac{e^{\frac{B_i c_2}{\hbar}}}{B_i}\right)x_i\, y_i}{-1+\frac{1}{B_i}}$$

In[•]:=
```
Sanify[{{v_ → s_}}] := v → PowerExpand[Normal[s] /. c_ → 0]
```

In[•]:=
```
Solve[Coefficient[eqn, x_i y_i] == 0, c_2]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[•]=
$$\left\{\left\{c_2 \to \frac{e^{\hbar b_i}\,\hbar\,\text{Log}\left[e^{\hbar b_i}\right]}{-1+e^{\hbar b_i}}\right\}\right\}$$

In[•]:=
```
LogKink[1] = Block[{$k = 1}, Log_dm[Kink_i]]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[•]=
$$\mathbb{U}_{\{\}\to\{i\}}\left[\frac{\hbar\, b_i}{2}+\hbar\, a_i\, b_i-\frac{\hbar^2\, b_i\, x_i\, y_i}{-1+B_i},\right.$$
$$\left.\frac{\hbar\, a_i}{2}-\frac{\hbar^2\, a_i\,(-1+B_i+\hbar\, b_i\, B_i)\, x_i\, y_i}{(-1+B_i)^2}+\frac{\hbar^3\,(-2+\hbar\, b_i+2\, B_i+\hbar\, b_i\, B_i)\, x_i^2\, y_i^2}{2\,(-1+B_i)^3}\right]$$

In[•]:=
```
(LogKink[1] /. 𝕌 → 𝔼) // b2t_i
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\, t_i}{2}-\hbar\, a_i\, t_i+\frac{\hbar^2\, t_i\, x_i\, y_i}{-1+T_i},\right.$$
$$\left.\hbar\, a_i+\hbar\, a_i^2+\frac{2\,\hbar^2\, a_i\,(1-T_i+\hbar\, t_i\, T_i)\, x_i\, y_i}{(-1+T_i)^2}-\frac{\hbar^3\,(2+\hbar\, t_i-2\, T_i+\hbar\, t_i\, T_i)\, x_i^2\, y_i^2}{2\,(-1+T_i)^3}\right]$$