Pensieve header: Exponentiation in ybax algebras.

Startup

In[•]:=

```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 2;
<< Engine.m
<< Objects.m
<< KT.m
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Out[•]=

```
{2021, 8, 19, 6, 50, 26.1964674}
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

```
This version: April 2020. Original version: July 1994.
```

» CCFLogFile is `CCFLog-2021-08-19T06-50-26.m`

Exponentials

Task. Define $\text{Exp}_m[U:\mathbb{U}_{\{\_\_\}\to\{i\_\}}[\_\_]]$ to compute $e^{\mathbb{O}(U)}$ to order $\epsilon^{\text{Length@}\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where U is an ϵ-dependent sub-balanced near-docile element, giving the answer in $\mathbb{E}$-form.
Example: $\text{Exp}_{\text{dm},1}[\mathbb{U}_{\{\}\to\{2\}}[b_2\, a_2 + y_2\, x_2, 0]]$ is the exponential of the arrow on strand 2, computed to degree 1.

In[•]:=
```
Exp_m_[U : 𝕌_{is_→{i1_}}[__]] :=
 Module[{λ, μ, k, n, F, f, i, j, lhs, rhs, U1, MI (*multi-index*), mis, mi, yax},
  MI /: Coefficient[ℰ_, MI[p_, n_, q_]] :=
   Coefficient[Coefficient[Coefficient[ℰ, y_i, p], a_i, n], x_i, q];
  yax /: yax^MI[p_,n_,q_] := y_i^p a_i^n x_i^q;
  U1 = U /. (v : (y | b | t | a | x | B | T | 𝒜))_i1 → v_i;
  F = 𝔼_{is→{i}}[];
  Do[AppendTo[F, 0]; Do[
    mis = Flatten@Table[MI[p, n, q],
       {p, 0, Min[k + 1, 2 k + 2 - 2 n]}, {q, 0, Min[k + 1, 2 k + 2 - 2 n - p]}];
    F⟦-1⟧ += Sum[f_mi[λ] yax^mi, {mi, mis}];
    lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // m_{i,j→i}]) /. μ → 0 /. f_[0] → 0 /.
      Table[f_mi'[0] → Coefficient[U1⟦k + 1⟧, mi], {mi, mis}];
    rhs = ∂_λ U2l@Last[F];
    F =
     l2U[F /. First@DSolve[Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_mi[0] == 0, {mi, mis}],
         Table[f_mi, {mi, mis}], λ]],
    {n, k + 1, 0, -1}], {k, 0, Length[U1] - 1}];
  CF@l2U[F /. {λ → 1, i → i1}] ]
```

In[•]:=
```
Exp_cm[𝕌_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, c (x_i + y_i)]]
```

» 0.016

» 0.031

» 0.063

» 0.078

» 0.125

» 0.157

Out[•]=

$$\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i},\; c\, x_i + c\, y_i + \frac{(\hbar - \hbar B_i)\, x_i\, y_i}{b_i} + \frac{a_i\,(-1 + B_i + \hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1 - 4 B_i + 3 B_i^2 + 2\hbar\, b_i\, B_i^2\right) x_i^2\, y_i^2}{2\, b_i^3}\right]$$

In[∘]:= **PrintProfile[]**

```
Out[∘]= ProfileRoot is root. Profiled time: 5.593
  (    1)  0.265/ 0.265 above Boot
  (   18)  0.079/ 0.079 above CF
  (    5)  0.078/ 2.593 above EZip3
  (    5)  0.125/ 0.188 above Zip1
  (    5)  0.108/ 0.468 above Zip2
  (    5)  0.689/ 2.000 above Zip3
CF: called 1562 times, time in 2.391/4.047
  (   13)  1.172/ 2.030 under EZip3
  (   18)  0.079/ 0.079 under ProfileRoot
  (   10)  0.047/ 0.063 under Zip1
  (  373)  0.203/ 0.360 under Zip2
  ( 1148)  0.890/ 1.515 under Zip3
  (  791)  1.656/ 1.656 above CCF
CCF: called 791 times, time in 1.656/1.656
  (  791)  1.656/ 1.656 under CF
Zip3: called 10 times, time in 0.97/2.485
  (    5)  0.281/ 0.485 under EZip3
  (    5)  0.689/ 2.000 under ProfileRoot
  ( 1148)  0.890/ 1.515 above CF
Boot: called 1 times, time in 0.265/0.265
  (    1)  0.265/ 0.265 under ProfileRoot
Zip1: called 5 times, time in 0.125/0.188
  (    5)  0.125/ 0.188 under ProfileRoot
  (   10)  0.047/ 0.063 above CF
Zip2: called 5 times, time in 0.108/0.468
  (    5)  0.108/ 0.468 under ProfileRoot
  (  373)  0.203/ 0.360 above CF
EZip3: called 5 times, time in 0.078/2.593
  (    5)  0.078/ 2.593 under ProfileRoot
  (   13)  1.172/ 2.030 above CF
  (    5)  0.281/ 0.485 above Zip3
```

In[∘]:= $\mathrm{Exp}_{cm}[\mathbb{U}_{\{\}\to\{i\}}[\hbar\, a_i\, b_i + \hbar\, x_i\, y_i,\ c_1\,(x_i + y_i),\ 0]]$

» 0.422

» 1.531

» 5.125

» 8.547

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\Big[\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i},$

$c_1\, x_i + c_1\, y_i + \frac{(\hbar - \hbar\, B_i)\, x_i\, y_i}{b_i} + \frac{a_i\,(-1 + B_i + \hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1 - 4\, B_i + 3\, B_i^2 + 2\,\hbar\, b_i\, B_i^2\right)\, x_i^2\, y_i^2}{2\, b_i^3},$

$\frac{1}{2}\, b_i\, c_1^2 + \frac{1}{2}\,\hbar\, c_1\, x_i + \frac{1}{2}\,\hbar\, c_1\, y_i + \frac{\left(\hbar^2 - \hbar^2\, B_i\right)\, x_i\, y_i}{2\, b_i} + \frac{a_i\,\left(-\hbar + \hbar\, B_i + \hbar^2\, b_i\, B_i\right)\, x_i\, y_i}{b_i^2} +$

$\frac{a_i^2\,\left(2 - 2\, B_i - 2\,\hbar\, b_i\, B_i - \hbar^2\, b_i^2\, B_i\right)\, x_i\, y_i}{2\, b_i^3} + \frac{a_i\,\left(-3 + 12\, B_i + 4\,\hbar\, b_i\, B_i - 9\, B_i^2 - 10\,\hbar\, b_i\, B_i^2 - 4\,\hbar^2\, b_i^2\, B_i^2\right)\, x_i^2\, y_i^2}{2\, b_i^4} +$

$\frac{\left(1 + 2\,\hbar\, b_i - 8\, B_i - 8\,\hbar\, b_i\, B_i + 7\, B_i^2 + 12\,\hbar\, b_i\, B_i^2 + 6\,\hbar^2\, b_i^2\, B_i^2\right)\, x_i^2\, y_i^2}{2\, b_i^4} +$

$\frac{\left(2 - 15\, B_i + 30\, B_i^2 + 12\,\hbar\, b_i\, B_i^2 - 17\, B_i^3 - 18\,\hbar\, b_i\, B_i^3 - 6\,\hbar^2\, b_i^2\, B_i^3\right)\, x_i^3\, y_i^3}{3\, b_i^5}\Big]$

In[∘]:=
```
PrintProfile[]
```

Out[∘]=
```
ProfileRoot is root. Profiled time: 228.938
   (    1)   0.265/   0.265 above Boot
   (   61)   0.252/   0.346 above CF
   (   14)   0.326/ 147.160 above EZip3
   (   14)   0.249/   0.438 above Zip1
   (   14)   0.311/   1.280 above Zip2
   (   14)  19.171/  79.452 above Zip3
CF: called 4587 times, time in 123.579/207.345
   (   42)  88.470/ 143.910 under EZip3
   (   61)   0.252/   0.346 under ProfileRoot
   (   28)   0.126/   0.189 under Zip1
   ( 1050)   0.576/   0.969 under Zip2
   ( 3406)  34.155/  61.936 under Zip3
   ( 4249)  83.766/  83.766 above CCF
CCF: called 4249 times, time in 83.766/83.766
   ( 4249)  83.766/  83.766 under CF
Zip3: called 28 times, time in 20.442/82.378
   (   14)   1.271/   2.926 under EZip3
   (   14)  19.171/  79.452 under ProfileRoot
   ( 3406)  34.155/  61.936 above CF
EZip3: called 14 times, time in 0.326/147.157
   (   14)   0.326/ 147.160 under ProfileRoot
   (   42)  88.470/ 143.910 above CF
   (   14)   1.271/   2.926 above Zip3
Zip2: called 14 times, time in 0.311/1.28
   (   14)   0.311/   1.280 under ProfileRoot
   ( 1050)   0.576/   0.969 above CF
Boot: called 1 times, time in 0.265/0.265
   (    1)   0.265/   0.265 under ProfileRoot
Zip1: called 14 times, time in 0.249/0.438
   (   14)   0.249/   0.438 under ProfileRoot
   (   28)   0.126/   0.189 above CF
```

Testing dS

In[∘]:= $\eta_1 \left(\left(\partial_{\eta_1}\#\right) /. \{(\eta \mid \beta \mid \alpha \mid \xi)_1 \to 0, \mathcal{A}_1 \to 1\}\right) \& /@ dS_1 /. \mathbb{E} \to \mathbb{U}$

Out[∘]= $\mathbb{U}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1}, \frac{\hbar\, y_1\,\eta_1}{B_1}, -\frac{\hbar^2\, y_1\,\eta_1}{2\,B_1}\right]$

In[∘]:= $\mathrm{Exp}_{dm}\left[\mathbb{U}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1}\right]\right]$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1}\right]$

In[ ]:= $\text{Exp}_{dm}\left[\mathbb{U}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\ \frac{\hbar\,y_1\,\eta_1}{B_1}\right]\right]$

Out[ ]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\ \frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2}\right]$

In[ ]:= $\text{Exp}_{dm}\left[\mathbb{U}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\ \frac{\hbar\,y_1\,\eta_1}{B_1},\ -\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}\right]\right]$

Out[ ]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\ \frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2},\ -\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{5\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\hbar^2\,y_1^3\,\eta_1^3}{2\,B_1^3}\right]$

In[ ]:= $\mathbb{E}_{\{1\}\to\{1\}}[\eta_1\,y_1,\ 0,\ 0]$ // dS_1

Out[ ]= $\mathbb{E}_{\{1\}\to\{1\}}\left[-\frac{y_1\,\eta_1}{B_1},\ \frac{\hbar\,y_1\,\eta_1}{B_1}-\frac{\hbar\,y_1^2\,\eta_1^2}{2\,B_1^2},\ -\frac{\hbar^2\,y_1\,\eta_1}{2\,B_1}+\frac{5\,\hbar^2\,y_1^2\,\eta_1^2}{4\,B_1^2}-\frac{\hbar^2\,y_1^3\,\eta_1^3}{2\,B_1^3}\right]$

In[ ]:= **PrintProfile**[]

```
Out[ ]= ProfileRoot is root. Profiled time: 242.083
  (    2)   0.265/  8.969 above Boot
  (  130)   0.409/  0.566 above CF
  (   31)   0.421/ 148.580 above EZip3
  (   31)   0.499/  0.734 above Zip1
  (   31)   0.561/  2.108 above Zip2
  (   31)  19.658/ 81.126 above Zip3
CF: called 23122 times, time in 129.905/215.043
  (  124)   0.172/  0.232 under Boot
  (  187)  88.672/ 144.390 under EZip3
  (  130)   0.409/  0.566 under ProfileRoot
  (  112)   0.204/  0.313 under Zip1
  ( 5461)   1.923/  2.566 under Zip2
  (17108)  38.525/ 66.977 under Zip3
  (11651)  85.138/ 85.138 above CCF
CCF: called 11651 times, time in 85.138/85.138
  (11651)  85.138/ 85.138 under CF
Zip3: called 112 times, time in 23.733/90.71
  (   25)   1.004/  2.988 under Boot
  (   56)   3.071/  6.596 under EZip3
  (   31)  19.658/ 81.126 under ProfileRoot
  (17108)  38.525/ 66.977 above CF
Zip1: called 56 times, time in 1.251/1.564
  (   25)   0.752/  0.830 under Boot
  (   31)   0.499/  0.734 under ProfileRoot
  (  112)   0.204/  0.313 above CF
Zip2: called 56 times, time in 1.153/3.719
  (   25)   0.592/  1.611 under Boot
  (   31)   0.561/  2.108 under ProfileRoot
  ( 5461)   1.923/  2.566 above CF
EZip3: called 56 times, time in 0.514/151.499
  (   25)   0.093/  2.919 under Boot
  (   31)   0.421/ 148.580 under ProfileRoot
  (  187)  88.672/ 144.390 above CF
  (   56)   3.071/  6.596 above Zip3
Boot: called 21 times, time in 0.389/31.642
  (   19)   0.124/ 22.673 under Boot
  (    2)   0.265/  8.969 under ProfileRoot
  (   19)   0.124/ 22.673 above Boot
  (  124)   0.172/  0.232 above CF
  (   25)   0.093/  2.919 above EZip3
  (   25)   0.752/  0.830 above Zip1
  (   25)   0.592/  1.611 above Zip2
  (   25)   1.004/  2.988 above Zip3
```

Exp step by step #1

Exp step by step #2

In[∘]:= $\{m = cm, is = \{\}, i1 = i, U = \mathbb{E}_{\{\}\to\{i\}}[g_{MI[0,1,0]}\, a_i + g_{MI[0,0,0]} + g_{MI[0,0,1]}\, x_i + g_{MI[0,0,2]}\, x_i^2 + g_{MI[1,0,0]}\, y_i + g_{MI[1,0,1]}\, x_i\, y_i + g_{MI[2,0,0]}\, y_i^2]\}$

Out[∘]= $\{cm, \{\}, i, \mathbb{E}_{\{\}\to\{i\}}[g_{MI[0,0,0]} + a_i\, g_{MI[0,1,0]} + g_{MI[0,0,1]}\, x_i + g_{MI[0,0,2]}\, x_i^2 + g_{MI[1,0,0]}\, y_i + g_{MI[1,0,1]}\, x_i\, y_i + g_{MI[2,0,0]}\, y_i^2]\}$

Generic exponentiation

In[∘]:= $CF\ @@\ Exp_{cm}[\mathbb{U}_{\{\}\to\{i\}}[c_1\, a_i\, b_i + c_2\, x_i\, y_i + c_3\, x_i + c_4\, y_i]]$

» 0.016

Out[∘]= $$a_i\, b_i\, c_1 + \frac{B_i^{-\frac{c_2}{\hbar}} \left(B_i^{\frac{c_1}{\hbar}} - B_i^{\frac{c_2}{\hbar}} + b_i\, B_i^{\frac{c_2}{\hbar}}\, c_1 - b_i\, B_i^{\frac{c_2}{\hbar}}\, c_2\right) c_3\, c_4}{b_i\, (c_1 - c_2)^2} + \frac{B_i^{-\frac{c_2}{\hbar}} \left(-B_i^{\frac{c_1}{\hbar}} + B_i^{\frac{c_2}{\hbar}}\right) c_3\, x_i}{b_i\, (c_1 - c_2)} + \frac{B_i^{-\frac{c_2}{\hbar}} \left(-B_i^{\frac{c_1}{\hbar}} + B_i^{\frac{c_2}{\hbar}}\right) c_4\, y_i}{b_i\, (c_1 - c_2)} - \frac{B_i^{\frac{c_1}{\hbar} - \frac{c_2}{\hbar}} \left(-1 + B_i^{\frac{c_2}{\hbar}}\right) x_i\, y_i}{b_i}$$

In[∘]:= $Block[\{CCF = FullSimplify\}, Exp_{cm}[\mathbb{U}_{\{\}\to\{i\}}[c_0 + c_1\, a_i\, b_i + c_2\, x_i\, y_i + c_3\, x_i + c_4\, y_i]]]$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[c_0 + a_i\, b_i\, c_1 + \frac{\left(-1 + B_i^{\frac{c_1 - c_2}{\hbar}} + b_i\, (c_1 - c_2)\right) c_3\, c_4}{b_i\, (c_1 - c_2)^2} + \frac{\left(1 - B_i^{\frac{c_1 - c_2}{\hbar}}\right) c_3\, x_i}{b_i\, (c_1 - c_2)} + \frac{\left(1 - B_i^{\frac{c_1 - c_2}{\hbar}}\right) c_4\, y_i}{b_i\, (c_1 - c_2)} + \frac{B_i^{\frac{c_1}{\hbar}} \left(-1 + B_i^{-\frac{c_2}{\hbar}}\right) x_i\, y_i}{b_i}\right]$$

Logarithms

Task. Define $\text{Log}_m[\mathcal{E} : \mathbb{E}_{\{\_\_\}\to\{i\_\}}[\_\_]]$ to compute $\text{Log}@\mathbb{O}[e^{\mathcal{E}}]$ to order $\epsilon^{\text{Length}@\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where $\mathcal{E}$ is an ϵ-dependent sub-balanced docile element, giving the answer in $\mathbb{U}$-form.

In[•]:=
```
Log_m_[ℰ : 𝔼_{is_→{i1_}}[__]] :=
 Module[{e, k, n, G, g, lhs, rhs, MI (*multi-index*), mis, mi, yax, p, q, pq},
  MI /: Coefficient[e_, MI[p_, n_, q_]] :=
   Coefficient[Coefficient[Coefficient[e, y_i, p], a_i, n], x_i, q];
  yax /: yax^MI[p_,n_,q_] := y_i^p a_i^n x_i^q;
  G = 𝕌_{is→{i}}[];
  Do[AppendTo[G, 0]; Do[ Do[
      mis = Echo@Flatten@Table[MI[p, n, pq - p], {p, Max[0, pq - 1 - k], Min[k + 1, pq]}];
      G⟦-1⟧ += Sum[g_mi yax^mi, {mi, mis}];
      lhs = Last[Exp_m[G]];
      rhs = ℰ⟦k + 1⟧;
      G = CF[G /. First@Solve[
           Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]]],
      {pq, 2 k + 2 - 2 n, 0, -1}], {n, k + 1, 0, -1}], {k, 0, Length[ℰ] - 1}];
  G
 ]
```

In[•]:= $\mathrm{Log}_{cm}\Big[\mathbb{E}_{\{\}\to\{i\}}\Big[\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i},$

$c_1 x_i + c_1 y_i + \frac{(\hbar - \hbar B_i)\, x_i\, y_i}{b_i} + \frac{a_i\, (-1 + B_i + \hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1 - 4 B_i + 3 B_i^2 + 2 \hbar\, b_i\, B_i^2\right) x_i^2\, y_i^2}{2 b_i^3},$

$\frac{1}{2} b_i\, c_1^2 + \frac{1}{2} \hbar\, c_1\, x_i + \frac{1}{2} \hbar\, c_1\, y_i + \frac{\left(\hbar^2 - \hbar^2 B_i\right) x_i\, y_i}{2 b_i} +$

$\frac{a_i \left(-\hbar + \hbar B_i + \hbar^2 b_i B_i\right) x_i\, y_i}{b_i^2} + \frac{a_i^2 \left(2 - 2 B_i - 2 \hbar\, b_i B_i - \hbar^2 b_i^2 B_i\right) x_i\, y_i}{2 b_i^3} +$

$\frac{a_i \left(-3 + 12 B_i + 4 \hbar\, b_i B_i - 9 B_i^2 - 10 \hbar\, b_i B_i^2 - 4 \hbar^2 b_i^2 B_i^2\right) x_i^2\, y_i^2}{2 b_i^4} +$

$\frac{\left(1 + 2 \hbar\, b_i - 8 B_i - 8 \hbar\, b_i B_i + 7 B_i^2 + 12 \hbar\, b_i B_i^2 + 6 \hbar^2 b_i^2 B_i^2\right) x_i^2\, y_i^2}{2 b_i^4} +$

$\frac{\left(2 - 15 B_i + 30 B_i^2 + 12 \hbar\, b_i B_i^2 - 17 B_i^3 - 18 \hbar\, b_i B_i^3 - 6 \hbar^2 b_i^2 B_i^3\right) x_i^3\, y_i^3}{3 b_i^5}\Big]\Big]$

» {MI$66498[0, 1, 0]}

» {MI$66498[1, 0, 1]}

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

» {MI$66498[0, 0, 1], MI$66498[1, 0, 0]}

» {MI$66498[0, 0, 0]}

» {MI$66498[0, 2, 0]}

```
» 0.047
» 0.063
» 0.11
» 0.141
» {MI$66498[0, 1, 2], MI$66498[1, 1, 1], MI$66498[2, 1, 0]}
» {MI$66498[0, 1, 1], MI$66498[1, 1, 0]}
» {MI$66498[0, 1, 0]}
» {MI$66498[2, 0, 2]}
» {MI$66498[1, 0, 2], MI$66498[2, 0, 1]}
» {MI$66498[0, 0, 2], MI$66498[1, 0, 1], MI$66498[2, 0, 0]}
» {MI$66498[0, 0, 1], MI$66498[1, 0, 0]}
» {MI$66498[0, 0, 0]}
» {MI$66498[0, 3, 0]}
» 0.422
» 1.641
» 5.359
» 7.109
» {MI$66498[0, 2, 2], MI$66498[1, 2, 1], MI$66498[2, 2, 0]}
» 7.375
» {MI$66498[0, 2, 1], MI$66498[1, 2, 0]}
» {MI$66498[0, 2, 0]}
» {MI$66498[1, 1, 3], MI$66498[2, 1, 2], MI$66498[3, 1, 1]}
» {MI$66498[0, 1, 3], MI$66498[1, 1, 2], MI$66498[2, 1, 1], MI$66498[3, 1, 0]}
» 7.968
» {MI$66498[0, 1, 2], MI$66498[1, 1, 1], MI$66498[2, 1, 0]}
» {MI$66498[0, 1, 1], MI$66498[1, 1, 0]}
» {MI$66498[0, 1, 0]}
» {MI$66498[3, 0, 3]}
» {MI$66498[2, 0, 3], MI$66498[3, 0, 2]}
» {MI$66498[1, 0, 3], MI$66498[2, 0, 2], MI$66498[3, 0, 1]}
» {MI$66498[0, 0, 3], MI$66498[1, 0, 2], MI$66498[2, 0, 1], MI$66498[3, 0, 0]}
» {MI$66498[0, 0, 2], MI$66498[1, 0, 1], MI$66498[2, 0, 0]}
» {MI$66498[0, 0, 1], MI$66498[1, 0, 0]}
» {MI$66498[0, 0, 0]}
```

Out[◦]= $\mathbb{U}_{\{\}\to\{i\}}[\hbar\, a_i\, b_i + \hbar\, x_i\, y_i,\ c_1\, x_i + c_1\, y_i,\ 0]$

Logarithms step by step

In[∘]:= $\left\{m = cm,\ \mathcal{E} = \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i},\right.\right.$
$\left.\left. c_1\, x_i + c_1\, y_i + \frac{(\hbar - \hbar\, B_i)\, x_i\, y_i}{b_i} + \frac{a_i\, (-1 + B_i + \hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1 - 4\, B_i + 3\, B_i^2 + 2\, \hbar\, b_i\, B_i^2\right)\, x_i^2\, y_i^2}{2\, b_i^3}\right]\right\}$

Out[∘]= $\left\{cm,\ \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i},\right.\right.$
$\left.\left. c_1\, x_i + c_1\, y_i + \frac{(\hbar - \hbar\, B_i)\, x_i\, y_i}{b_i} + \frac{a_i\, (-1 + B_i + \hbar\, b_i\, B_i)\, x_i\, y_i}{b_i^2} + \frac{\left(1 - 4\, B_i + 3\, B_i^2 + 2\, \hbar\, b_i\, B_i^2\right)\, x_i^2\, y_i^2}{2\, b_i^3}\right]\right\}$

In[∘]:=
```
MI /: Coefficient[e_, MI[p_, n_, q_]] :=
  Coefficient[Coefficient[Coefficient[e, y_i, p], a_i, n], x_i, q];
yax /: yax^MI[p_,n_,q_] := y_i^p a_i^n x_i^q;
G = 𝕌_{}→{i}[]
```

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}[\,]$

In[∘]:= `Length[ℰ] - 1`

Out[∘]= 1

In[∘]:= `k = 0`

Out[∘]= 0

In[∘]:= `AppendTo[G, 0]`

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}[0]$

In[∘]:= `n = 1`

Out[∘]= 1

In[∘]:=
```
mis =
 Flatten@Table[MI[p, n, q], {p, 0, Min[k + 1, 2 k + 2 - 2 n]}, {q, 0, Min[k + 1, 2 k + 2 - 2 n - p]}]
```

Out[∘]= {MI[0, 1, 0]}

In[∘]:= $G[\![-1]\!]$ += $\mathrm{Sum}\left[g_{mi}\, \mathrm{yax}^{mi},\ \{mi, mis\}\right]$

Out[∘]= $a_i\, g_{MI[0,1,0]}$

In[∘]:= `G`

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}\left[a_i\, g_{MI[0,1,0]}\right]$

In[∘]:= $\mathrm{lhs} = \mathrm{Last}[\mathrm{Exp}_m[G]]$

Out[∘]= $a_i\, g_{MI[0,1,0]}$

In[∘]:= `rhs = ℰ⟦k + 1⟧`

Out[∘]= $\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i}$

In[∘]:= `Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}]`

Out[∘]= $\{-\hbar\, b_i + g_{MI[0,1,0]} == 0\}$

In[∘]:= `First@Solve[Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]]`

Out[∘]= $\{g_{MI[0,1,0]} \to \hbar\, b_i\}$

In[∘]:= `G = CF[G /. First@Solve[Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]]]`

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}[\hbar\, a_i\, b_i]$

In[∘]:= `n = 0`

Out[∘]= 0

In[∘]:= `mis = Flatten@Table[MI[p, n, q], {p, 0, Min[k + 1, 2 k + 2 - 2 n]}, {q, 0, Min[k + 1, 2 k + 2 - 2 n - p]}]`

Out[∘]= {MI[0, 0, 0], MI[0, 0, 1], MI[1, 0, 0], MI[1, 0, 1]}

In[∘]:= `G⟦-1⟧ += Sum[g_mi yax^mi, {mi, mis}]`

Out[∘]= $\hbar\, a_i\, b_i + g_{MI[0,0,0]} + g_{MI[0,0,1]}\, x_i + g_{MI[1,0,0]}\, y_i + g_{MI[1,0,1]}\, x_i\, y_i$

In[∘]:= `lhs = Last[Exp_m[G]]`

» 0.016

Out[∘]= $\hbar\, a_i\, b_i + \frac{1}{b_i\,(\hbar - g_{MI[1,0,1]})^2} B_i^{-\frac{g_{MI[1,0,1]}}{\hbar}} \Big(\hbar^2\, b_i\, B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,0]} + B_i\, g_{MI[0,0,1]}\, g_{MI[1,0,0]} - B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,1]}\, g_{MI[1,0,0]} + \hbar\, b_i\, B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,1]}\, g_{MI[1,0,0]} - 2\,\hbar\, b_i\, B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,0]}\, g_{MI[1,0,1]} - b_i\, B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,1]}\, g_{MI[1,0,0]}\, g_{MI[1,0,1]} + b_i\, B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\, g_{MI[0,0,0]}\, g_{MI[1,0,1]}^2\Big) + \frac{B_i^{-\frac{g_{MI[1,0,1]}}{\hbar}}\Big(-B_i + B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\Big)\, g_{MI[0,0,1]}\, x_i}{b_i\,(\hbar - g_{MI[1,0,1]})} + \frac{B_i^{-\frac{g_{MI[1,0,1]}}{\hbar}}\Big(-B_i + B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\Big)\, g_{MI[1,0,0]}\, y_i}{b_i\,(\hbar - g_{MI[1,0,1]})} - \frac{B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}\Big(-1 + B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\Big)\, x_i\, y_i}{b_i}$

In[∘]:= `rhs = ℰ[[k + 1]]`

Out[∘]= $\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i}$

In[∘]:= `Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}] // FullSimplify`

Out[∘]= $\left\{g_{MI[0,0,1]}\, g_{MI[1,0,0]} + g_{MI[0,0,0]}\, (\hbar - g_{MI[1,0,1]}) == \frac{\left(1 - B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}\right) g_{MI[0,0,1]}\, g_{MI[1,0,0]}}{b_i\,(\hbar - g_{MI[1,0,1]})},\right.$
$\left.\frac{\left(1 - B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}\right) g_{MI[0,0,1]}}{b_i\,(\hbar - g_{MI[1,0,1]})} == 0,\ \frac{\left(1 - B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}\right) g_{MI[1,0,0]}}{b_i\,(\hbar - g_{MI[1,0,1]})} == 0,\ \frac{-1 + B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}}{b_i} == 0\right\}$

In[∘]:= `First@Solve[Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]]`

First: {} has zero length and no first element.

Out[∘]= `First[{}]`

In[∘]:= `{solxy} = Block[{mis = {MI[1, 0, 1]}},`
`First@Solve[Echo@Table[Coefficient[lhs - rhs, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]] ]`

» $\left\{-\frac{1 - B_i}{b_i} - \frac{B_i^{1-\frac{g_{MI[1,0,1]}}{\hbar}}\left(-1 + B_i^{\frac{g_{MI[1,0,1]}}{\hbar}}\right)}{b_i} == 0\right\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[∘]= $\{g_{MI[1,0,1]} \to \hbar\}$

In[∘]:= `lhs /. solxy`

Power: Infinite expression $\frac{1}{0^2}$ encountered.

Infinity: Indeterminate expression $\frac{0\ \text{ComplexInfinity}}{b_i\, B_i}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression $\frac{0\ \text{ComplexInfinity}\ g_{MI[0,0,1]}\, x_i}{b_i\, B_i}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

General: Further output of Power::infy will be suppressed during this calculation.

Infinity: Indeterminate expression $\frac{0\ \text{ComplexInfinity}\ g_{MI[1,0,0]}\, y_i}{b_i\, B_i}$ encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

Out[∘]= `Indeterminate`

In[∘]:=
```
rhs /. solxy
```

Out[∘]= $\hbar\, a_i\, b_i + \frac{(1 - B_i)\, x_i\, y_i}{b_i}$

In[∘]:=
```
Block[{mis = {MI[0, 0, 1]}},
 First@Solve[
   Echo@Table[Coefficient[(lhs - rhs) /. solxy, mi] == 0, {mi, mis}], Table[g_mi, {mi, mis}]] ]
```

Power: Infinite expression $\frac{1}{0^2}$ encountered.

Infinity: Indeterminate expression $\frac{0\,\text{ComplexInfinity}}{b_i\, B_i}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression $\frac{0\,\text{ComplexInfinity}\, g_{MI[0,0,1]}\, x_i}{b_i\, B_i}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

General: Further output of Power::infy will be suppressed during this calculation.

Infinity: Indeterminate expression $\frac{0\,\text{ComplexInfinity}\, g_{MI[1,0,0]}\, y_i}{b_i\, B_i}$ encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

» {True}

Out[∘]= { }