Pensieve header: Full testing of the sl_2 portfolio. Continues pensieve://Projects/SL2Portfolio2/PortfolioTesting.nb. Time: 262.768.

Startup

```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine-210104.m
<< Objects.m
<< KT.m
```

(Alt) Out[∘]= {2021, 1, 4, 8, 32, 36.3874662}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

(Alt) In[∘]:=

```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

(Alt) In[∘]:=

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

(Alt) In[∘]:=
```
Block[{$k = 1}, {
    am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
    aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
    dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

(Alt) Out[∘]=

$am \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\left(\alpha_i+\alpha_j\right)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\ 0\right]$

$bm \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\left(\beta_i+\beta_j\right)+y_k\left(\eta_i+\eta_j\right),\ -y_k\,\beta_i\,\eta_j\right]$

$dm \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\left(\alpha_i+\alpha_j\right)+b_k\,\beta_i+b_k\,\beta_j+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$
$\left.-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(1-3\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(1-3\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{\left(1-4\,B_k+3\,B_k^2\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right]$

$R \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\hbar\,x_j\,y_i,\ -\frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right]$

$\overline{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i-\frac{\hbar\,x_j\,y_i}{B_i},\ -\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right]$

$P \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar}+\frac{\eta_i\,\xi_j}{\hbar},\ \frac{\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$

$aS \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$

$\overline{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ \hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$

$bS \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ -\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$

$\overline{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ \frac{\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$

$dS \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-b_i\,\beta_i-\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},\right.$
$\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\mathcal{A}_i+B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+$
$\left.\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\left(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,B_i^2}-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,B_i}+\frac{\left(-3\,\mathcal{A}_i^2+4\,B_i\,\mathcal{A}_i^2-B_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\right]$

$a\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+x_j\,\xi_i+x_k\,\xi_i,\ -\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$

$b\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[\left(b_j+b_k\right)\beta_i+B_k\,y_j\,\eta_i+y_k\,\eta_i,\ \frac{1}{2}\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\right]$

$d\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+\left(b_j+b_k\right)\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$
$\left.\frac{1}{2}\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$

$C \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2},\ -\frac{\hbar\,a_i}{2}\right]$

$\overline{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2},\ \frac{\hbar\,a_i}{2}\right]$

$Kink \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2}+\hbar\,a_i\,b_i+\hbar\,x_i\,y_i,\ \frac{\hbar\,a_i}{2}-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2\right]$

$\overline{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2}-\hbar\,a_i\,b_i-\frac{\hbar\,x_i\,y_i}{B_i},\ -\frac{\hbar\,a_i}{2}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right]$

$b2t \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-t_i\,\beta_i+y_i\,\eta_i+x_i\,\xi_i,\ a_i\,\beta_i\right]$

$t2b \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+y_i\,\eta_i+x_i\,\xi_i-b_i\,\tau_i,\ a_i\,\tau_i\right]$

Check that on the generators this agrees with our conventions in the handout:

(Alt) In[∘]:=
```
𝔼2Λ[ℰ_] := Module[{k}, Sum[ℰ[k] ϵ^(k-1), {k, 0, ℰ[$]}]];
Timing@Block[{$k = 2}, {
   {
    "[a,x]" → 𝔼2Λ[𝔼_{}→{1,2}[0, a₂ x₁] // am_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, a₁ x₂] // am_{1,2→1}],
    "[b,y]" → 𝔼2Λ[𝔼_{}→{1,2}[0, y₂ b₁, 0] // bm_{1,2→1}] - 𝔼2Λ[𝔼_{}→{1,2}[0, y₁ b₂, 0] // bm_{1,2→1}]
   } /. z_1 → z,
   {
    "Δ[y]" → Last[𝔼_{}→{1}[0, y₁] // bΔ_{1→1,2}],
    "Δ[b]" → Last[𝔼_{}→{1}[0, b₁] // bΔ_{1→1,2}],
    "Δ[a]" → Last[𝔼_{}→{1}[0, a₁] // aΔ_{1→1,2}],
    "Δ[x]" → Last[𝔼_{}→{1}[0, x₁] // aΔ_{1→1,2}]},
   {
    "S(a)" → ((𝔼_{}→{1}[0, a₁] // aS₁)[1]),
    "S(x)" → ((𝔼_{}→{1}[0, x₁] // aS₁)[1]),
    "S(b)" → ((𝔼_{}→{1}[0, b₁] // bS₁)[1]),
    "S(y)" → ((𝔼_{}→{1}[0, y₁] // bS₁)[1])
   } /. z_1 → z
  }]
```

(Alt) Out[∘]=
$$\Big\{4.48438, \Big\{\{[a,x] \to -x,\ [b,y] \to -y\,\epsilon\},\ \{\Delta[y] \to B_2\, y_1 + y_2,\ \Delta[b] \to b_1 + b_2,\ \Delta[a] \to a_1 + a_2,\ \Delta[x] \to x_1 + x_2\},\ \Big\{S(a) \to -a,\ S(x) \to -x,\ S(b) \to -b,\ S(y) \to -\frac{y}{B}\Big\}\Big\}\Big\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

(Alt) Out[∘]= {0.4375, {True, True}}

R and *P* are inverses:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ aσ_{k→j}]}]
```

(Alt) Out[∘]=
$$\Big\{0.578125, \Big\{\mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar\, a_j\, b_i + \hbar\, x_j\, y_i,\ -\frac{1}{4}\hbar^3 x_j^2 y_i^2,\ \frac{1}{9}\hbar^5 x_j^3 y_i^3,\ \frac{1}{48}\left(\hbar^5 x_j^2 y_i^2 - 3\hbar^7 x_j^4 y_i^4\right)\Big],$$
$$\mathbb{E}_{\{i,k\}\to\{\}}\Big[\frac{\alpha_k \beta_i}{\hbar} + \frac{\eta_i \xi_k}{\hbar},\ \frac{\eta_i^2 \xi_k^2}{4\hbar},\ \frac{1}{8}\eta_i^2 \xi_k^2 + \frac{5\eta_i^3 \xi_k^3}{36\hbar},\ \frac{1}{24}\hbar\,\eta_i^2 \xi_k^2 + \frac{1}{6}\eta_i^3 \xi_k^3 + \frac{5\eta_i^4 \xi_k^4}{48\hbar}\Big],\ \text{True}\Big\}\Big\}$$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

(Alt) In[∘]:=
```
Timing[HL /@ {(aS̄₁ // aS₁) ≡ aσ_{1→1}, (bS̄₁ // bS₁) ≡ bσ_{1→1}}]
```

(Alt) Out[∘]= {1.23438, {True, True}}

(co)-associativity on both sides

```
(Alt) In[ ]:= Timing[
  HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
    (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
(Alt) Out[ ]= {1.09375, {True, True, True, True}}
```

Δ is an algebra morphism

```
(Alt) In[ ]:= Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
    (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
(Alt) Out[ ]= {1.92188, {True, True}}
```

An explicit formula for aS_i

(Alt) In[∘]:=
```
Timing@Block[{$k = 4}, HL[aS_i ≡ (𝔼_{i}→{i,j}[-α_i a_j, -ξ_i x_i,
    Sum[Expand[(e^(ξ_i x_i) (-ℏ γ ϵ)^k)/(2^k k!) Nest[Expand[x_i^2 ∂_{x_i,2} #] &, e^(-ξ_i e^(ℏ ϵ a_i) x_i), k]], {k, 0, $k}]]_$k //
  am_{i,j→i})]]
```

(Alt) Out[∘]=

$$\Bigl\{6.21875,\ \mathbb{E}_{\{i\}\to\{i\}}\Bigl[-a_i\alpha_i - x_i\mathcal{A}_i\xi_i,\ -\hbar a_i x_i\mathcal{A}_i\xi_i - \frac{1}{2}\hbar x_i^2\mathcal{A}_i^2\xi_i^2,\ -\frac{1}{2}\hbar^2 a_i^2 x_i\mathcal{A}_i\xi_i + \frac{1}{4}\hbar^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^2 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{2}\hbar^2 x_i^3\mathcal{A}_i^3\xi_i^3,\ -\frac{1}{6}\hbar^3 a_i^3 x_i\mathcal{A}_i\xi_i - \frac{1}{12}\hbar^3 x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{1}{2}\hbar^3 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^3 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{2}{3}\hbar^3 x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{3}{2}\hbar^3 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{2}{3}\hbar^3 x_i^4\mathcal{A}_i^4\xi_i^4,\ -\frac{1}{24}\hbar^4 a_i^4 x_i\mathcal{A}_i\xi_i + \frac{1}{48}\hbar^4 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{6}\hbar^4 a_i x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{1}{2}\hbar^4 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{2}{3}\hbar^4 a_i^3 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{13}{24}\hbar^4 x_i^3\mathcal{A}_i^3\xi_i^3 + 2\hbar^4 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{9}{4}\hbar^4 a_i^2 x_i^3\mathcal{A}_i^3\xi_i^3 + \frac{13}{8}\hbar^4 x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{8}{3}\hbar^4 a_i x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{25}{24}\hbar^4 x_i^5\mathcal{A}_i^5\xi_i^5\Bigr] \equiv$$

$$\mathbb{E}_{\{i,j\}\to\{i\}}\Bigl[a_i(\alpha_i+\alpha_j) + x_i\Bigl(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\Bigr),\ 0,\ 0,\ 0,\ 0\Bigr]\Bigl[$$

$$\mathbb{E}_{\{i\}\to\{i,j\}}\Bigl[-a_j\alpha_i,\ -x_i\xi_i,\ e^{x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i} - \frac{1}{2}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma\epsilon\hbar x_i^2\xi_i^2 + \frac{1}{4}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^2\xi_i^2 - \frac{1}{12}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^2\xi_i^2 + \frac{1}{48}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^2\xi_i^2 - \frac{1}{2}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^3\xi_i^3 + \frac{2}{3}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^3\xi_i^3 - \frac{13}{24}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^3\xi_i^3 + \frac{1}{8}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^4\xi_i^4 - \frac{19}{24}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^4\xi_i^4 + \frac{163}{96}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^4\xi_i^4 + \frac{1}{4}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^5\xi_i^5 - \frac{3}{2}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^5\xi_i^5 - \frac{1}{48}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^6\xi_i^6 + \frac{47}{96}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^6\xi_i^6 - \frac{1}{16}e^{7\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^7\xi_i^7 + \frac{1}{384}e^{8\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^8\xi_i^8\Bigr]_4\Bigr]\Bigr\}$$

S is convolution inverse of id

(Alt) In[∘]:=
```
Timing[HL[# ≡ sϵ_1 sη_1] & /@ {
   (aΔ_{1→1,2} // aS_1) // am_{1,2→1}, (aΔ_{1→1,2} // aS_2) // am_{1,2→1},
   (bΔ_{1→1,2} // bS_1) // bm_{1,2→1}, (bΔ_{1→1,2} // bS_2) // bm_{1,2→1}}]
```

(Alt) Out[∘]= {1.8125, {True, True, True, True}}

But not with the opposite product:

(Alt) In[∘]:=
```
Timing[Short[# ≡ sϵ₁ sη₁] & /@ {
   (aΔ₁→₁,₂ ~ B₁ ~ aS₁) ~ B₁,₂ ~ am₂,₁→₁, (aΔ₁→₁,₂ ~ B₂ ~ aS₂) ~ B₁,₂ ~ am₂,₁→₁,
   (bΔ₁→₁,₂ ~ B₁ ~ bS₁) ~ B₁,₂ ~ bm₂,₁→₁, (bΔ₁→₁,₂ ~ B₂ ~ bS₂) ~ B₁,₂ ~ bm₂,₁→₁}]
```

(Alt) Out[∘]=
```
{0.015625,
 {B₁,₂[B₁[𝔼{1}→{1,2}[a₁ α₁ + a₂ α₁ + x₁ ξ₁ + x₂ ξ₁, <<1>> + <<1>>, <<1>>], <<1>>], <<1>>] ≡ <<1>>,
  B₁,₂[B₂[𝔼{1}→{1,2}[a₁ α₁ + a₂ α₁ + x₁ ξ₁ + x₂ ξ₁, <<1>> + <<1>>, <<1>>], <<1>>], <<1>>] ≡ <<1>>,
  B₁,₂[B₁[𝔼{1}→{1,2}[<<1>>], 𝔼{1}→{1}[-b₁ β₁ - <<1>>/<<1>>, -<<1>> - <<1>>, <<1>>]], <<1>>] ≡ <<1>>,
  B₁,₂[B₂[𝔼{1}→{1,2}[<<1>>], 𝔼{2}→{2}[-b₂ β₂ - <<1>>/<<1>>, -<<1>> - <<1>>, <<1>>]], <<1>>] ≡
   <<1>>}}
```

S is an algebra anti-(co)morphism

(Alt) In[∘]:=
```
Timing[HL /@ {(am₁,₂→₁ // aS₁) ≡ ((aS₁ aS₂) // am₂,₁→₁), (bm₁,₂→₁ // bS₁) ≡ ((bS₁ bS₂) // bm₂,₁→₁),
   (aS₁ // aΔ₁→₁,₂) ≡ (aΔ₁→₂,₁ // (aS₁ aS₂)), (bS₁ // bΔ₁→₁,₂) ≡ (bΔ₁→₂,₁ // (bS₁ bS₂))}]
```

(Alt) Out[∘]=
```
{2.23438, {True, True, True, True}}
```

Pairing axioms

(Alt) In[∘]:=
```
Timing[HL /@ {((bm₁,₂→₁ sϒ₃→₀,₀,₃,₃ // sϵ₀) // P₁,₃) ≡
    (((sϒ₁→₁,₁,₀,₀ // sϵ₀) (sϒ₂→₂,₂,₀,₀ // sϵ₀) aΔ₃→₄,₅) // P₁,₄ // P₂,₅),
   ((bΔ₁→₁,₂ (sϒ₃→₀,₀,₃,₃ // sϵ₀) (sϒ₄→₀,₀,₄,₄ // sϵ₀)) // P₁,₃ // P₂,₄) ≡
    (((sϒ₁→₁,₁,₀,₀ // sϵ₀) am₃,₄→₃) // P₁,₃)}]
```

(Alt) Out[∘]=
```
{2.09375, {True, True}}
```

(Alt) In[∘]:=
```
Timing[HL /@ {((bS₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ aS₂) // P₁,₂),
   ((b̅S̅₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ a̅S̅₂) // P₁,₂)}]
```

(Alt) Out[∘]=
```
{1.125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

(Alt) In[∘]:=
```
(*Timing@{{
    "[a,y]"→ ((𝔼{}→{1,2}[0,0,y2a1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,y1a2]~B1,2~dm1,2→1)[3]),
    "[b,x]"  →
     ((𝔼{}→{1,2}[0,0,x2b1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,x1b2]~B1,2~dm1,2→1)[3]),"xy-qyx"→
     ((𝔼{}→{1,2}[0,0,x1y2]~B1,2~dm1,2→1)[3]-(1+∊)(𝔼{}→{1,2}[0,0,y1x2]~B1,2~dm1,2→1)[3])
   }/.{z_1→z}//Expand//Factor,
  {
   "Δ(a)"→((𝔼{}→{1}[0,0,a1]~B1~dΔ1→1,2)[3]),
   "Δ(x)"→((𝔼{}→{1}[0,0,x1]~B1~dΔ1→1,2)[3]),
   "Δ(b)"→((𝔼{}→{1}[0,0,b1]~B1~dΔ1→1,2)[3]),
   "Δ(y)"→((𝔼{}→{1}[0,0,y1]~B1~dΔ1→1,2)[3])
  }//Simplify,
  {
    "S(a)"→((𝔼{}→{1}[0,0,a1]~B1~dS1)[3]),
    "S(x)"→((𝔼{}→{1}[0,0,x1]~B1~dS1)[3]),
    "S(b)"→((𝔼{}→{1}[0,0,b1]~B1~dS1)[3]),
    "S(y)"→((𝔼{}→{1}[0,0,y1]~B1~dS1)[3])
   }/.{z_1→z}//Simplify
 }*)
```

(Alt) In[∘]:=
```
{HL[((sϒ1→0,0,1,1 // sϵ0) (sϒ2→0,0,2,2 // sϵ0) // dm1,2→1) ≡ am1,2→1],
 HL[((sϒ1→1,1,0,0 // sϵ0) (sϒ2→2,2,0,0 // sϵ0) // dm1,2→1) ≡ bm1,2→1]}
```

(Alt) Out[∘]= {True, True}

(co)-associativity

(Alt) In[∘]:=
```
Timing[Block[{$k = 1},
  HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
]
```

(Alt) Out[∘]= {0.84375, {True, True}}

Δ is an algebra morphism

(Alt) In[∘]:=
```
Timing@HL[(dm1,2→1 // dΔ1→1,2) ≡ ((dΔ1→1,3 dΔ2→2,4) // (dm3,4→2 dm1,2→1))]
```

(Alt) Out[∘]= {3.9375, True}

dS and $\overline{\text{dS}}$ are inverses:

(Alt) In[∘]:=
```
Timing@HL[(d̅S̅1 // dS1) ≡ dσ1→1]
```

(Alt) Out[∘]= {3.84375, True}

S_2 inverts R, but not S_1:

(Alt) In[◦]:=
```
Timing@{(R1,2 // dS1) ≡ R̄1,2, HL[(R1,2 // dS2) ≡ R̄1,2]}
```

(Alt) Out[◦]=

$$\left\{0.65625, \left\{\frac{\hbar^2 x_2 y_1}{B_1} - \frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} == -\frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} \;\&\&\; -\frac{\hbar^3 x_2 y_1}{2 B_1} + \frac{\hbar^3 a_2 x_2 y_1}{B_1} - \frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{2\hbar^4 x_2^2 y_1^2}{B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3} == -\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{\hbar^4 x_2^2 y_1^2}{2 B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3}, \text{True}\right\}\right\}$$

dS is convolution inverse of id

(Alt) In[◦]:=
```
Timing[HL[# ≡ dϵ1 dη1] & /@ {(dΔ1→1,2 // dS1) // dm1,2→1, (dΔ1→1,2 // dS2) // dm1,2→1}]
```

(Alt) Out[◦]= {4.95313, {True, True}}

dS is a (co)-algebra anti-morphism

(Alt) In[◦]:=
```
Timing[HL /@
  Expand /@ {(dm1,2→1 // dS1) ≡ ((dS1 dS2) // dm2,1→1), (dS1 // dΔ1→1,2) ≡ (dΔ1→2,1 // (dS1 dS2))}]
```

(Alt) Out[◦]= {11.0469, {True, True}}

Quasi-triangular axiom 1:

(Alt) In[◦]:=
```
Timing[
 HL /@ {(R1,3 // dΔ1→1,2) ≡ ((R1,4 R2,3) // dm3,4→3), (R1,2 // dΔ2→2,3) ≡ ((R1,2 R4,3) // dm1,4→1)}]
```

(Alt) Out[◦]= {1.0625, {True, True}}

Quasi-triangular axiom 2:

(Alt) In[◦]:=
```
Timing@HL[((dΔ1→1,2 R3,4) // (dm1,3→1 dm2,4→2)) ≡ ((R1,2 dΔ1→3,4) // (dm1,4→1 dm2,3→2))]
```

(Alt) Out[◦]= {3.28125, True}

The Drinfel’d element inverse property, $(u_1\,\overline{u}_2)$ // $dm_{1,2\to 1} \equiv d\epsilon_i$:

(Alt) In[◦]:=
```
Timing@HL[(((R1,2 // dS1 // dm2,1→i) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i) ≡ dηi]
```

(Alt) Out[◦]=

$$\left\{5.67188, \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - \text{Log}\left[B_i^2\right]\right) == 0\right\}$$

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner $C=uv^{-1}$. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

(Alt) In[◦]:=
```
Timing@
 Block[{$k = 2}, (((R1,2 // dS1 // dm2,1→i) // dSi) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i]
```

(Alt) Out[◦]=

$$\left\{10.2188, \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right), \hbar\, a_i, 0\right]\right\}$$

(Alt) In[∘]:= $$\text{Timing@Block}\left[\{\$k = 2\},\ \text{HL} /@ \left\{\left(\left(C_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv \left(((R_{1,2} // dS_1 // dm_{2,1\to i}) // dS_i)\ \left(R_{1,2} // dS_2 // dS_2 // dm_{2,1\to j}\right) // dm_{i,j\to i}\right)\right\}\right]$$

(Alt) Out[∘]= $$\left\{10.1719,\ \left\{\text{True},\ \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$$

(Alt) In[∘]:= $$\text{Timing@Block}\left[\{\$k = 2\},\ \text{HL} /@ \left\{\left(\left(C_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv d\eta_i,\ \left(\left(\overline{C}_i\,\overline{C}_j\right) // dm_{i,j\to i}\right) \equiv \left(\left(((R_{1,2} // dS_1 // dm_{2,1\to i}) // dS_i)\ \left(R_{1,2} // dS_2 // dS_2 // dm_{2,1\to j}\right)\right) // dm_{i,j\to i}\right)\right\}\right]$$

(Alt) Out[∘]= $$\left\{10.7813,\ \left\{\text{True},\ \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$$

Reidemeister 2:

(Alt) In[∘]:= $$\text{Timing}\left[\text{HL}[\# \equiv d\eta_1\, d\eta_2]\ \&\ /@ \left\{\left(\overline{R}_{1,2}\, R_{3,4}\right) // (dm_{1,3\to 1}\, dm_{2,4\to 2}),\ \left(R_{1,2}\, \overline{R}_{3,4}\right) // (dm_{1,3\to 1}\, dm_{2,4\to 2})\right\}\right]$$

(Alt) Out[∘]= $\{2.53125, \{\text{True}, \text{True}\}\}$

Cyclic Reidemeister 2:

(Alt) In[∘]:= $$\text{Timing@HL}\left[\left(\left(R_{1,4}\, \overline{R}_{5,2}\, \overline{C}_3\right) // dm_{2,4\to 2} // dm_{1,3\to 1} // dm_{1,5\to 1}\right) \equiv \overline{C}_1\, d\eta_2\right]$$

(Alt) Out[∘]= $\{1.64063, \text{True}\}$

Reidemeister 3:

(Alt) In[∘]:= $$\text{Timing@HL}[(R_{1,2}\, R_{6,3}\, R_{4,5} // dm_{1,6\to 1}\, dm_{2,4\to 2}\, dm_{3,5\to 3}) \equiv (R_{2,3}\, R_{1,4}\, R_{5,6} // dm_{1,5\to 1}\, dm_{2,6\to 2}\, dm_{3,4\to 3})]$$

(Alt) Out[∘]= $\{4.95313, \text{True}\}$

Relations between the four kinks:

(Alt) In[∘]:= $$\text{Timing}\left[\text{HL} /@ \left\{\text{Kink}_i \equiv ((R_{3,1}\, C_2) // dm_{1,2\to 1} // dm_{1,3\to i}),\ \overline{\text{Kink}}_j \equiv \left(\left(\overline{R}_{3,1}\, \overline{C}_2\right) // dm_{1,2\to 1} // dm_{1,3\to j}\right),\ \left(\left(\text{Kink}_i\, \overline{\text{Kink}}_j\right) // dm_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$$

(Alt) Out[∘]= $$\left\{6.71875,\ \left\{\frac{\hbar\, b_i}{2} + \hbar\, a_i\, b_i + \hbar\, x_i\, y_i == \hbar\, a_i\, b_i + \frac{1}{2}\left(-\text{Log}\left[B_i^2\right] - \hbar\, b_i\right) + \hbar\, x_i\, y_i,\right.\right.$$
$$\left.\left. -\frac{\hbar\, b_j}{2} - \hbar\, a_j\, b_j - \frac{\hbar\, x_j\, y_j}{B_j} == -\hbar\, a_j\, b_j + \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_j^2}\right] + \hbar\, b_j\right) - \frac{\hbar\, x_j\, y_j}{B_j},\ \text{True}\right\}\right\}$$

The Trefoil

(Alt) In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
  Do[Z31 = Z31 // dm_{1,r→1}, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31 // b2t_1 /. T_1 → T)}]
```

(Alt) Out[∘]=
$$\left\{6.65625, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\,\text{Log}\left[\left(1-B_1+B_1^2\right)^2\right]-\hbar\, b_1,\right.\right.\right.$$
$$\left.-\frac{\hbar\left(B_1-2B_1^2-2B_1^4-a_1\left(-1+B_1-B_1^3+B_1^4\right)+2\hbar\, x_1 y_1+B_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-B_1+B_1^2\right)^2}\right],$$
$$\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\,\text{Log}\left[\left(1-T_1+T_1^2\right)^2\right]+\hbar\, t_1,\right.$$
$$\left.\left.\left.-\frac{\hbar\left(T_1-2T_1^2-2T_1^4-2a_1\left(-1+T_1-T_1^3+T_1^4\right)+2\hbar\, x_1 y_1+T_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-T_1+T_1^2\right)^2}\right]\right\}\right\}$$

b2t, t2b, knot tensors.

(Alt) In[∘]:=
```
HL[(b2t_i // t2b_i) ≡ dσ_{i→i}]
```

(Alt) Out[∘]= True

(Alt) In[∘]:=
```
t2b_i // b2t_i
```

(Alt) Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}[a_i\,\alpha_i + y_i\,\eta_i + x_i\,\xi_i + t_i\,\tau_i,\ 0,\ 0]$

Reidemeister 2:

(Alt) In[∘]:=
```
Timing[HL[# ≡ dη_1 dη_2] & /@ {(k̄R_{1,2} kR_{3,4}) // (km_{1,3→1} km_{2,4→2}), (kR_{1,2} k̄R_{3,4}) // (km_{1,3→1} km_{2,4→2})}]
```

(Alt) Out[∘]= {3.875, {True, True}}

Cyclic Reidemeister 2:

(Alt) In[∘]:=
```
Timing@HL[((kR_{1,4} k̄R_{5,2} k̄C_3) // km_{2,4→2} // km_{1,3→1} // km_{1,5→1}) ≡ k̄C_1 dη_2]
```

(Alt) Out[∘]= {1.32813, True}

Reidemeister 3:

(Alt) In[∘]:=
```
Timing@HL[(kR_{1,2} kR_{4,3} kR_{5,6} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3}) ≡
   (kR_{1,6} kR_{2,3} kR_{4,5} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3})]
```

(Alt) Out[∘]= {2.60938, True}

Relations between the four kinks:

(Alt) In[∘]:=

$$\text{Timing}\left[\text{HL} \mathbin{/@} \left\{\text{kKink}_i \equiv ((\text{kR}_{3,1}\,\text{kC}_2) \mathbin{//} \text{km}_{1,2\to 1} \mathbin{//} \text{km}_{1,3\to i}),\ \overline{\text{kKink}}_j \equiv \left(\left(\overline{\text{kR}}_{3,1}\,\overline{\text{kC}}_2\right) \mathbin{//} \text{km}_{1,2\to 1} \mathbin{//} \text{km}_{1,3\to j}\right),\ \left(\left(\text{kKink}_i\,\overline{\text{kKink}}_j\right) \mathbin{//} \text{km}_{i,j\to 1}\right) \equiv d\eta_1\right\}\right]$$

(Alt) Out[∘]=

$$\left\{3.53125, \left\{-\frac{t\hbar}{2} - t\hbar a_i + \hbar x_i y_i == \frac{1}{2}\left(t\hbar - \text{Log}\left[T^2\right]\right) - t\hbar a_i + \hbar x_i y_i,\ \frac{t\hbar}{2} + t\hbar a_j - \frac{\hbar x_j y_j}{T} == \frac{1}{2}\left(-t\hbar - \text{Log}\left[\frac{1}{T^2}\right]\right) + t\hbar a_j - \frac{\hbar x_j y_j}{T},\ \text{True}\right\}\right\}$$

The Trefoil

(Alt) In[∘]:=

$$\text{Timing@Block}\left[\{\$k = 1\},\ Z31 = \text{kR}_{1,5}\,\text{kR}_{6,2}\,\text{kR}_{3,7}\,\overline{\text{kC}}_4\,\overline{\text{kKink}}_8\,\overline{\text{kKink}}_9\,\overline{\text{kKink}}_{10};\ \text{Do}[Z31 = Z31 \mathbin{//} \text{km}_{1,r\to 1}, \{r, 2, 10\}];\ \text{Simplify} \mathbin{/@} Z31\right]$$

(Alt) Out[∘]=

$$\left\{5.375, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \frac{1}{2}\text{Log}\left[\left(1 - T + T^2\right)^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\times\left(-1 + T - T^3 + T^4\right)a_1 + 2\times\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

(Alt) In[∘]:=

$$\text{Timing@Block}\left[\{\$k = 1\},\ Z31 = \text{kR}_{1,5}\,\text{kR}_{6,2}\,\text{kR}_{3,7}\,\overline{\text{kC}}_4\,\overline{\text{kKink}}_8\,\overline{\text{kKink}}_9\,\overline{\text{kKink}}_{10};\ \text{Do}[Z31 = Z31 \mathbin{//} \text{km}_{1,r\to 1}, \{r, 2, 10\}];\ \text{Simplify} \mathbin{/@} Z31\right]$$

(Alt) Out[∘]=

$$\left\{4.32813, \mathbb{E}_{\{\}\to\{1\}}\left[t\hbar - \frac{1}{2}\text{Log}\left[\left(1 - T + T^2\right)^2\right], -\frac{\hbar\left(T - 2T^2 + 3T^3 - 2T^4 - 2\times\left(-1 + T - T^3 + T^4\right)a_1 + 2\times\left(1 + T^3\right)\hbar x_1 y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$$

(Alt) In[∘]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[∘]=

$$\left\{101.578, \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-2\,t\,\hbar-\text{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right]-\text{Log}\left[\left(1+\frac{T}{1-4T+6T^2-5T^3+T^4}-\frac{2T^2}{1-4T+6T^2-5T^3+T^4}+\frac{T^3}{1-4T+6T^2-5T^3+T^4}\right)^2\right]-\text{Log}\left[\left(1-\frac{T}{1-3T+4T^2-4T^3+T^4}+\frac{4T^2}{1-3T+4T^2-4T^3+T^4}-\frac{7T^3}{1-3T+4T^2-4T^3+T^4}+\frac{7T^4}{1-3T+4T^2-4T^3+T^4}-\frac{4T^5}{1-3T+4T^2-4T^3+T^4}+\frac{T^6}{1-3T+4T^2-4T^3+T^4}\right)^2\right]\right),\ \frac{-3\hbar+8T\hbar-8T^2\hbar+8T^4\hbar-8T^5\hbar+3T^6\hbar}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{a\left(-6\hbar+16T\hbar-16T^2\hbar+16T^4\hbar-16T^5\hbar+6T^6\hbar\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{x\,y\left(-6\hbar^2+10T\hbar^2-6T^2\hbar^2-6T^3\hbar^2+10T^4\hbar^2-6T^5\hbar^2\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}\right]\right\}$$

CU

Associativity of CU:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3}, HL[(cm_{1,2→1} // cm_{1,3→1}) ≡ (cm_{2,3→2} // cm_{1,2→1})]]
```

(Alt) Out[∘]= {2.01563, True}

Associativity, co-associativity, and Δ is an algebra morphism:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3}, HL /@ {(cm_{1,2→1} // cm_{1,3→1}) ≡ (cm_{2,3→2} // cm_{1,2→1}),
    (cΔ_{1→1,2} // cΔ_{2→2,3}) ≡ (cΔ_{1→1,3} // cΔ_{1→1,2}),
    (cm_{1,2→1} // cΔ_{1→1,2}) ≡ ((cΔ_{1→1,3} cΔ_{2→2,4}) // (cm_{3,4→2} cm_{1,2→1}))}]
```

(Alt) Out[∘]= {3.54688, {True, True, True}}

S is convolution inverse of id:

(Alt) In[∘]:=
```
Timing@Block[{$k = 3}, HL[# ≡ cϵ_1 cη_1] & /@ {
    (cΔ_{1→1,2} // cS_1) // cm_{1,2→1}, (cΔ_{1→1,2} // cS_2) // cm_{1,2→1}}]
```

(Alt) Out[∘]= {3.45313, {True, True}}

S is an algebra anti-(co)morphism

(Alt) In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(cm_{1,2→1} // cS_1) ≡ ((cS_1 cS_2) // cm_{2,1→1}), (cS_1 // cΔ_{1→1,2}) ≡ (cΔ_{1→2,1} // (cS_1 cS_2))}]
```

(Alt) Out[∘]= {5.53125, {True, True}}

Classical is the $\hbar \to 0$ limit of quantum:

```
(Alt) In[∘]:= ClassicalLimit[f_] := Normal@Series[Normal[f] // U2l, {ℏ, 0, 0}] // l2U;
        Timing[HL /@ Simplify /@
           {cm1,2→3 ≡ ClassicalLimit /@ dm1,2→3,
            (cΔ1→2,3 /. τ1 → 0) ≡ ClassicalLimit /@ dΔ1→2,3, cS1 ≡ ClassicalLimit /@ dS1}]
(Alt) Out[∘]= {1.95313, {True, True, True}}
```

```
(Alt) In[∘]:= PrintProfile[]
(Alt) Out[∘]= ProfileRoot is root. Profiled time: 262.768
   (      1)    0.360/ 101.530 above Z
   (     59)    1.189/  33.596 above Boot
   (   1314)    3.086/   9.406 above CF
   (    197)    6.817/  31.713 above EZip3
   (      1)        0/       0 above RVK
   (    197)    4.110/   5.906 above Zip1
   (    394)    4.843/  41.981 above Zip2
   (    197)   10.741/  38.635 above Zip3
CCF: called 115780 times, time in 105.884/105.884
   ( 115780)  105.880/ 105.880 under CF
CF: called 88372 times, time in 91.391/197.275
   (    214)    1.390/   3.066 under Z
   (    407)    0.421/   1.481 under Boot
   (   1263)    8.786/  20.797 under EZip3
   (   1314)    3.086/   9.406 under ProfileRoot
   (    680)    1.402/   3.360 under Zip1
   (   2526)   35.363/  97.405 under Zip2
   (  81968)   40.943/  61.760 under Zip3
   ( 115780)  105.880/ 105.880 above CCF
Zip3: called 680 times, time in 28.242/90.002
   (     57)    3.268/  17.156 under Z
   (     86)    4.596/  11.610 under Boot
   (    340)    9.637/  22.601 under EZip3
   (    197)   10.741/  38.635 under ProfileRoot
   (  81968)   40.943/  61.760 above CF
EZip3: called 340 times, time in 18.696/62.094
   (     57)   10.861/  22.017 under Z
   (     86)    1.018/   8.364 under Boot
   (    197)    6.817/  31.713 under ProfileRoot
   (   1263)    8.786/  20.797 above CF
   (    340)    9.637/  22.601 above Zip3
Zip2: called 680 times, time in 8.489/105.894
   (    114)    1.602/  56.540 under Z
   (    172)    2.044/   7.373 under Boot
   (    394)    4.843/  41.981 under ProfileRoot
   (   2526)   35.363/  97.405 above CF
Zip1: called 340 times, time in 8.344/11.704
   (     57)    1.246/   2.033 under Z
```

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  (    86)   2.988/  3.765 under Boot
  (   197)   4.110/  5.906 under ProfileRoot
  (   680)   1.402/  3.360 above CF
Boot: called 86 times, time in 1.362/48.422
  (     3)       0/  0.359 under Z
  (    24)   0.173/ 14.467 under Boot
  (    59)   1.189/ 33.596 under ProfileRoot
  (    24)   0.173/ 14.467 above Boot
  (   407)   0.421/  1.481 above CF
  (    86)   1.018/  8.364 above EZip3
  (    86)   2.988/  3.765 above Zip1
  (   172)   2.044/  7.373 above Zip2
  (    86)   4.596/ 11.610 above Zip3
Z: called 1 times, time in 0.36/101.531
  (     1)   0.360/ 101.530 under ProfileRoot
  (     3)       0/  0.359 above Boot
  (   214)   1.390/  3.066 above CF
  (    57)  10.861/ 22.017 above EZip3
  (    57)   1.246/  2.033 above Zip1
  (   114)   1.602/ 56.540 above Zip2
  (    57)   3.268/ 17.156 above Zip3
RVK: called 1 times, time in 0./0.
  (     1)       0/     0 under ProfileRoot
```