Pensieve header: Exponentiation in ybax algebras.

Startup

In[∘]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine.m
<< Objects.m
<< KT.m
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Out[∘]= {2021, 8, 12, 8, 30, 14.4571939}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

Exponentials

Task. Define $\text{Exp}_m[\mathbb{U}_{\{\_\_\}\to\{i\_\}}[U\_\_]]$ to compute $e^{\mathbb{O}(U)}$ to order $\epsilon^{\text{Length@}\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where U is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.

Example: $\text{Exp}_{\text{dm},1}[\mathbb{U}_{\{\}\to\{2\}}[b_2\, a_2 + y_2\, x_2, 0]]$ is the exponential of the arrow on strand 2, computed to degree 1.

In[∘]:=
```
m = dm; i = 2; U = Sequence[b₂ a₂ + y₂ x₂, 0]
```

Out[∘]= $\text{Sequence}[a_2\, b_2 + x_2\, y_2,\, 0]$

In[∘]:=
```
F[λ_, i_] := 𝔼_{}→{i}[f₁[λ] + f₂[λ] a_i + f₃[λ] x_i + f₄[λ] y_i + f₅[λ] x_i y_i];
F[λ, i]
F[μ, j]
```

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}[f_1[\lambda] + a_2\, f_2[\lambda] + x_2\, f_3[\lambda] + y_2\, f_4[\lambda] + x_2\, y_2\, f_5[\lambda]]$

Out[∘]= $\mathbb{E}_{\{\}\to\{j\}}[f_1[\mu] + a_j\, f_2[\mu] + x_j\, f_3[\mu] + y_j\, f_4[\mu] + x_j\, y_j\, f_5[\mu]]$

In[∘]:= `l1 = (∂_μ List @@ (F[λ, i] F[μ, j] // m_{i,j→i})) /. μ → 0`

Out[∘]=
$$\Big\{a_2 f_2'[0] + \frac{e^{-f_2[0]} x_2 \left(-e^{f_2[0]} \hbar f_3[0] - \hbar f_3[\lambda] - e^{f_2[0]} f_3[\lambda] f_5[0] + e^{f_2[0]} B_2 f_3[\lambda] f_5[0]\right) f_2'[0]}{\hbar} +$$
$$\frac{1}{\hbar} e^{-f_2[0]-f_2[\lambda]} x_2 y_2 \left(-e^{f_2[0]} \hbar f_5[0] - e^{f_2[\lambda]} \hbar f_5[\lambda] - e^{f_2[0]+f_2[\lambda]} f_5[0] f_5[\lambda] + e^{f_2[0]+f_2[\lambda]} B_2 f_5[0] f_5[\lambda]\right) f_2'[0] +$$
$$\frac{\hbar f_1'[0] + f_3[\lambda] f_4'[0] - B_2 f_3[\lambda] f_4'[0]}{\hbar} +$$
$$\frac{e^{-f_2[\lambda]} y_2 \left(\hbar f_4'[0] + e^{f_2[\lambda]} f_5[\lambda] f_4'[0] - e^{f_2[\lambda]} B_2 f_5[\lambda] f_4'[0]\right)}{\hbar} -$$
$$\frac{1}{\hbar} e^{-f_2[0]} x_2 \left(-e^{f_2[0]} \hbar f_3[0] f_2'[0] - e^{f_2[0]} f_3[\lambda] f_5[0] f_2'[0] + e^{f_2[0]} B_2 f_3[\lambda] f_5[0] f_2'[0] - e^{f_2[0]} \hbar f_3'[0] - e^{f_2[0]} f_3[\lambda] f_5'[0] + e^{f_2[0]} B_2 f_3[\lambda] f_5'[0]\right) -$$
$$\frac{1}{\hbar} e^{-f_2[0]-f_2[\lambda]} x_2 y_2 \left(-e^{f_2[0]} \hbar f_5[0] f_2'[0] - e^{f_2[0]+f_2[\lambda]} f_5[0] f_5[\lambda] f_2'[0] + e^{f_2[0]+f_2[\lambda]} B_2 f_5[0] f_5[\lambda] f_2'[0] - e^{f_2[0]} \hbar f_5'[0] - e^{f_2[0]+f_2[\lambda]} f_5[\lambda] f_5'[0] + e^{f_2[0]+f_2[\lambda]} B_2 f_5[\lambda] f_5'[0]\right)\Big\}$$

In[∘]:= `r1 = (∂_μ List @@ F[λ + μ, i]) /. μ → 0`

Out[∘]= $\{f_1'[\lambda] + a_2 f_2'[\lambda] + x_2 f_3'[\lambda] + y_2 f_4'[\lambda] + x_2 y_2 f_5'[\lambda]\}$

In[∘]:= `eqs1 = And @@ ((# == 0) & /@ Flatten@CoefficientList[l1 - r1, {a_i, x_i, y_i}]) /. f_[0] → 0`

Out[∘]=
$$-f_1'[\lambda] + \frac{\hbar f_1'[0] + f_3[\lambda] f_4'[0] - B_2 f_3[\lambda] f_4'[0]}{\hbar} == 0 \;\&\&$$
$$\frac{e^{-f_2[\lambda]} \left(\hbar f_4'[0] + e^{f_2[\lambda]} f_5[\lambda] f_4'[0] - e^{f_2[\lambda]} B_2 f_5[\lambda] f_4'[0]\right)}{\hbar} - f_4'[\lambda] == 0 \;\&\&$$
$$-f_3[\lambda] f_2'[0] - f_3'[\lambda] - \frac{-\hbar f_3'[0] - f_3[\lambda] f_5'[0] + B_2 f_3[\lambda] f_5'[0]}{\hbar} == 0 \;\&\&$$
$$-f_5[\lambda] f_2'[0] - \frac{e^{-f_2[\lambda]} \left(-\hbar f_5'[0] - e^{f_2[\lambda]} f_5[\lambda] f_5'[0] + e^{f_2[\lambda]} B_2 f_5[\lambda] f_5'[0]\right)}{\hbar} - f_5'[\lambda] == 0 \;\&\&$$
$$f_2'[0] - f_2'[\lambda] == 0$$

In[∘]:= `l2 = Take[{U}, 1]`

Out[∘]= $\{a_2 b_2 + x_2 y_2\}$

In[∘]:= `r2 = (∂_μ List @@ F[μ, i]) /. μ → 0`

Out[∘]= $\{f_1'[0] + a_2 f_2'[0] + x_2 f_3'[0] + y_2 f_4'[0] + x_2 y_2 f_5'[0]\}$

In[∘]:= `eqs2 = And @@ ((# == 0) & /@ Flatten@CoefficientList[l2 - r2, {a_i, x_i, y_i}])`

Out[∘]= $-f_1'[0] == 0 \;\&\&\; -f_4'[0] == 0 \;\&\&\; -f_3'[0] == 0 \;\&\&\; 1 - f_5'[0] == 0 \;\&\&\; b_2 - f_2'[0] == 0$

In[∘]:= `eqs3 = eqs1 /. {f5'[0] → 1, f2'[0] → b2, f_'[0] → 0}`

Out[∘]= $$-f_1'[\lambda] == 0 \,\&\&\, -f_4'[\lambda] == 0 \,\&\&\, -b_2 f_3[\lambda] - \frac{-f_3[\lambda] + B_2 f_3[\lambda]}{\hbar} - f_3'[\lambda] == 0 \,\&\&\, -b_2 f_5[\lambda] - \frac{e^{-f_2[\lambda]}\left(-\hbar - e^{f_2[\lambda]} f_5[\lambda] + e^{f_2[\lambda]} B_2 f_5[\lambda]\right)}{\hbar} - f_5'[\lambda] == 0 \,\&\&\, b_2 - f_2'[\lambda] == 0$$

In[∘]:= `DSolve[f1[0] == 0 ∧ f2[0] == 0 ∧ f3[0] == 0 ∧ f4[0] == 0 ∧ f5[0] == 0 ∧ eqs3, {f1[λ], f2[λ], f3[λ], f4[λ], f5[λ]}, λ]`

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $-\mathrm{Log}[e^{c_4}] == 0$.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[∘]= $$\left\{\left\{f_1[\lambda] \to 0,\, f_4[\lambda] \to 0,\, f_3[\lambda] \to 0,\, f_2[\lambda] \to \lambda b_2,\, f_5[\lambda] \to \frac{e^{-\frac{\lambda}{\hbar} - \frac{\lambda(-1+\hbar b_2 + B_2)}{\hbar}}\left(-e^{\lambda/\hbar} + e^{\frac{\lambda B_2}{\hbar}}\right)\hbar}{-1 + B_2}\right\}\right\}$$

In[∘]:= `ans = FullSimplify[f5[λ] /. DSolve[-b2 f5[λ] - (e^(-λ b2) (-ℏ - e^(λ b2) f5[λ] + e^(λ b2) B2 f5[λ]))/ℏ - f5'[λ] == 0 ∧ f5[0] == 0, f5[λ], λ]]`

$$\texttt{ans} = \texttt{FullSimplify}\left[f_5[\lambda] \,/.\, \texttt{DSolve}\left[-b_2 f_5[\lambda] - \frac{e^{-\lambda b_2}\left(-\hbar - e^{\lambda b_2} f_5[\lambda] + e^{\lambda b_2} B_2 f_5[\lambda]\right)}{\hbar} - f_5'[\lambda] == 0 \wedge f_5[0] == 0,\, f_5[\lambda],\, \lambda\right]\right]$$

Out[∘]= $$\left\{\frac{e^{-\frac{\lambda(-1+\hbar b_2+B_2)}{\hbar}}\left(-1 + e^{\frac{\lambda(-1+B_2)}{\hbar}}\right)\hbar}{-1 + B_2}\right\}$$

In[∘]:= `FullSimplify[ans /. b2 → 0]`

Out[∘]= $$\left\{\frac{\left(1 - e^{\frac{\lambda - \lambda B_2}{\hbar}}\right)\hbar}{-1 + B_2}\right\}$$