

Pensieve header: The Engine, with Zip3 encapsulation.

Canonical Forms:

```
CCF[ $\mathcal{E}$ _] := PPCCF@ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ]; (*Coefficient Canonical Form *)
CF[ $\mathcal{E}$ _] := PPCF@Module[
  {vs = Cases[ $\mathcal{E}$ , (y | a | x |  $\eta$  |  $\beta$  |  $\tau$  |  $\xi$ )_,  $\infty$ ] U {y, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\xi$ }},
  Total[(CCF[#][2]] (Times@@vs#[[1]]) & /@ CoefficientRules[ $\mathcal{E}$ , vs]]
];
CF[ $\mathcal{E}$ _E] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[Esp___[ $\mathcal{E}$ S___]] := CF /@ Esp[ $\mathcal{E}$ S];
```

Variables and their duals:

```
In[ $e$ ]:=
{t*, b*, y*, a*, x*, z*,  $\tau$ *,  $\beta$ *,  $\eta$ *,  $\alpha$ *,  $\xi$ *,  $\zeta$ *} = { $\tau$ ,  $\beta$ ,  $\eta$ ,  $\alpha$ ,  $\xi$ ,  $\zeta$ , t, b, y, a, x, z};
(vs_List)* := (v  $\mapsto$  v*) /@ vs;
(u_i)* := (u*)i;
F[u_i] := F[ui] = ToExpression["F" <> ToString[u]]i
```

Weights:

```
Clear[Wt];
Evaluate[Wt /@ {y, b, t, a, x,  $\eta$ ,  $\beta$ ,  $\tau$ ,  $\alpha$ ,  $\xi$ }] = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[u_i] := Wt[u];
```

The maximal weight \$n, i.e. the \$n\$ of \$gl(n)\$. Initially and for a long while this will not be tested beyond \$n == 2\$.

```
In[ $e$ ]:=
$n = 2;
```

Upper to lower and lower to Upper:

```
U21[ $\mathcal{E}$ _] :=  $\mathcal{E}$  /. {Bip -> e-p h bi, Bp -> e-p h b, Tip -> ep h ti, Tp -> ep h t, Aip -> ep  $\alpha$ i, Ap -> ep  $\alpha$ };
l2U[ $\mathcal{E}$ _] :=  $\mathcal{E}$  //. {ec- bi+d- -> Bi-c/h ed, ec- b+d- -> B-c/h ed, ec- ti+d- -> Tic/h ed, ec- t+d- -> Tc/h ed,
ec-  $\alpha$ i+d- -> Aic ed, ec-  $\alpha$ +d- -> Ac ed, e $\mathcal{X}$  -> eExpand@ $\mathcal{X}$ };
l2U[r_Rule] := Module[{U = r[[1]] /. {b -> B, t -> T,  $\alpha$  -> A}}, U -> l2U[U21[U] /. r]];
AlsoUpper[rs_List] := rs U (l2U /@ rs);
```

Derivatives in the presence of exponentiated variables:

```
In[ $e$ ]:=
Db[f_] :=  $\partial_b$  f - h B  $\partial_B$  f; Dbi[f_] :=  $\partial_{b_i}$  f - h Bi  $\partial_{B_i}$  f;
Dt[f_] :=  $\partial_t$  f + h T  $\partial_T$  f; Dti[f_] :=  $\partial_{t_i}$  f + h Ti  $\partial_{T_i}$  f;
D $\alpha$ [f_] :=  $\partial_\alpha$  f + A  $\partial_A$  f; D $\alpha$ i[f_] :=  $\partial_{\alpha_i}$  f + Ai  $\partial_{A_i}$  f;
Dv[f_] :=  $\partial_v$  f;
```

E operations:

```

 $\mathcal{E}_E[\$] := \text{Length}[\mathcal{E}] - 1; \mathbb{E}_E[\mathcal{ES}\_\_\_] [\$] := \mathbb{E}[\mathcal{ES}][\$];$ 
 $\mathcal{E}_E[k\_Integer] := \mathcal{E}[[k + 1]]; \mathbb{E}_E[\mathcal{ES}\_\_\_] [k\_Integer] := \{\mathcal{ES}[[k + 1]]\};$ 
 $\mathbb{E} /: \mathcal{E1}_E \equiv \mathcal{E2}_E := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \ \&, \mathcal{E1}, \mathcal{E2}, \text{And}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}\_\_\_] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E1S}] \equiv \mathbb{E}[\mathcal{E2S}]);$ 
 $\mathbb{E} /: \mathcal{E1}_E * \mathcal{E2}_E := \mathbb{E}@\text{Table}[\text{CF}[\mathcal{E1}[\text{kk}] + \mathcal{E2}[\text{kk}]], \{\text{kk}, \emptyset, \text{Min}[\mathcal{E1}[\$], \mathcal{E2}[\$]]\}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}\_\_\_] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}\_\_\_] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}@\text{Table}[\text{CF}[\mathcal{E1S} \mathcal{E2S}]];$ 

```

```

 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}\_\_\_] // \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}\_\_\_] := \text{Module}[\{\text{is} = r1 \cap d2, \text{lvs}\},$ 
 $\text{lvs} = \text{Flatten}@\text{Table}[\{\text{y}_{\$ei}, \text{b}_{\$ei}, \text{t}_{\$ei}, \text{a}_{\$ei}, \text{x}_{\$ei}\}, \{\text{i}, \text{is}\}];$ 
 $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])}@\text{Table}[\{\text{Zip}_{\text{lvs} \cup \text{lvs}^*}[\{(F / @ \text{lvs}^*) \cdot (F / @ \text{lvs})\}, \text{Times}[\mathbb{E}[\mathcal{E1S}] / . \text{Table}[(\text{v} : \text{b} | \text{B} | \text{t} | \text{T} | \text{a} | \text{x} | \text{y})_i \rightarrow \text{v}_{\$ei}, \{\text{i}, \text{is}\}], \mathbb{E}[\mathcal{E2S}] / . \text{Table}[(\text{v} : \beta | \tau | \alpha | \mathcal{A} | \xi | \eta)_i \rightarrow \text{v}_{\$ei}, \{\text{i}, \text{is}\}]\}]\}]]$ 

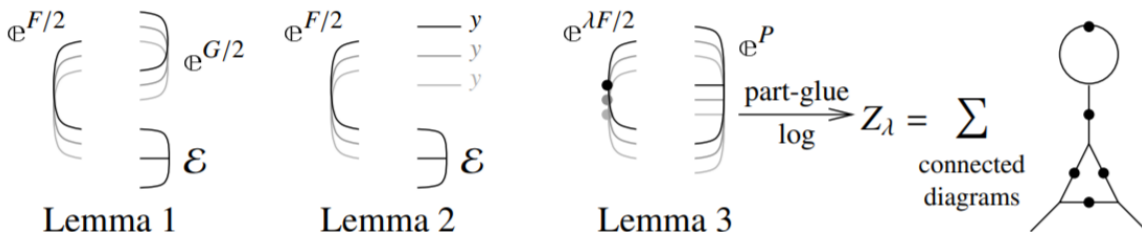
```

```

 $\Delta \mathbb{E}_{d \rightarrow r}[\mathcal{A}_\_] := \text{Module}[\{\text{k}\}, \mathbb{E}_{d \rightarrow r}@\text{Table}[\text{SeriesCoefficient}[\mathcal{A}, \{\epsilon, \emptyset, \text{k}\}], \{\text{k}, \emptyset, \text{\$k}\}];$ 

```

Ziping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



```

In[ ] :=  $\text{Zip}_{\text{vs}}[\{\mathcal{F}_\_, \mathcal{E}_\-\}] := \{\mathcal{F}, \mathcal{E}\} // \text{Zip1}_{\text{vs}} // \text{Zip2}_{\text{Select}}[\text{vs}, (\emptyset < \text{Wt}[\#] < \text{\$n}) \&] // \text{EZip3}_{\text{Select}}[\text{vs}, (\emptyset < \text{Wt}[\#] < \text{\$n}) \&] //$ 
 $\text{Zip2}_{\text{Select}}[\text{vs}, (\text{Wt}[\#] == \emptyset \vee \text{Wt}[\#] == \text{\$n}) \&] // \text{Zip3}_{\text{Select}}[\text{vs}, (\text{Wt}[\#] == \emptyset \vee \text{Wt}[\#] == \text{\$n}) \&] // \text{Last};$ 

```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \mathbb{E}^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```

In[ ] :=  $\text{Zip1}_{\{\}} = \text{Identity};$ 
 $\text{Zip1}_{\text{vs}}@\{\mathcal{F}_\_, \mathbb{E}[\mathcal{Q}_\_, \mathcal{P}\_\_\_]\} := \text{PPZip1}@\text{Module}[\{\mathcal{I}, \mathcal{F}, \mathcal{G}, \text{u}, \text{v}\},$ 
 $\mathcal{I} = \text{IdentityMatrix}@\text{Length}@\text{vs};$ 
 $\mathcal{F} = \text{Table}[\text{If}[\text{Wt}[\text{u}] + \text{Wt}[\text{v}] == \text{\$n}, \partial_{\mathcal{F}[\text{u}], \mathcal{F}[\text{v}]} \mathcal{F}, \emptyset], \{\text{u}, \text{vs}\}, \{\text{v}, \text{vs}\}];$ 
 $\mathcal{G} = \text{Table}[\text{If}[\text{Wt}[\text{u}] + \text{Wt}[\text{v}] == \text{\$n}, \partial_{\text{u}, \text{v}} \mathcal{Q}, \emptyset], \{\text{u}, \text{vs}\}, \{\text{v}, \text{vs}\}];$ 
 $\{\text{CF}[(F / @ \text{vs}) \cdot (F.\text{Inverse}[\mathcal{I} - \mathcal{G}.\mathcal{F}]) \cdot (F / @ \text{vs}) / 2],$ 
 $\mathbb{E}[\text{CF}[\mathcal{Q} - \text{Log}[\text{Det}[\mathcal{I} - \mathcal{G}.\mathcal{F}]] / 2 - \text{vs}.\mathcal{G}.\text{vs} / 2], \mathcal{P}]\}$ 

```

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E} \mathbb{E}^{\frac{1}{2} \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{E}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```

Zip2_{ } = Identity;
Zip2_{vs_} @ { F_, E[Q_, P_] } := PPZip2@Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == $n, D_{F[u], F[v]} F, 0], {u, vs}, {v, vs}];
  Y = Table[D_v Q, {v, vs}] /. AlsoUpper@Table[v -> 0, {v, vs}];
  CF /@ ({F, E[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs -> vs + F.Y])
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{C}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

Zip3_{vs_} @ { F_, E[E_] } := PPZip3@Module[{F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[E] - 1;
  Do[Z[0, kk] = E[[kk + 1]], {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n, D_{F[u], F[v]} F, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = D_u[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = D_v[Z[m, kk, u]];
  For[m = 0, m <= 2 $m, ++m, For[kk = 0, kk <= $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0, F[u, v] / (2 (m + 1))
        (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1];
  ]];
  CF /@ ({
    F - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
  }) /. AlsoUpper@Table[v -> 0, {v, vs}]
]

```

Encapsulation.

```

EZip3vs @ { $\mathcal{F}$ _,  $\mathcal{E}$ _E} := PPEZip3@Module [
  {n $\mathcal{E}$ , n $\mathcal{F}$ , rc, ps, rr = {(*release rules*)}, FVS},
  rc = 0; n $\mathcal{E}$  = Total [
    CoefficientRules[#, vs] /. (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{E}$ [++rc] → c]; c $\mathcal{E}$ [rc] (Times @@ vsps))
  ] & /@  $\mathcal{E}$ ;
  rc = 0; FVS = F /@ vs;
  n $\mathcal{F}$  = Total [CoefficientRules[ $\mathcal{F}$ , FVS] /.
    (ps_ → c_) ⇒ (AppendTo[rr, c $\mathcal{F}$ [++rc] → c]; c $\mathcal{F}$ [rc] (Times @@ FVSps))];
  CF [Expand[{n $\mathcal{F}$ , n $\mathcal{E}$ } // Zip3vs] /. rr]
]

```