Quillen's theorem - degree 2 testing

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Goal. Construct an expansion $Z: G \to \mathcal{U}(L_*(G))$.

$G_1 = G$; $G_{n+1} = (G, G_n)$ $\quad L_n := \mathbb{Q} \otimes (G_n / G_{n+1})$

$L_n \longrightarrow \mathcal{P}_n(\mathcal{A}) =: \mathcal{L}_n$ by $c \longmapsto \bar{c} = c - 1$

There is
$\pi: A = \mathcal{U}(L_*) \twoheadrightarrow \mathcal{A}$

surjective because it is surjective in degree 1.

Can I construct an A-expansion?

$G_1/G_2 \underset{\pi_1}{\overset{\sigma_1}{\rightleftarrows}} G_1 \qquad G_2/G_3 \underset{\pi_2}{\overset{\sigma_2}{\rightleftarrows}} G_2$

$$\zeta(g) = \left(0, \pi_0(G), \pi_1\left(g\sigma_1(\pi_0(G))^{-1}\right), \ldots\right) \in L_*$$

So there is $\quad$ And also

$\zeta: G \to L_*$. $\quad \sigma: L_* \to G$

<u>Claim</u> $Z = e^\zeta$ is an A-expansion. (?)

I.e., $Z(c) = e^{\zeta(c)} = \left(1, \zeta_1(c), \frac{1}{2}\zeta_1(c)^2 + \zeta_2(c), \ldots\right) = \#$

if $c = \sigma_1(\gamma)$ then $\zeta(c) = (\gamma, 0, 0 \ldots)$ so

$\# = \left(1, \gamma, \frac{1}{2}\gamma\gamma, \ldots\right)$

Check in degree 2:

Need to consider $Z(\pi(ab))$ ($\&$ $Z(\pi([a,b]))$?)

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$Z(\pi(ab))$: (assume $a=\sigma_1(\alpha)$, $b=\sigma_1(\beta)$)

$$ab \xrightarrow{\pi} \pi(a)\pi(b) = (a-1)(b-1) = 1-a-b+ab$$

$$\xrightarrow{Z} (1,0,0,\dots)-(1,\alpha,\tfrac12\alpha\alpha)-(1,\beta,\tfrac12\beta\beta)$$

$$+\left(1,\alpha+\beta,\frac{(\alpha+\beta)(\alpha+\beta)}{2}+\pi_2\left(ab\,\sigma_1(\alpha+\beta)^{-1}\right)\right)$$

$$=\left(0,0,\frac{\alpha\beta+\beta\alpha}{2}+\pi_2\left(ab\,\sigma_1(\alpha+\beta)^{-1}\right)\right)$$

$$=\left(0,0,\alpha\beta+\tfrac12[\beta,\alpha]+\pi_2\left(ab\,\sigma_1(\alpha+\beta)^{-1}\right)\right)$$

.....?