A: they may or may not separate group elements,

# EXPANSIONS AND QUADRATICITY FOR GROUPS

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Abstract. First year students learn that the Taylor expansion $Z^T$ carries functions into power series, and that it has some nice algebraic properties (e.g., $Z^T(fg) = Z^T(f)Z^T(g)$). It is less well known that the same game can be played within arbitrary groups: there is a natural way to say "a Taylor expansion $Z$ for elements of an arbitrary group $G$", and a natural way to carry the algebraic properties of the Taylor expansion to this more general context. In the case of a general $G$ "~~bi-multiplicative~~ expansions" (expansions with the same good properties as $Z^T$) may or may not exist, and a further good property which is hidden in the case of $Z^T$, "quadraticity", may or may not hold. Taylor

The purpose of this expository note is to properly define all the notions in the above paragraph, to enumerate some classes of groups ~~for which we know whether bi-multiplicative expansions exists and/or for which we know whether they are quadratic~~, to briefly indicate the relationship between these notions and the notions of "finite type invariants" and "unipotent" and "Mal'cev" completions, and to point out (with references) that our generalization of "expansions" to arbitrary groups is merely the tip of an iceberg, for almost everything we say can be generalized further to "expansions for arbitrary algebraic structures". whose theory of expansions we either understand or wish to understand,

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## 1. Introduction

1.1. **The Taylor Expansion.** Before the real thing, which commences in Section 1.2, we start with a brief reminder on the Taylor ~~polynomial~~, expansion which serves as a motivation for Section 1.2.

Let $R = C^\infty(\mathbb{R}^k)$ the the algebra of smooth real-valued functions on the $k$-dimensional Euclidean space $\mathbb{R}^k$, and let $I$ be the ideal within $R$ of functions that vanish at 0: $I := \{f \in R\colon f(0) = 0\}$. Let $I^0 := R$, $I^1 := I$, and for $n > 1$ let $I^n$ be the $n$th power of $I$: the set of all products of the form $f_1 f_2 \dots f_n$, where $f_i \in I$ for every $i$. Then $I^n$ is the space of smooth functions that "vanish at least $n$ times at 0", and hence the quotient $I^n/I^{n+1}$ is "functions vanishing $n$ times, while regarding as 0 functions that vanish more than $n$ times", which is precisely "homogeneous polynomials of degree $n$". Thus the space

$$\mathcal{A}(R) := \prod_{n\geq 0} I^n/I^{n+1} \tag{0}$$

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can be identified as the space of power series on $\mathbb{R}^k$. The "Taylor expansion" is a ~~linear~~ linear map $Z^T\colon R \to \mathcal{A}(R)$, and one may show that it is characterised by the following three properties (see Proposition 1.8 ~~below~~ on page ...):

(1) $Z^T$ *is an expansion:* If $f \in I^n$ then $Z^T(f)$ begins with $[f]$, the class of $f$ within $I^n/I^{n+1}$. Namely,

$$Z^T(f) = (0, \dots, 0, \underbrace{[f]}_{\text{in degree } n}, *, *, \dots), \qquad (1)$$

where "$*$" stands for "something arbitrary".

(2) $Z^T$ *is multiplicative:*

$$Z^T(fg) = Z^T(f)Z^T(g), \qquad (2)$$

where the product on the left is the pointwise product of functions and the product on the right is the product of power series.

(3) $Z^T$ *is co-multiplicative:* If $g$ is a smooth function~~s~~ of two variables $x, y \in \mathbb{R}^k$ then its Taylor expansion $Z^T_x(g)$ with respect to the variable $x$ is a smooth function of the variable $y$ (namely, each coefficient of each homogeneous polynomial appearing in $Z^T_x(g)$ is a smooth function of $y$). Thus there is an iterated Taylor expansion $Z^T_{x,y}(g) := Z^T_y(Z^T_x(g))$, and it can be interpreted as taking values in the space of power series in two variables $x, y$. With all this and with $f \in R$, we have that

$$Z^T_{x,y}(f(x+y)) = (Z^T(f))(x+y).$$

Alternatively, with $\square$ denoting the operation $f(x) \mapsto f(x+y)$, defined on both functions and power series (and doubling the number of variables in each case), we have that

$$Z^T_{x,y} \circ \square = \square \circ Z^T. \qquad (3)$$

1.2. **General Expansions.** A space "of power series" $\mathcal{A}(R)$ can be defined whenever $R$ is a ring and $I$ is an ideal in $R$. Yet in this paper we restrict to the case when the ring $R$ is the group ring of a group $G$.[1] So let $G$ be an arbitrary discrete group whose identity element is denoted $e$.[2] Let $R := \mathbb{Q}G = \{\sum_{i=1}^k a_i g_i \colon a_i \in \mathbb{Q},\ g_i \in G\}$ be the group ring of $G$ over the rational numbers $\mathbb{Q}$, and let $I = I_G$ be the augmentation ideal of $\mathbb{Q}G$:

$$I = \left\{ \sum a_i g_i \colon \sum a_i = 0 \right\} = \langle g - e \colon g \in G \rangle .$$

We declare that $I^0$ is $R = \mathbb{Q}G$ and also consider all higher powers $I^n$ of $I$.

**Definition 1.1.** The power series ring Foot A $\mathcal{A}(G)$ of the group $G$ is the direct product

$$\mathcal{A}(G) := \prod_{n \geq 0} I^n/I^{n+1}. \qquad (4)$$

We note that M! the product $I^m \otimes I^n \to I^{m+n}$ induced by the product in $R$ descends to a product $(I^m/I^{m+1}) \otimes (I^n/I^{n+1}) \to I^{m+n}/I^{m+n+1}$ and hence $\mathcal{A}(G)$ is in fact a (completed) graded ring. We note also that $\mathcal{A}$ is a functor: a group homomorphism $\phi\colon G \to H$ induces a morphism $\phi\colon \mathbb{Q}G \to \mathbb{Q}H$ for which $\phi(I_G) \subset I_H$ and hence $\phi(I^n_G) \subset I^n_H$ for all $n$. Hence we get an induced map $\phi\colon \mathcal{A}(G) \to \mathcal{A}(H)$ which is easily seen to be a morphism of graded rings.

$\mathcal{A}(G)$ sometimes remembers much of the structure of $G$, and sometimes forgets much of it, as we shall see below. It Yet ~~it~~ always, for any group $G$ whatsoever, makes sense to seek a "Taylor expansion for $G$" — a map $Z\colon G \to \mathcal{A}(G)$ satisfying the three properties that characterize the ordinary Taylor expansion, as in the three definitions below.

[1] So in fact, our motivating example, the Taylor expansion, is strictly speaking not a special case for our definitions but only a close associate which is obtained when our definitions are restated ~~using~~ For arbitrary rings.

[2] In practice, the groups we tend to consider are infinite and have little torsion. A good example to keep in mind is the pure braid group $PuB_n$ on $n$ strands. Many further example are in Table 1.



Foot A: Quillen has a paper [Qu] devoted to the study of $\mathcal{A}(G)$, but he never names that ring beyond "the associated graded ring of a group ring".

**Definition 1.2.** A map $Z\colon G \to \mathcal{A}(G)$ is called "an expansion" if its homonymous (uniquely defined) linear extension $Z\colon \mathbb{Q}G \to$ if $\mathcal{A}(G)$ has the property that ~~whenever~~ $f \in I^n$ then $Z(f)$ begins with $[f]$, the class of $f$ within $I^n/I^{n+1}$. Namely,

$$Z(f) = (0, \dots, 0, \underbrace{[f]}_{\text{in degree } n}, *, *, \dots), \qquad (5)$$

where "$*$" stands for "something arbitrary".

The sequence $I^0 \supset I^1 \supset I^2 \dots$

Somewhat more abstractly, note that $\mathbb{Q}G$ is filtered by $(I^n)_n$ and that $\mathcal{A}(G)$ is the (completed) associated graded space of that filtration, $\mathcal{A}(G) = \operatorname{gr}\mathbb{Q}G$. Note that graded spaces are automatically also filtered, with the $n$th filtration space being the product of the degree $m$ subspaces over all $m \geq n$. Note also that gr is a functor on the category of filtered spaces and that $\operatorname{gr}\circ\operatorname{gr}$ is naturally equivalent to gr. With all this in mind, ~~the~~ condition (5) is equivalent to the following:

> $Z\colon \mathbb{Q}G \to \operatorname{gr}\mathbb{Q}G$ is a filtration preserving linear map so that $\operatorname{gr} Z\colon \operatorname{gr}\mathbb{Q}G \to \operatorname{gr}\operatorname{gr}\mathbb{Q}G = \operatorname{gr}\mathbb{Q}G$ is the identity map of $\operatorname{gr}\mathbb{Q}G = \mathcal{A}(G)$.

Even before completing the definition of a "Taylor" expansion for a general group, we can already ponder whether a group has "powerful" expansions.

**Definition 1.3.** We say that a group $G$ has a faithful expansion if it has an injective expansion $Z\colon G \to \mathcal{A}(G)$.

Proposition **??** in Section 2 implies that if one expansion for a group $G$ is injective, then so is every other expansion for $G$.

A summary of what we know about the faithfulness of expansions for specific groups is in Table 1.

Next is the analogue of (2):

**Definition 1.4.** An expansion $Z\colon G \to \mathcal{A}(G)$ is said to be "multiplicative" if $Z(g_1g_2) = Z(g_1)Z(g_2)$ for every $g_{1,2} \in G$.

Before we can state Definition 1.6, the analogue of (3), we need the following proposition:

**Proposition 1.5** (Proof in MORE). *If $G$ and $H$ are groups, then $\mathcal{A}(G\times H) \cong \mathcal{A}(G)\otimes\mathcal{A}(H)$, where the $\otimes$ symbol is understood in the completed graded sense:*

$$\mathcal{A}(G)\otimes\mathcal{A}(H) = \prod_n \bigoplus_{n_1+n_2=n} \mathcal{A}(G)_{n_1}\otimes\mathcal{A}(H)_{n_2}.$$

*(More precisely, $\mathcal{A}$ is a "monoidal functor").*

Now given a group $G$ let $\Box\colon G \to G$ be the "diagonal" map $g \mapsto (g,g)$ and let the same symbol $\Box$ also denote the functorially-induced morphism $\Box\colon \mathcal{A}(G) \to \mathcal{A}(G\times G) \cong \mathcal{A}(G)\otimes\mathcal{A}(G)$. The analogue of (3) is:

**Definition 1.6.** An expansion $Z\colon G \to \mathcal{A}(G)$ is said to be "co-multiplicative" if the following equality of maps $G \to \mathcal{A}(G)\otimes\mathcal{A}(G)$ holds true:

$$\Box\circ Z = (Z\otimes Z)\circ\Box.$$

Finally, we come to ~~our first main definition~~ the definition of "Taylor".:

**Definition 1.7.** We say that a group $G$ is "Taylor" if it has a Taylor expansion — an $\mathcal{A}(G)$-valued multiplicative and co-multiplicative expansion.

A summary of what we know about the Taylor property for specific groups is in Table 1.

1.3. **Quadraticity.** MORE.

1.4. **Back to Taylor.** For the sake of completeness, we conclude this introduction with the following proposition:

**Proposition 1.8.** *The three properties of the Taylor expansion enumerated at the beginning of the introduction characterize the Taylor expansion. In other words, if $Z'\colon R \to \mathcal{A}(R)$ is linear and satisfies Equations (1), (2), and (3), then $Z' = Z^T$*

move to an earlier page.

| Group(s) $G$ | Faithful $Z$? | Taylor $Z$? | Quadratic? | See |
|---|---|---|---|---|
| Finite / torsion groups | No (except $G = \{e\}$) | Yes (trivially) | Yes (trivially) | |
| Knot and link groups | No (except $G = \mathbb{Z}^n$) | Yes (trivially) | Yes (trivially) | |
| $\mathbb{Z}^n$ | Yes | Yes | Yes | |
| Free groups $FG_n$ | Yes | Yes | Yes | |
| Pure braid groups $PuB_n$ | Yes | Yes | Yes | |
| indent Reduced (homotopy) pure braid groups $RPuB_n$ | Yes | Yes | Yes | |
| Pure v-braid groups $PvB_n$ | Unknown | No | Yes | |
| Pure w-braid groups $PwB_n$ | Yes | Yes | Yes | |
| Elliptic pure braid groups $PuB_n^1$ (braids on the torus) | Unknown | Yes | No | |
| Higher genus pure braid groups $PuB_n^{>1}$ (braids on high genus surfaces) | Unknown | Unknown | No | |
| Fundamental groups of surfaces | | Yes | Yes | |
| Right-angled Artin groups | | Yes | Yes | |

MORE: Sort in: Brochier's arXiv:1209.0417, make sure that all statements are referenced.

**Table 1.** A summary of the groups considered in this paper.

Disclaimer: "Unknown" is to the author.

*Proof.* The Taylor expansion is elementary and well known and for us it is merely a motivating example. Hence we only indicate the main steps of the proof and leave the details to ~~for~~ the reader. As $Z'$ is an expansion (1), it is enough to show that $Z'(p) = p$ whenever $p$ is a polynomial. As it is multiplicative (2), it is enough to show that $Z'(x_i) = x_i$ for $i = 1, \dots, k$. Let $f_i := Z'(x_i)$. By (1), $f_i = x_i + (\text{higher degrees})$. Also, $x_i$ satisfies $\square(x_i) = x_i + y_i$ and so by (3) ~~and with $f_i := Z'(x_i)$~~ we have that $f_i(x+y) = f_i(x) + f_i(y)$. It is easy to check that the only power series $f_i$ satisfying these two conditions are ~~is~~ $f_i = x_i$. $\square$

We note that all three conditions are necessary for Proposition 1.8. Indeed if $\Upsilon\colon \mathcal{A}(R) \to \mathcal{A}(R)$ is an arbitrary degree-increasing linear operator (no conditions at all, so there are plenty of choices) then $Z' := \Upsilon \circ Z^T$ is an expansion (satisfies (1)) while generally breaking (2) and (3), and if $f_i$ are of the form $f_i = x_i + (\text{higher degrees})$ then setting $Z'(x_i) := f_i$ defines a multiplicative expansion uniquely, and unless $f_i = x_i$, it will not be co-multiplicative.

## 2. Some General Facts about Power Series and Expansions

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