REPORT ON "RIBBON 2-KNOTS, 1+1=2 AND DUFLO'S THEOREM FOR ARBITRARY LIE ALGEBRAS"

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic 0. Duflo's theorem is a fundamental result in Lie theory stating the existence of an algebra isomorphism from the center $Z(U(\mathfrak{g}))$ of the enveloping algebra of $\mathfrak{g}$, and the invariant part of the symmetric algebra $S(\mathfrak{g})$ under the adjoint action of $\mathfrak{g}$. In 1978, Kashiwara and Vergne made a remarkable conjecture which implies Duflo's theorem, as well as many other important results in harmonic analysis and representation theory. This conjecture was proved only fairly recently, and it turns out it is best understood in the realm of deformation quantization. Indeed, the first proof by Alekseev-Meinrenken relies on Kontsevich's theorem on deformation quantization of Poisson manifold, while the second by Alekseev-Torrossian (which is the one relevant for the paper under review) uses the theory of Drinfeld associators which is the key ingredient tying low dimensional topology and quantum algebra.

A general philosophy, of which the first-named author of the paper under review has always been a strong advocate, is that many hard theorems in quantum algebra are, in fact, images of fairly simple statements about topological gadgets, through a well-behaved topological invariant. The prominent example of this philosophy is precisely the theory of Drinfeld associators, which on the one hand provides a combinatorial construction of the famous Kontsevich integral of ordinary braids and knots, and on the other hand leads to universal quantizations of metrizable Lie algebras, i.e. Lie algebra equipped with a non-degenerate invariant symmetric pairing.

In a series of articles, the first two named authors of the paper under review gave such a topological interpretation of the Kashiwara–Vergne conjecture by showing solutions of this conjectures are in one to one correspondence with well-behaved topological invariants of so-called welded knots, braids and tangles. Those are interesting topological objects in their own right, being naturally related to the theory of knotted surfaces in a $\mathbb{R}^4$.

The main goal of the paper under review is to give a new proof of Duflo's theorem using the above-mentioned topological invariant as a black box. More precisely the interpret the map $Z(U(\mathfrak{g})) \to S(\mathfrak{g})^{\mathfrak{g}}$ as an element in (a certain completion of) $S(\mathfrak{g}^*)_{\mathfrak{g}} \otimes U(\mathfrak{g})$ where the under script means taking coinvariants. They show this element is the image through this well-behaved invariant of a simple topological object, and then that the condition that this induces an algebra isomorphism is the image of a simple topological statement that they call "1+1=2".

I should stress this is, as far as I can tell, a genuinely new proof of Duflo's theorem, not just a graphical reformulation of the known implication from the KV conjecture. Roughly speaking, this is because the equivalence between solutions of the KV conjecture, and the existence of a well-behaved invariant for those topological objects is itself quite non trivial and the proof at hand uses the latter.

The paper is clearly and well written, with many pictures and intuitive explanations.

In conclusion, this is an important and very interesting result, which provides a clear and nice interpretation of Duflo's theorem. I strongly recommend for this paper to be accepted for publication in AGT.

A question and a minor comment:

- I understand of course that the whole point is to give an elementary proof of this theorem using the topological invariant constructed by the first two-named authors without going into the details of that construction. Yet, there are in fact many Duflo isomorphisms, parametrized by what Alekseev-Torrossian call a Duflo function ; the one given in the paper under review coincides with Duflo's original one and it seems to me that it means the underlying Drinfeld associator is even. It would be interesting to know if this is indeed the case, and whether it is a deliberate choice made for the sake of simplicity or if there is some actual technical obstruction to using a more general associator.
- Despite being well written the paper has a lot of small annoying typos, in particular a lot of g's that should be $\mathfrak{g}$'s. It also seems to me that the $S(\mathfrak{g})$ in the introduction should be $S(\mathfrak{g}^*)$'s.