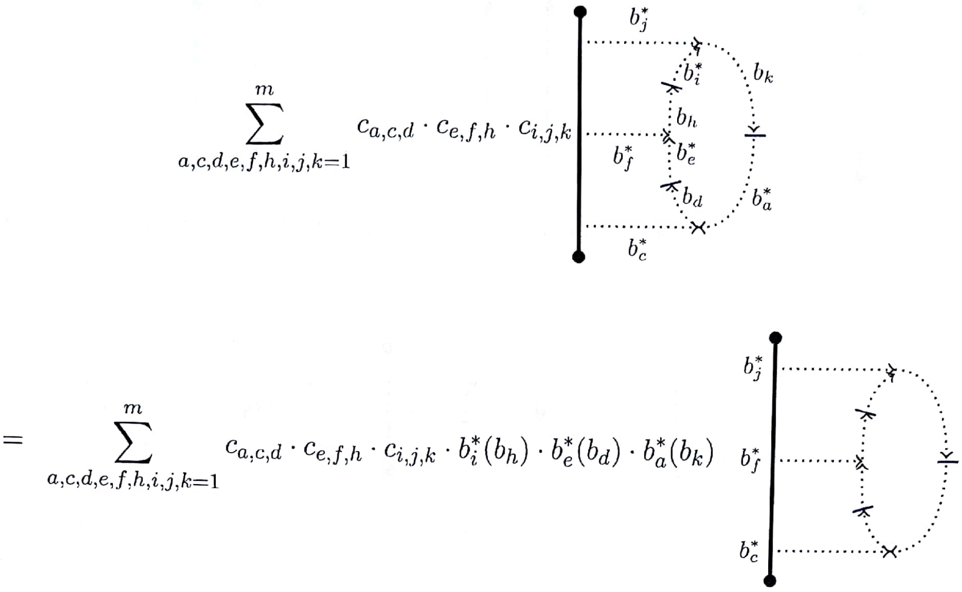

$$= \sum_{c,e,f,i,j,k=1}^{m} c_{k,c,e} \cdot c_{e,f,i} \cdot c_{i,j,k} \cdot b_c^* \cdot b_f^* \cdot b_j^* \in \mathcal{T}\mathfrak{g}^*$$

FIGURE 21. Example computation of T for a wheel with three spokes.

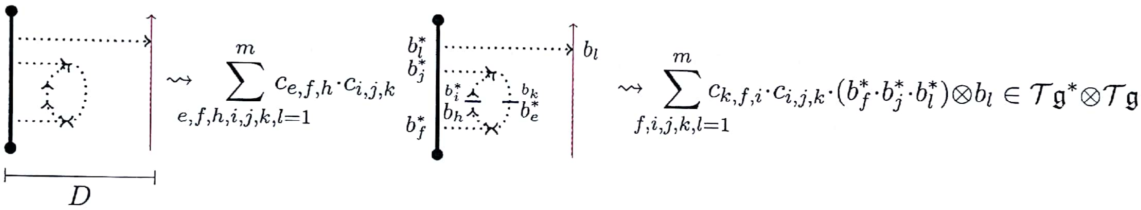

FIGURE 22. An example for computing T.

Proposition 4.2. *T descends to a well defined map on $\mathcal{A}^{sw}(\updownarrow^{k_1} \uparrow^{k_2}) \to (\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes (\hat{U}\mathfrak{g})^{\otimes k_2}$, where $(S\mathfrak{g}^*)_{\mathfrak{g}}$ denotes invariants under the co-adjoint action.*

Proof. Given and arrow diagram D, the ~~algorithm~~ construction above gives an element of $(\mathcal{T}\mathfrak{g}^*)^{\otimes k_1} \otimes (\mathcal{T}\mathfrak{g})^{\otimes k_2}$. We need to show that, modulo the relations of $\mathcal{A}^{sw}(\updownarrow^{k_1} \uparrow^{k_2})$, we obtain a well-defined map into $(\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes (\hat{U}\mathfrak{g})^{\otimes k_2}$.

Along the red strings, the $\overrightarrow{STU}$ relation translates to the defining relation $[g,h] = gh - hg$ of $U\mathfrak{g}$. Along the capped black strands, there are two further CP relations at the caps.

Due to the first CP relation only tails lie on the capped strands, and the TC relation then translates to the defining relation $\phi\psi = \psi\phi$ of $S\mathfrak{g}^*$. We need to analyze the effect of the second CP relation. First, we assume that only one arrow tail ends on a given capped black strand.

Prop 4.2 is Lie-algebraic, not diagrammatic. It is the statement:

$$\mathfrak{g}\backslash U(I\mathfrak{g})/\mathfrak{g} \cong S(\mathfrak{g}^*)_{\mathfrak{g}}$$

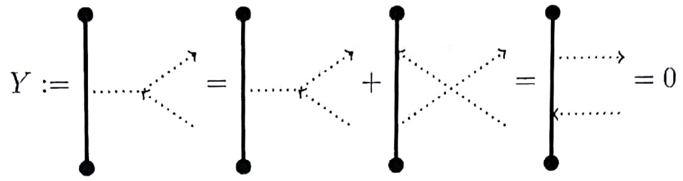

FIGURE 23. A single trivalent vertex on a twice-capped strand is zero.

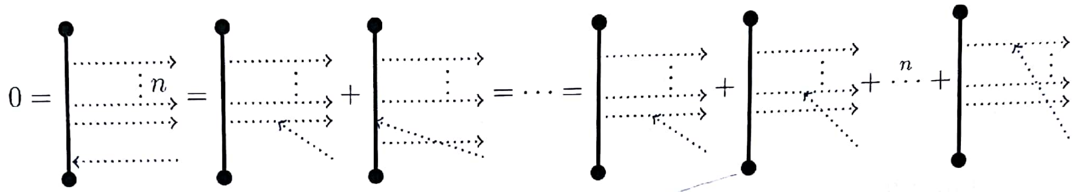

FIGURE 24. Commuting an arrow head up to the top cap, creating a relation on $S\mathfrak{g}^*$.

For diagrams which have a single arrow tail ending on a double-capped strand, the second cap relation is equivalent to stating that an arrow diagram where a single trivalent arrow vertex is attached to a capped strand is zero in $\mathcal{A}^{sw}(\uparrow^{k_1} \uparrow^{k_2})$, as shown in Figure 23.

Using the basis $\{b_1, \cdots b_m\}$ and $\{b_1^*, \cdots b_m^*\}$ as before, apply T to the arrow diagram Y of Figure 23. We only know part of the diagram Y, but we can compute the corresponding part of the image. In particular, the structure tensor $\sum_{i,j,k}$ is placed on the single trivalent vertex, the index j does not appear anywhere else, and there are no other factors on the shown capped strand:

$$T(Y) = \sum_{\substack{\cdots \\ i,j,k}} c_{i,j,k} \cdot \underline{\qquad\qquad} \cdot b_j^* \otimes \underline{\qquad\qquad} = \left(\sum_j c_{i,j,k} b_j^* \right) \otimes \sum_{i,k,\ldots} \underline{\qquad\qquad}$$

way too complicated.

Since $Y = 0$ and hence $T(Y) = 0$ for all such diagrams Y, we need

$$\sum_j c_{i,j,k} b_j^* = 0 \text{ for all } i, k.$$

A short calculation shows that this is precisely the defining relation for the co-invariant space for the co-adjoint action. In other words the first tensor factor of $T(Y)$ has a well-defined value in $(\mathfrak{g}^*)_{\mathfrak{g}}$.

In general, if several arrows end on a double-capped strand, having two CP relations means that any arrow heads can be commuted to either cap and be killed. The resulting relation is shown in Figure 24. After applying T, this translates to exactly how the co-adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ extends to $\hat{S}\mathfrak{g}^*$, hence we obtain the quotient $(\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}$. □

The following Proposition will play a crucial role later; we present it here as it is based on a similar principle as the proof above:

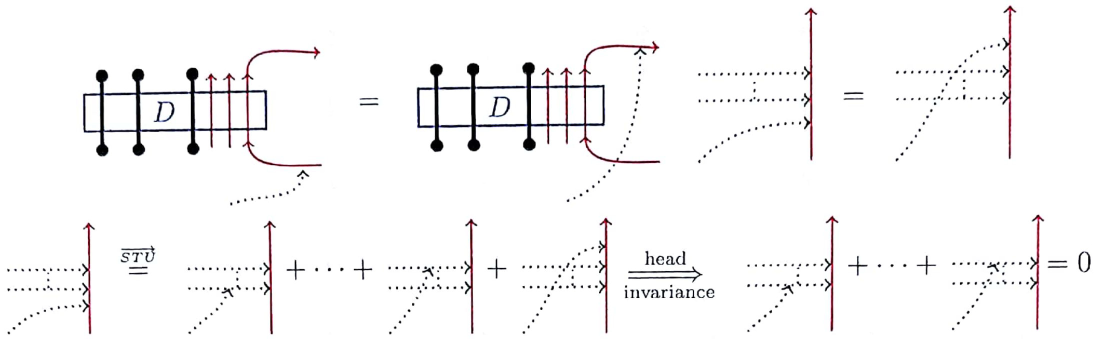

FIGURE 25. Above, the *head invariance* property in $\mathcal{A}^{sw}(\uparrow^{k_1}_{\bullet} \uparrow^{k_2})$. Below, head invariance and $\overrightarrow{STU}$ implies a relation in $\mathcal{A}^{sw}(\uparrow^{k_1}_{\bullet} \uparrow^{k_2})$.

Proposition 4.3. *The image of T is $\mathfrak{g}$-invariant, where $\mathfrak{g}$ acts via the adjoint action on the i-th $U\mathfrak{g}$ tensor factor, for any $i = 1, ..., k_2$. In other words, $T(D) \in ((\hat{S}\mathfrak{g}^*)_{\mathfrak{g}}^{\otimes k_1} \otimes ((\hat{U}\mathfrak{g}))^{\otimes k_2})^{\mathfrak{g}_i}$ for any $D \in \mathcal{A}^{sw}(\uparrow^{k_1}_{\bullet} \uparrow^{k_2})$.*

Proof. This follows from the *head invariance* property of arrow diagrams: the implication of this property relevant here is shown in Figure 25; the property in general is discussed in [BD2, Remark 3.14]. In short, attaching an additional arrow head at the bottom of a red strand gives the same result as attaching it at the top. Commuting the arrow head from bottom to top gives a relation similar to the one used in the previous proof and shown in Figure 24.

Now assume that D is an arrow diagram in $\mathcal{A}^{sw}(\uparrow^{k_1}_{\bullet} \uparrow^{k_2})$, with r arrow heads ending on the i-th red string, for $r, i \in \mathbb{N}$. The tensor $T(D)$ is a sum of terms of the form $P_1 \otimes ... \otimes P_{k_1} \otimes x_1 \otimes ... \otimes x_{k_2}$, where $P_\alpha \in (S\mathfrak{g}^*)_{\mathfrak{g}}$ and $x_\beta \in U\mathfrak{g}$ for $\alpha = 1, ..., k_1$ and $\beta = 1, ..., k_2$. We need to show that for each of these terms x_i is $\mathfrak{g}$-invariant; and enough to show that it is annihilated by each basis element b_s of $\mathfrak{g}$. In such a term, the tensor factor x_i is a product of r basis elements $b_{j_1} \cdot b_{j_2} \cdot ... \cdot b_{j_r}$.

Denote by D_γ the arrow diagram obtained from D by adding an extra balck capped strand, and connecting this with a single arrow to one of the γ-th arrow ending on the i-th red string, as shown Figure 26. The sum $\sum_{\gamma=1}^{r} D_\gamma$ is zero by the head invariance property.

In $T(\sum_{\gamma=1}^{r} D_\gamma)$, the term $P_1 \otimes ... \otimes P_{k_1} \otimes x_1 \otimes ... \otimes x_{k_2}$ of $T(D)$ is replaced with

$$\sum_{s=1}^{m} b_s^* \otimes P_1 \otimes ... \otimes P_{k_1} \otimes x_1 \otimes ...x_{i-1} \otimes (\sum_{\gamma=1}^{m} b_{j_1} \cdot ... \cdot [b_s, b_{j_\gamma}] \cdot ...b_{j_2} \cdot ... \cdot b_{j_r}) \otimes x_{i+1} ... \otimes x_{k_2}$$

$$= \sum_{s=1}^{m} b_s^* \otimes P_1 \otimes ... \otimes P_{k_1} \otimes x_1 \otimes ...x_{i-1} \otimes (b_s * x_i) \otimes x_{i+1} ... \otimes x_{k_2},$$

where $*$ in the second line denotes the adjoint action. So

$$0 = T\left(\sum_{\gamma=1}^{r} D_\gamma\right) = \sum_{s=1}^{m} b_s^* \otimes (b_s *_i T(D)),$$

where $*_i$ denotes the adjoint action on the i-th $U\mathfrak{g}$ tensor factor. Hence, $b_s *_i T(D) = 0$ for all $s = 1, ..., m$, and this completes the proof. □

False as written — $T(D)$ is not invariant under $\mathfrak{g}_i$; only under the joint action on all red strands at once. And probably too complicated.

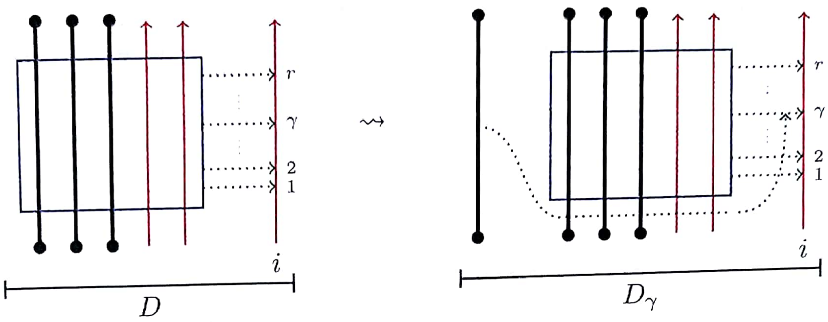

FIGURE 26. Attaching an extra arrow head to the γ-th arrow ending on the i-th red string.

4.3. **The tensor statement.** To obtain the tensor statement, we simply apply the map T to the diagrammatic statement of Figure 18:

$$Z(\text{-}|\text{-})\,\#Z(\text{-}|\text{-}) = u^2 Z(\diamond).$$

Since $Z(\text{-}|\text{-})$ is an element of $\mathcal{A}^{sw}(\uparrow\uparrow)$, there exists $\Pi_i \in S(\mathfrak{g}^*)_\mathfrak{g}$ and $W_i \in U(\mathfrak{g})$ so that

Avoid ∞ summations?
simply call the tensor Υ

$$T(Z(\text{-}|\text{-})) = \sum_{i=0}^{\infty} \Pi_i \otimes W_i \in \hat{S}(\mathfrak{g}^*)_\mathfrak{g} \otimes \hat{U}(\mathfrak{g}).$$

} completions are wrong (CAW)

The connect sum operation in $\mathcal{A}^{sw}$ is concatenation along the red strings. Under the tensor interpretation map T, this translates to multiplication in $U(\mathfrak{g})$, while the $S(\mathfrak{g}^*)$ components remain separate tensor factors. Hence,

$\Upsilon^{13}\Upsilon^{23}$

$$T(Z(\text{-}|\text{-})\#Z(\text{-}|\text{-})) = \sum_{i,j=0}^{\infty} \Pi_i \otimes \Pi_j \otimes W_i W_j \in \hat{S}(\mathfrak{g}^*)_\mathfrak{g}^{\otimes 2} \otimes \hat{U}(\mathfrak{g}).$$

(CAW)

On the right side, the unzip operation sends an arrow ending on the unzipped strand to a sum of two arrows ending on either daughter strand. Under the tensor interpretation map T, this is sent to the Hopf algebra coproduct Δ of $\hat{S}(\mathfrak{g}^*)_\mathfrak{g}$ given by $\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi$ for primitive elements $\varphi \in \mathfrak{g}^*_\mathfrak{g}$, as shown in Figure 27. In other words, $T \circ u = \Delta \circ T$.

Examining the right side of the diagrammatic statement in Figure 18, note that it is simply the diagrammatic unzip along the twice-capped strand of $Z(\text{-}|\text{-})$. This is because all arrows of $Z(\diamond)$ end on the capped strands, none on ~~the "bubble" in the~~ middle. Therefore,

remove...

$(\Delta\otimes 1)(\Upsilon)$

$$T\left(u^2 Z(\diamond)\right) = \sum_{i=0}^{\infty} \cancel{\Delta(\Pi_i) \otimes W_i} \in \hat{S}(\mathfrak{g}^*)_\mathfrak{g}^{\otimes 2} \otimes \hat{U}(\mathfrak{g}).$$

(CAW)

In summary, we obtain the Tensor Statement of Figure 28.

5. THE DUFLO ISOMORPHISM

use Υ for tensor, call the isomorphism Du or something like 17.

There is a pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{K}$ given by the evaluation. This extends to a pairing $S\mathfrak{g}^* \times S\mathfrak{g} \to \mathbb{K}$ in the following way. Given a monomial $\varphi_1 \cdot \ldots \cdot \varphi_k \in S\mathfrak{g}^*$ with $\varphi_i \in \mathfrak{g}^*$ and a monomial of

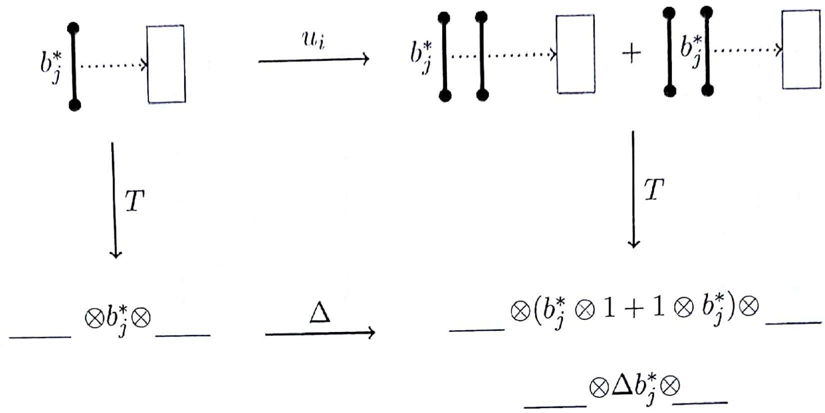

FIGURE 27. The tensor interpretation map intertwines unzip with co-multiplication.

Tensor Statement

$$\cancel{\sum_{i,j=0}^{\infty} \Pi_i \otimes \Pi_j \otimes W_i W_j = \sum_{i=0}^{\infty} \Delta \Pi_i \otimes W_i}$$

$\Upsilon^{13}\Upsilon^{23} = (\Delta \otimes 1)\Upsilon$ in $S(\mathfrak{y})_{\mathfrak{y}}^{\otimes 2} \hat{\otimes} U(\mathfrak{y})$

FIGURE 28. The tensor statement.

[note the similarity w/ the triangularity eqn $R^{13}R^{23} = (\Delta \otimes 1)R$], yet note that this isn't a triangularity eqn as $S(\mathfrak{y})_{\mathfrak{y}}$ and $U(\mathfrak{y})$ are not a dual pair.

the same degree $x_1 \cdot \ldots \cdot x_k \in S\mathfrak{g}$ with $x_j \in \mathfrak{g}$,

better write pairings as $\langle \varphi_1 \varphi_2 \ldots \varphi_k, x_1 \ldots x_k \rangle$

$$\underline{(\varphi_1 \cdot \ldots \cdot \varphi_k)\Big(x_1 \cdot \ldots \cdot x_k\Big)} := \sum_{\sigma \in S_k} \varphi_1(x_{\sigma(1)}) \cdot \ldots \cdot \varphi_k(x_{\sigma(k)}),$$

where the sum is over all permutations of the k indices. Monomials pair as zero with any monomial of a different degree, and the pairing is then extended bilinearly.

Alternatively, given a basis $\{b_1, ..., b_m\}$ of $\mathfrak{g}$ and dual basis $\{b_1^*, ..., b_m^*\}$ of $\mathfrak{g}^*$, $S\mathfrak{g}$ and $S\mathfrak{g}^*$ are spanned linearly by monomials in the basis elements b_i and b_j^* respectively. The monomial $(b_1^*)^{\alpha_1} \cdot \ldots \cdot (b_m^*)^{\alpha_m}$ pairs as zero with every monomial in the basis vectors $\{b_i\}$ except $b_1^{\alpha_1} \cdot \ldots \cdot b_m^{\alpha_m}$, and

$$((b_1^*)^{\alpha_1} \cdot \ldots \cdot (b_m^*)^{\alpha_m})\Big(b_1^{\alpha_1} \cdot \ldots \cdot b_m^{\alpha_m}\Big) = \prod_{i=1}^{m} \alpha_i! \tag{5.1}$$

This descends to a pairing $(Sg^*)_{\mathfrak{g}} \times (S\mathfrak{g})^{\mathfrak{g}} \to \mathbb{K}$. For $\Pi \in (Sg^*)_{\mathfrak{g}}$ and $P \in (S\mathfrak{g})^{\mathfrak{g}}$ we will denote the value of this pairing by by $\Pi(P)$. Finally, one can extend to a pairing $(Sg^*)_{\mathfrak{g}}^{\otimes n} \times ((S\mathfrak{g})^{\mathfrak{g}})^{\otimes n} \to \mathbb{K}$ by simply multiplying the pairings of tensor factors. This satisfies the equality

$$\Pi(PQ) = \Delta\Pi(P \otimes Q) \tag{5.2}$$

$\langle \Pi, PQ \rangle = \langle \Delta\Pi, P \otimes Q \rangle$

for any $P, Q \in (S\mathfrak{g})^{\mathfrak{g}}$, where Δ is the co-product on $(Sg^*)_{\mathfrak{g}}$ induced by the co-product on (Sg^*).

Definition 5.1. We define $\Upsilon : S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$ by pairing with the first tensor factor of $T(Z(\text{-}\!\!\uparrow\!\!\text{-})) = \sum_{i=0}^{\infty} \Pi_i \otimes W_i$. That is, or $P \in S(\mathfrak{g})^{\mathfrak{g}}$, define

$$\Upsilon(P) := \sum_{i=0}^{\infty} \Pi_i(P) \cdot W_i.$$

convergence!

rewrite with $\sum \Pi_i \otimes W_i$ ↓ Υ

The fact that $\Upsilon(P) \in U(\mathfrak{g})^{\mathfrak{g}}$ is a direct consequence of Proposition 4.3.

Theorem 5.2. *The map Υ is an algebra homomorphism.*

Proof. By definition, Υ is linear. The heart of the proof is the multiplicativity of Υ, which is a direct consequence of the Tensor Statement. Let $P, Q \in S(\mathfrak{g})^{\mathfrak{g}}$, then

$$\Upsilon(PQ) = \sum_i \Pi_i(PQ) \cdot W_i \stackrel{1}{=} \sum_i \Delta\Pi_i(P \otimes Q) \cdot W_i \stackrel{2}{=} \sum_{i,j} (\Pi_i \otimes \Pi_j)(P \otimes Q) \cdot W_i W_j,$$

$$\stackrel{3}{=} \left(\sum_i \Pi_i(P) \cdot W_i\right)\left(\sum_j \Pi_j(Q) \cdot W_j\right) = \Upsilon(P)\Upsilon(Q)$$

Here Equality 1 is Equation (5.2) above. Equality 2 is the Tensor Statement, and Equality 3 is the associativity of product. □

Proposition 5.3. *The map Υ is an algebra isomorphism.*

no need to work in coordinates!

Proof. In light of Theorem 5.2 we only ~~need to~~ prove that Υ is bijective. This follows from inspection of $T(Z(\text{-}\!\!\uparrow\!\!\text{-}))$; we use the bases $\{b_i\}_{i=1}^m$ for $\mathfrak{g}$ and $\{b_i^*\}_{i=1}^m$ for $\mathfrak{g}^*$. Recall that $Z(\text{-}\!\!\uparrow\!\!\text{-})$ consists of a value C^2 on the twice-capped strand followed by an exponential e^a of an arrow a from the capped startand to the red string, as shown in Figure 18. Hence,

$$T(Z(\text{-}\!\!\uparrow\!\!\text{-})) = (T(C^2) \otimes 1) \cdot T(e^a).$$

$\otimes \tau$, where τ is the identity tensor in $\mathfrak{g}^* \otimes \mathfrak{g}$

really you are proving here that $Id_{S(\mathfrak{g})} = e^{\tau}$.

Since $T(a) = \sum_{i=1}^m b_i^* \otimes b_i$, we have

$$T(e^a) = e^{\sum_{i=1}^m b_i^* \otimes b_i} = \sum_{d=0}^{\infty} \frac{1}{d!} \left(\sum_{i=1}^m b_i^* \otimes b_i\right)^d.$$

We will analyse the value C later, for now it is enough to note that since Z is group-like, C is an exponential, and hence $T(C^2) = 1 +$ higher degree terms.

Perhaps instead, where C is first mentioned it should be written out to degree 4. (or 2)

Now let $P \in S(\mathfrak{g})^{\mathfrak{g}}$ be homogeneous of degree d. Observe that $\Upsilon(P)$ is a sum of terms of degree *at most* d, and the highest order term arises from pairing with the term

$$(1 \otimes 1) \cdot \frac{1}{d!} \left(\sum_{i=1}^m b_i^* \otimes b_i\right)^d$$

of T(Z(-↑-)). Specifically, P is given by a finite sum $\sum_r b_1^{\alpha_{1,r}} \cdot \ldots \cdot b_m^{\alpha_{m,r}}$, where $\sum_{i=1}^m \alpha_{i,r} = d$ for each r. Then, using Equation 5.1, the highest order term of $\Upsilon(P)$ is

$$\frac{1}{d!} \sum_r \left(\prod_{i=1}^m \alpha_{i,r}!\right) W(b_1^{\alpha_{1,r}} \cdot \ldots \cdot b_m^{\alpha_{m,r}}),$$

where $W(b_1^{\alpha_{1,r}} \cdot \ldots \cdot b_m^{\alpha_{m,r}})$ denotes the sum of all words in $U\mathfrak{g}$ which use each letter b_i $\alpha_{i,r}$ times. As this sum is non-zero, we have shown that Υ is injective.

I'd organize the whole proof differently: to show D_u is a bijection, enough to show $gr(D_u)$ is a bijection, but $gr(D_u) = e^{\tau} = Id$.