RIBBON 2-KNOTS, 1+1=2, AND DUFLO'S THEOREM FOR ARBITRARY LIE ALGEBRAS

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ABSTRACT. We explain the most direct topological proof we know for the Duflo isomorphism for arbitrary finite dimensional Lie algebras. The proof follows a series of implications, starting with "the calculation 1+1=2 on a 4D abacus", using the study of *homomorphic expansions* (aka universal finite type invariants) for ribbon 2-knots, and the relationship between the corresponding associated graded space of *arrow diagrams* and universal enveloping algebras. This complements the results of the first author, Le and Thurston, where similar arguments using a "3D abacus" and the Kontsevich Integral was used to deduce Duflo's theorem for *metrized* Lie algebras; and results of the first two authors on finite type invariants of w-knotted objects, which also imply a relation of 2-knots with the Duflo theorem in full generality, though via a lengthier path.

1. INTRODUCTION

For a finite dimensional Lie algebra $\mathfrak{g}$, the famous Duflo isomorphism is an algebra isomorphism $\Upsilon : S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$, where $U(\mathfrak{g})^{\mathfrak{g}}$ and $S(\mathfrak{g})^{\mathfrak{g}}$ are the the $\mathfrak{g}$ invariant subspaces of the adjoint action on the universal enveloping algebra and the symmetric algebra . (Recall x is called invariant if $g \cdot x = 0$ for every $g \in \mathfrak{g}$.) The map Υ is given by an explicit formula.

This isomorphism was first described for semi-simple Lie algebras by Harish-Chandra in 1951 [HC]. Kirillov conjectured that a formulation of Harish-Chandra's map was an algebra isomorphism for all finite dimensional Lie algebras. Duflo proved Kirillov's conjecture in 1977 [D], which is now referred to as Duflo's Theorem. Since then, there have been many proofs of Duflo's theorem using techniques outside the setting of the originally formulated problem. For metrized Lie algebras, a topological proof was found by the first author, Le and Thurston in 2009 [BLT] using the Kontsevich integral and a knot theoretic interpretation of "$1+1=2$ on an abacus". In this paper we give a new topological proof of Duflo's theorem for *arbitrary finite dimensional Lie algebras* using a "4-dimensional abacus" instead of an ordinary 3-dimensional one.

The Dulfo isomorphism is also implied by the Kashiwara–Vergne (KV) conjecture [REF KV 1970's]. The KV-conjecture states that a certain set of equations has a solution in the group of *tangential automorphisms* of the degree completed free Lie algebra on 2 generators. One can extract the Duflo isomorphism from such a solution. The KV-conjecture is now a theorem and was first proved by [AM] in 2006 using deformation quantization. New proofs exploiting the relationship between the KV equations and Drinfeld associators were found by Alekseev, Torossian and Enriquez shortly thereafter [AT, AET]. A topological context and solution in terms of the 4-dimensional knot theory of *w-foams* was established by the first two authors in [BD2, BD3]. In this context, the KV-conjecture is equivalent to the existence of a *homomorphic expansion* for w-Foams. In this paper, we directly address how such a

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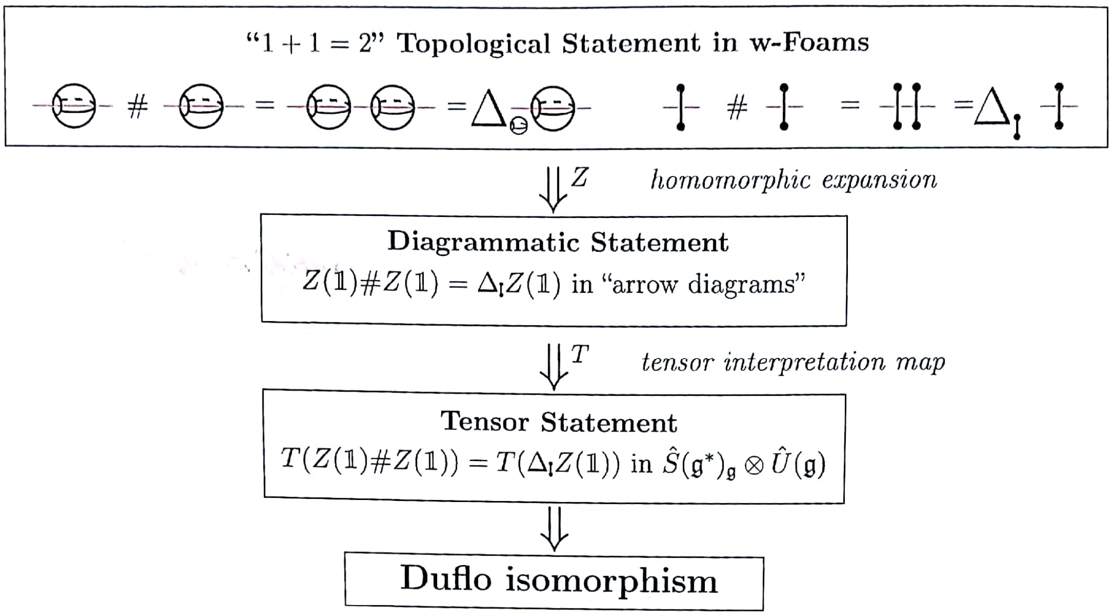

FIGURE 1. The rough sketch of the proof.

FIGURE 2. The threaded sphere as a movie of a circle and a point in $\mathbb{R}^3$.

homomorphic expansion gives rise to a proof of the mult. of Duflo isomorphism and a formula for Υ, and thus completing a topological proof of the Duflo isomorphism in full generality.

This paper is structured to follow the implications shown in the Figure 1. We start with an intuitive topological statement "$1+1=2$" and interpret this in the more sophisticated setting of w-Foams. Using the homomorphic expansion Z and the tensor interpretation map T, we can re-interpret "$1+1=2$" as an equality in $\hat{S}(\mathfrak{g}^*)_{\mathfrak{g}} \otimes \hat{U}(\mathfrak{g})$. This will imply that our formulation of the Duflo isomorphism is an algebra homomorphism. The essential ingredient in this process is the homomorphic expansion Z of [BD2, BD3].

2. Understanding the Topological Statement and w-Foams

2.1. "4D Abacus Arithmetic". The "threaded sphere" or "abacus bead" shown in Figure 2 is a knotted object in $\mathbb{R}^4$, and an element of the space of w-Foams studied in [BD2]. To understand this 4D object, we describe it as a sequence of 3D slices, or "frames of a 3D movie". The movie starts with two points A and B. Point B opens up to a circle, A flies through the circle, and B closes to a point again. In 4 dimensions this is a line threaded through a sphere with no intersections; and embedded pair. We depict this object as  ; this is a *broken surface diagram* in the sense of [CS].

We can interpret "addition on the 4D abacus" by iteratively threading embedded spheres on a single thread, or in other words, connecting along the threads, as shown in Figure 3.

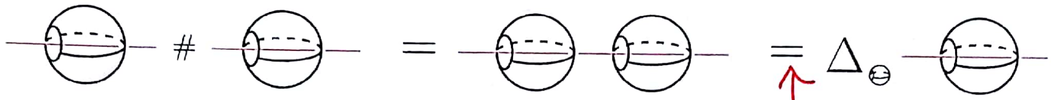

FIGURE 3. "$1+1=2$" on the 4D abacus.

However, there are two ways to obtain the number 2 from the number 1: by addition – which is represented by iterative threading on the "abacus" thread as above, or by doubling, as explained below.

Assuming there is a framing along the sphere (we will define such a framing later), it is well defined to double the sphere along the framing in 4-dimensional space. This operation will be denoted by $\Delta_{\ominus}$ ⦵ . Given an appropriate framing, doubling the sphere on the thread has the same effect as the connected sum of two threaded spheres, as shown in Figure 3. To simplify notation, we will denote the threaded sphere by $\mathbb{1}$, and write $\mathbb{1}\#\mathbb{1} = \Delta_{\ominus}\mathbb{1}$.

2.2. **w-Foams.** In order to introduce the main ingredient Z, the homomorphic expansion, we need to place the threaded sphere in the more complex space of w-Foams. We will briefly describe this space here and for more detail refer to [BD3, Section 2].

The space of w-Foams, denoted $\widetilde{wTF}$, is a *circuit algebra*, as defined in [BD2, Section 2.4]. In short, circuit algebras are similar to the planar algebras of Jones [J] but without the planarity requirement for the connection diagrams. For an example of a circuit algebra connection diagram, see Figure 6. Circuit algebras are also close relatives of modular operads. Each generator and relation of $\widetilde{wTF}$ has a local topological interpretation in terms of certain *ribbon knotted tubes with foam vertices and strings* in $\mathbb{R}^4$. Note that one dimensional strands cannot be knotted in $\mathbb{R}^4$, however, they can be knotted *with* two-dimensional tubes. In the diagrams, two-dimensional tubes will be denoted by **thick lines** and one dimensional strings by thin red lines.

With this in mind, we define $\widetilde{wTF}$ as a circuit algebra given in terms of generators and relations, and with some extra operations beyond circuit algebra composition. The generators, relations and operations are explained in detail below. The local topological interpretation of the generators and relations provides much of the intuition for this paper.

$\widetilde{wTF} = CA\Big\langle$  $1, 2, 3, 4, 5, 6, 7, 8, 9$ $\Big|$ $R1^s$, R2, R3, R4, OC, CP $\Big|$ u_e $\Big\rangle$

In [BD3] $\widetilde{wTF}$ appears in its larger unoriented version (includes a *wen* and relations describing its behaviour) and it is equipped with more auxiliary operations (eg punctures, orientation switches). The expansion Z constructed there is *homomorphic* with respect to all of the operations in the appropriate sense. Here we focus only on orientable surfaces and the operations strictly needed for the Duflo isomorphism – the restriction of the Z of [BD3] is a homomorphic expansion for this structure. In the following sections we will provide brief descriptions of $\widetilde{wTF}$, its associated graded space of arrow diagrams, and the homomorphic expansion, to make this paper more self-contained.

2.2.1. *The generators of* $\widetilde{wTF}$. We begin by discussing the local topological meaning of each generator shown above. For more details, see [BD2, Sections 4.1.1 and 4.5]

Knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying circles in $\mathbb{R}^3$. The two crossings – generators 1 and 2 – stand for movies where two circles trade places by the circle of the under strand flying through the circle of the over

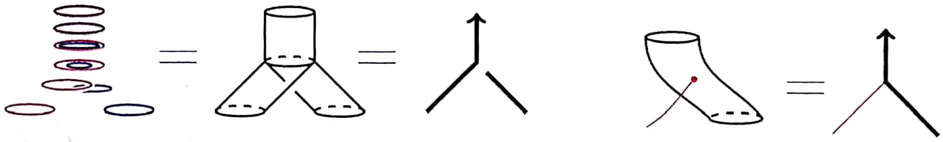

FIGURE 4. The trivalent vertices of $\widetilde{wTF}$.

strand from below. The bulleted end in generator 3 represents a tube "capped off" by a disk, or alternatively the movie where a circle shrinks to a point and disappears.

Generators 4 and 5 stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown in Figure 4: the right circle approaches the left circle from below, flies inside it and merges with it. The negative vertex represents a circle splitting and the inner circle flying out below and to the right.

The thin red strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (generators 6 and 7) represent "points flying through circles". For example, generator 6 stands for "the point on the right approaches the circle on the left from below, flies through the circle and out to the left above it". This explains why there are no generators with a thick strand crossing under a thin red strand: a circle cannot fly through a point.

Generator 8 is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Finally, generator 9 is a "mixed vertex", in other words a one-dimensional string attached to the wall of a 2-dimensional tube. This is shown in Figure 4.

An important notion for later use is the *skeleton* of a w-foam. In general, viewing knotted objects as embeddings of circles, manifolds, graphs, etc, the skeleton is the object embedded. In other words, the skeleton is the combinatorial object one obtains by forgetting the topology of a knotted object, or equivalently, by allowing arbitrary crossing changes, or still equivalently, by replacing all crossings with virtual (circuit) crossings. For example, the skeleton of an ordinary knot is a circle. The skeleton of the threaded sphere described above is a sphere and a string. The skeleton of a classical braid on n strands is an element of the permutation group S_n. This intuitive definition is sufficient for reading the rest of this paper; for a formal definition see [BD2, Section 2.4].

2.2.2. *The relations for* $\widetilde{wTF}$. The list of relations for $\widetilde{wTF}$ is {R1^s, R2, R3, R4, OC, CP}. Here, R1^s is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual Reidemeister moves; R4 allows moving a strand over or under a vertex. OC stands for *Over-corssings Commute*, and CP for *Cap Pullout*. Each of these relations has a 4D topological meaning: Figure 5 shows R1^s and OC, and explains CP. For more detail, see [BD2, Section 4.5]. All relations should be interpreted in all sensible combinations of strand types: tube or string, and all orientations.

Note that all relations are circuit algebra relations. For example, the relation OC is understood as a relationship between two specific circuit diagram compositions of and , as shown in Figure 6.

The circuit algebra $\widetilde{wTF}$ is conjectured to be a Reidemeister theory for *ribbon* knotted tubes in $\mathbb{R}^4$ with caps, singular foam vertices and strings. Here *ribbon* means that the tubes have "filling" in $\mathbb{R}^4$ with only restricted types of singularities, for details see [BD1, Section 2.2.2]. All the relations represent local topological statements: for example, Reidemeister 2

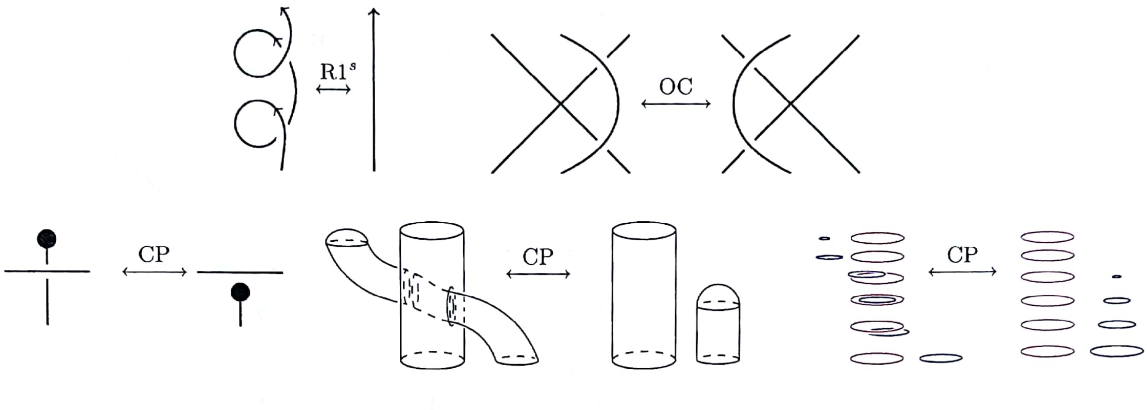

FIGURE 5. The relations R1^s and OC are shown. CP is explained with broken surface diagrams and as a movie of flying circles.

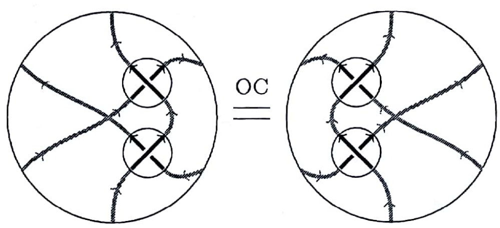

FIGURE 6. The OC relation written as a circuit algebra relation between two crossings.

with a thin red bottom strand is imposed because [The movie consisting of] a point flying in through a circle and then immediately flying back out is isotopic ~~to not having any interaction between the point and circle at all.~~ [the constant movie in which... stay in place.] However, it is an open question whether the known relations are sufficient. A similar Reidemeister theory has been proven for w-braids in [REF] which exhibits a simpler structure than $\widetilde{wTF}$. For a more detailed explanation of the difficulties that arise for knots and tangles, see [BD2, Introduction].

2.2.3. *The operations on* $\widetilde{wTF}$. In addition to the circuit algebra structure, $\widetilde{wTF}$ is equipped with a set of auxiliary operations. Of these, in this paper we only use *disc unzip*.

The *disc unzip operation* u_e is defined for a capped strand labeled by e. Using the blackboard framing, u_e doubles the capped strand e and then attaches the ends of the doubled strand to the connecting ones, as shown Figure 7. Topologically, the blackboard framing of the diagram induces a framing of the corresponding tubes and discs in $\mathbb{R}^4$ via Satoh's tubing map [BD1, Section 3.1.1] and [S]. Unzip is the operation "pushing the disc off of itself slightly in the framing direction". See [BD2, Section 4.1.3] for details on framings and unzips.

A related operation not strictly necessary for this proof is *strand unzip*, is defined for strands which end in two vertices are of opposite signs, as shown in the right of Figure 7. For the interested reader a detailed definition of crossing and vertex signs is in [BD2, Sections 3.4 and 4.1]. Strand unzip doubles the strand in the direction of the blackboard framing, and connects the ends of the doubled strands to the corresponding edge strands. Topologically,