Knot Theory as an Excuse **Discovery Grant Notice of Intent**

I'm considered an expert on Knot Theory, yet I don't understand knot theory at all. From a certain perspective, Knot Theory is the study of some silly combinatorial objects, that are considered modulo equally silly relations. My intuition as a student told me it must be a shallow topic, and there's still a remnant of that intuition in me.

Yet time after time this intuition is proven wrong and instead of shallow, Knot Theory is very deep. So much so, that Knot Theory sometimes serves to validate that other topics are interesting: **if it has applications to Knot Theory, it must be good.** (Historically, number theory's raison d'être had been similar; recently cryptography became a further bonus).

My plan over the grant period would be to continue to use knot theory as an excuse and as a benchmark to study several other topics, mostly in algebra:

1. I plan to continue to study, along with Roland van der Veen and others, how "solvable approximation" of semisimple Lie algebras (Inonu-Wigner contractions of their lower Borel subalgebras) leads via perturbed Gaussian formulas (in spirit, QFT) to poly-time computable knot invariants that "behave well" under useful knot theoretic operations. I hope this sounds powerful; it certainly sounds highly technical. Can we make it less technical? Can we rely less on Lie algebra and quantum algebra techniques and instead make the topic intrinsic to knot theory? See ωεβ/SolvApp, ωεβ/PG, ωεβ/DaNang.

2. The simplest of these invariants, ρ_1, is ridiculously simple to define (ωεβ/APAI, ωεβ/Cars and it is perhaps even more ridiculous how much we fail to understand it. In short, ρ_1 is some quadratic expression in the entries of A^{-1}, where A is one of the standard matrices whose determinant is the Alexander polynomial Δ. Could we start from other matrices B whose determinants are Δ? Can we prove Alexander-like properties of ρ_1 using its similarity with Δ? By direct computations we observe many such properties, yet we still don't know how to prove them. And the $1M question: does ρ_1 have special properties on ribbon knots, similar to the Fox-Milnor property of Δ? If it does, it may lead to a new criteria to detect non-ribbon knots. Such criteria are in high demand for they may lead to the detection of counterexamples to the ribbon-slice conjecture, one of the greatest outstanding problems in knot theory.

3. Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich (ωεβ/PDS), I plan to continue to study knots and tangles in a "Pole Dancing Studio" (PDS, a cylinder with a few vertical lines removed) and their relationship with the Goldman-Turaev Lie bialgebra and Kashiwara-Vergne (KV) equations (ωεβ/AKKN). Are solutions of the KV equations sufficient to construct a homomorphic expansion of tangles in a PDS up to strand-strand degree 1? How is this related to my earlier work with Dancso (ωεβ/WKO1, ωεβ/WKO2) on welded knots? The subject is beautiful, yet it is a hard-to-penetrate patchwork of results and techniques and papers by different authors. In the past, this feeling that a subject's beauty is incongruous with its complexity had been a great motivator for me, often leading to deeper understanding. I have high hopes for this topic too.

4. Recently (ωεβ/PQ), along with Jessica Liu, we've found a truly elegant "signatures for tangles" invariant (sorry for complimenting ourselves, yet hey, it really is elegant). There is more to do before we can claim to fully understand these signatures. Will we be able to use our formalism to prove Kashaev's signatures conjecture (ωεβ/Kashaev)?

Knot Theory as an Excuse **Discovery Grant Proposal Summary**

One of the major triumphs of mathematics in the 1980s, which lead to at least 3 Fields medals (Jones, Drinfel'd, Witten) was the unexpected realization that low dimensional topology, and in particular knot theory, is closely related to quantum field theory and to the theory of quantum groups. Knot theory is mundane and ages-old; anything "quantum" seems hyper-modern. Why would the two have anything to do with each other?

The answer is long and complicated and has a lot to do with the "Yang-Baxter Equation" (YBE). The YBE on the one hand can be interpreted in knot theory as "the third Reidemeister move", or as "controlling the most basic interaction of 3 pieces of string" (this turns out to be a very crucial part of knot theory). On the other hand solutions of the YBE arise from "quantum" machinery. Hence the quantum is useful to the knotted, and by similar ways, to the rest of low dimensional topology.

But "quantum" has a caveat, which makes it super-exciting (to some) yet bounds its usefulness (to others). When quantum systems grow large (as they do when the knot or low-dimensional space we study grows complicated), their "state space" grows at an exponential rate. "Quantum computers" aim to exploit this fact and make large quantum systems performs overwhelmingly large computations by utilizing their vast state spaces. But quantum computers aren't here yet, may take many years to come, suffer from other limits on what they can do, and much of low-dimensional topology is anyway outside of these limits. So at least for now and likely forever, many things that have "quantum" in their description are exponentially-complex to compute, which in practice means that they cannot be computed beyond a few simple cases.

Recently Van der Veen and myself, following Rozansky and Overbay and Ohtsuki, found a corner (figuratively speaking) of the vast state space of the quantum machinery used in knot theory, which can be described extremely simply, which computes in just polynomial complexity, and which carries enough information to still speak to knot theory. The "knot invariants" ρ_d constructed that way seem to be the strongest invariants we know that are computable even for very large knots and they have the potential of relating to knot properties such that their genus and whether or not knots are slice or ribbon.

Our approach in itself comes from sophisticated quantum algebra, yet the results can be described using nothing more than first-year university mathematics. More often than not, when a result is simple there is also a simple way to derive it, and it is often crucial to find that simple way. We don't know yet how to tell the ρ_d story in a language as simple as the formulas at the end of that story, and we dream that over the grant period we will learn to do better.

We also dream to find topological applications of ρ_d and especially of ρ_1, and to continue our work on other topics within knot theory.

Knot Theory as an Excuse

Discovery Grant Proposal

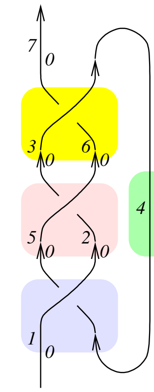
Grant proposals are often written as if they describe a victory parade. "The principal investigator will march from A to B to C collecting trophies along the way". I hope this one is written differently. There is a single "!" in it, and it is in quotes. There are plenty of "?" in it. Each one represents a dream. A question that I plan to study and that I hope I have the tools to address and to elucidate, if not resolve. Will the NSERC help?

I have chosen to concentrate in this proposal on what was topic #2 in my Notice of Intent (ωεβ/NOI) — the invariant ρ_1: a well-connected, strong, homomorphic, and ridiculously easy to define and to compute knot invariant, which is nevertheless far from being understood and utilized. Let me tell you some more about it.

ρ_1 is not new. It traces back to work by Rozansky [Ro3, Ro4] and Overbay [Ov] and to work by Ohtsuki [Oh], which in itself traces back to work by Garoufalidis and myself [BNG] proving the Melvin-Morton-Rozansky Conjecture [MM, Ro1, Ro2] which relates the Coloured Jones polynomial [Jo] with the Alexander polynomial [Al]. Yet my recent work with Roland van der Veen [BV3] makes it ridiculously easy to define and to compute and shows it to be "homomorphic" (see below) and hence suggests that ρ_1 may have far-reaching topological implications and applications. Does it?

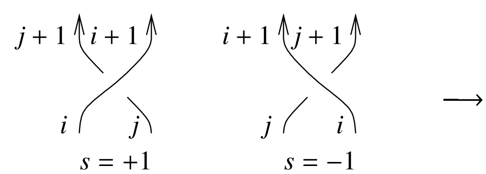
ρ_1 is thus dominated by the coloured Jones polynomial. That can be seen as a handicap, for supposedly we already "understand" the coloured Jones polynomial. But no, we don't really. The coloured Jones polynomial is complicated to define and nearly impossible to compute for knots with more than just a few crossings. A section of the coloured Jones that is simple and easy and which is more than the "classical" Alexander polynomial may well be the golden key that many have been looking for, that will finally bring the power of quantum invariants to use within classical topology. Is it? (It is a bit of an absurd, and a bit of a sore point, that quantum invariants that are so much stronger than the Alexander polynomial say so little, beyond what Alexander already knows, on classical properties of knots such as their genus, unknotting numbers, and whether or not they are slice or ribbon or fibred).

ρ_1 really is easy to define, so here's a definition, in full and with a worked-out example, following [BV3].



Preparation. Given an oriented knot K, we draw it in the plane as a long knot diagram D with n crossings in such a way that the two strands intersecting at each crossing are pointed up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

Making a matrix. We let A be the $(2n+1)\times(2n+1)$ matrix with entries in the ring $\mathbb{Z}[T^{\pm 1}]$ of Laurent polynomials in a formal variable T obtained by starting with the identity matrix I_{2n+1} and adding to it one contribution per crossing as follows (s is the sign of the crossing):



add at	column $i+1$	column $j+1$
row i	$-T^s$	T^s-1
row j	0	-1

(1)

For our example, A comes out to be:

$$A=\begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Please count everything so far as "trivial". The matrix A is a presentation matrix for the Alexander module of K, obtained by using Fox calculus on the lower Wirtinger presentation. Up to a unit $\pm T^\bullet$, it's determinant is the normalized Alexander polynomial Δ and there's nothing new about it. Note that in our example $\Delta = T - 1 + T^{-1}$.

Doing something new. Let $G = (g_{\alpha\beta}) = A^{-1}$ be the inverse matrix of A, so in our example, G is

$$\begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now define

$$\rho_1 := \Delta^2 \left(\sum_c R_1(c) - \sum_k \varphi_k \left(g_{kk} - 1/2\right) \right), \qquad (2)$$

where the first summation is over crossings c and the second is over edges k, and where

$$R_1(c) = R_1(s, i, j) :=$$
$$s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right) \qquad (3)$$

where for a crossing c, the parameters s, i, and j are as in (1). This completes the definition of ρ_1. It's invariance is proven by elementary means[1] in [BV3].

For our trefoil example, using the values of Δ and of $g_{\alpha\beta}$ established before,

$$\begin{aligned} \rho_1 &= \Delta^2(R_1(1,3,6) + R_1(1,5,2) + R_1(1,1,4) \\ &\qquad - (-1)(g_{44} - 1/2)) \\ &= \Delta^2(g_{63}(g_{76} + g_{67} - g_{36}) - g_{33}(g_{67} - 1) - 1/2 \\ &\qquad g_{25}(g_{32} + g_{23} - g_{52}) - g_{55}(g_{23} - 1) - 1/2 \\ &\qquad g_{41}(g_{54} + g_{45} - g_{14}) - g_{11}(g_{45} - 1) - 1/2 \\ &\qquad + g_{44} - 1/2) \\ &= -T^2 + 2T - 2 + 2T^{-1} - T^{-2}. \end{aligned}$$

But wait, what? Inverting a presentation matrix? What does it mean? Who does that? Forming a quadratic expression out of the entries of said inverse? Who does that? What does it mean?

ρ_1 really is easy to implement. As evidence for that, here is a complete implementation, written in Mathematica [Wo]. The only reason it is included to make a point: It is ridiculously short.

```
R₁[s_, i_, j_] :=
  s (g_{ji} (g_{j⁺,j} + g_{j,j⁺} - g_{ij}) - g_{ii} (g_{j,j⁺} - 1) - 1/2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :>
     (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                      0      -1    ))];
   Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   ρ1 = Sum[R₁ @@ Cs⟦k⟧, {k, 1, n}] - Sum[φ⟦k⟧ (g_{kk} - 1/2), {k, 1, 2 n}];
   Factor@
    {Δ, Δ^2 ρ1 /. α_⁺ :> α + 1 /. g_{α_,β_} :> G⟦α, β⟧}];
```

ρ_1 is easy to compute, also in a technical sense. Except for the computation of A^{-1}, the computation of ρ_1 takes only a linear number of additions and multiplications in the ring $\mathbb{Z}[T^{\pm 1}]$, as a function of the number of crossings n (and the degrees and the digit-lengths of the coefficients of all the polynomials that appear are easily linearly bounded by n, so ring operations are cheap). The hardest part of the computation of ρ_1, inverting a matrix with entries that are affine linear in $T^{\pm 1}$, is standard and efficient and takes polynomial time (in n), though it's better not to commit to a specific bound because the bounds on the complexity of matrix operations are still improving.

ρ_1 is strong. Direct computations show that, at least on knots with up to 12 crossings, it has more separation power then the HOMFLY-PT polynomial and Khovanov homology taken together. Both are considered rather strong, and both are much harder to compute: the programs are longer, and they run in non-polynomial time. To the best of my knowledge, presently ρ_1 is the strongest knot invariant we know, both per line of code and per CPU cycle.

ρ_1 has a home in quantum algebra. Indeed, in [BV1, BV2, BV3] and in future publications, Roland van der Veen and I explain how the formulas (2) and (3) arise in a natural way from the quantization of a natural contraction of the lie algebra sl_2. Very roughly, up to a central factor, sl_2 is the double of its half, its upper Borel subalgebra $\mathfrak{b}$. If one takes $\mathfrak{b}$, scales its cobracket by ϵ and then doubles, one gets a new algebra sl_{2+}^{ϵ}, which is isomorphic to sl_2 (plus a central factor) if ϵ is invertible, yet is solvable when $\epsilon = 0$. The algebra sl_{2+}^{ϵ} can be quantized using the Drinfel'd double procedure, and its uni-

[1] Elementary is better than fancy and complicated. If you do something new that an undergraduate student can understand you contribute more than if you do something new that only graduate students can understand.

versal quantum knot invariant Z_ϵ may be considered. It turns out that all the tensors that appear within the study of sl^ϵ_{2+} are "perturbed Gaussians", and can be effectively computed using techniques reminiscent of the techniques used in perturbative quantum field theory, with ϵ as the perturbation parameter. (Can we make this lofty quantum algebra / QFT discussion a lot easier?)

Thus if we expand $Z_\epsilon = \sum_{d\geq 0} Z^{(d)}\epsilon^d$, then $Z^{(0)}$ is computable using pure Gaussian techniques[2]. It turns out that $Z^{(0)}$ reduces to the Alexander polynomial. The computation of $Z^{(1)}$ then involves minimal perturbation theory, and when the dust settles, ρ_1 emerges from it.

So can we say we understand ρ_1? Oh no. Something so simple as formulas (2) and (3), and so close to the Alexander polynomial with its purely topological definitions, ought to have a much simpler home, hopefully within the hamlet of topology down the street from Alexander's. What does this home look like?

ρ_1 has neighbors. From quantum algebra it follows that there are also ρ_d for $d > 1$ that arise in a similar manner from $Z^{(d)}$. Quantum algebra gives as recipes for computing ρ_d, and Roland van der Veen and myself have implemented them and computed them. They are stronger than ρ_1, but get progressively harder to compute. For ρ_2, the bottleneck remains inverting A (so it is still "easy"). For ρ_3 and beyond the bottleneck moves to perturbation theory. Each ρ_d remains polynomial time, but the exponents get bigger and bigger. It also follows from quantum algebra that there should be similar poly-time computable $\rho_{\mathfrak{g},d}$ for other semisimple Lie algebras $\mathfrak{g}$. How can we compute them? Do they have homes in topology?

I think it is unreasonable to believe that looking from topology into perturbations of the Alexander polynomial the theory of semisimple Lie algebras will naturally emerge. I believe the Lie algebras appear in the collection $\{\rho_{\mathfrak{g},d}\}$ because we are searching under the existing lamppost of quantum algebra. Once we find the right vintage point, it will become a different collection, $\rho_{\clubsuit,d}$, parameterized by some unknown moduli $\clubsuit$ which will be meaningful in topology. What is it? Will it merely be a change of basis, or will it be stronger then $\{\mathfrak{g}\}$, the collection of all semisimple Lie algebras?

ρ_1 is "homomorphic", meaning that it extends to tangles, and that its value on a given tangle $\mathcal{T}$ determines its value on any tangle obtained from $\mathcal{T}$ by strand doubling or by composition with other tangles[3]. I've been shouting for a long time now [BN], that being homomorphic may be a very valuable property. For indeed, certain classes of knots that carry great interest, such as ribbon knots and slice knots and knots of a given genus, are definable in terms of tangles and tangle compositions and strand doublings. Invariants that respect tangle operations, namely which are homomorphic, thus have a greater a priori chance of "saying something" about these classes of knots: giving genus bounds, or slice or ribbon obstructions. I believe the Alexander polynomial has this kind of topological applications precisely because it is homomorphic [BNS], and I believe the homomorphic properties of ρ_1 mean that it is much more likely to be of interest in classical low dimensional topology than almost anything else quantum algebra ever produced. Am I right?

ρ_1 is far from understood. There are the heavy questions as above, but even the light requires further work. There are plenty of other matrices like A, whose determinant computes the Alexander polynomial (arising from the Dehn presentation, from Seifert surfaces and forms, from braid closures or plat closures of braids and the Burau representation, from arc presentations, from w-knots, from strange formulas by Kashaev and Liu [Ka, Li], and more). Are there formulas for ρ_1 in terms of the inverses of each of these matrices? In particular, will the formulas coming from Seifert surfaces produce genus bounds and ribbon obstructions, as they do for the Alexander polynomial? Will the arc presentation formula speak with knot Floer homology, as its Alexander counterpart does?

The formulas we've presented here for ρ_1 are directly related to the lower Wirtinger presentation, and there are similar formulas coming from the upper Wirtinger presentation. Is there an elementary proof that these formulas agree? Is there an elementary proof that ρ_1 is palindromic (satisfies $\rho_1(T) = \rho_1(T^{-1})$)? If we're having difficulty already with that, we are clearly missing something. What is it?

ρ_1 extends to tangles without closed components. Is there a natural extension of ρ_1 to links and to tangles that are allowed to have closed components?

[2] Stricktly speaking, "two-step Gaussians", but that need not concern us here.

[3] Reshetikhin-Turaev invariants with fixed representations (namely, those that are computable, even if in exponential time), do not have the "strand doubling" part of this property. In particular, the Jones polynomial and the HOMFLY-PT polynomial do not have the "strand doubling" part of this property.

As for the other topics within my Notice of Intent (ωεβ/NOI), topic #1 was mentioned in passing within the above, and will not be mentioned further. For topics #3 and #4, I will simply repeat ωεβ/NOI with some modifications:

#3 Along with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich (ωεβ/PDS), I plan to continue to study knots and tangles in a "Pole Dancing Studio" (PDS, a cylinder with a few vertical lines removed) and their relationship with the Goldman-Turaev Lie bialgebra and Kashiwara-Vergne (KV) equations [AKKN1, AKKN2]. Are solutions of the KV equations sufficient to construct a homomorphic expansion of tangles in a PDS up to strand-strand degree 1? How is this related to my earlier work with Dancso [BD1, BD2] on welded knots? The subject is beautiful, yet it is a hard-to-penetrate patchwork of results and techniques and papers by different authors. In the past, this feeling that a subject's beauty is incongruous with its complexity had been a great motivator for me, often leading to deeper understanding. I have high hopes for this topic too.

#4 Recently (ωεβ/PQ), along with Jessica Liu, we've found a truly elegant "signatures for tangles" invariant (sorry for complimenting ourselves, yet hey, it really is elegant). There is more to do before we can claim to fully understand these signatures. Is there an Alexander invariant for tangles obtained using the same "pushforward" techniques? Are its roots related to the jumping points of the signature? Does it generalize to the multivariable case? Within the Notice of Intent I also had a question about proving the Kashaev signatures conjecture [Ka], but that conjecture is by now my student's Jessica Liu's theorem [Li].

Knot Theory as an Excuse **Discovery Grant Proposal Budget Justification**

Salaries and Benefits. Since 2017 I have graduated three PhD students (Travis Ens, Jesse Frohlich, Huan Vo). I am presently working with three more (Leonard Afeke, Jessica Liu, Daniel Martchenkov). I plan to support each of those at around $10,000 per year. In addition I've had a number of master's students, I expect to have about two more per year, and to support each at about $5,000 per year. Likewise I've taken a number of undergraduate "summer project" students, and I hope to support about two such students per year, at about $2,500 each.

I hope to be able to support a postdoctoral fellow throughout the grant period, at about $50,000 per year.

Equipment or Facility. Many of my past projects required massive computations, often running for months at a time (e.g., the calculation of all the invariants appearing on the Knot Atlas, http://katlas.org), and many of the results are made available by means of a dedicated web server, http://drorbn.net, especially http://drorbn.net/ap. My current proposal will lead me to continue using computers in a similar way. This will be a lot more effective if I would be able to purchase and maintain current hardware. Hence the $3,200 allocated per year for purchase or rental of computers and peripherals, and the $700 allocated per year for the maintenance of those. Also, I will have to pay user fees for some of the programs I will be using (Mathematica, for example) and also to some shared facilities to be provided by my university — internet connection, backup services, etc. I am requesting an amount of $2,000 per year for these purposes.

Materials and Supplies. This amount of $700 per year will be used primarily to purchase office supplies and printer paper and ink.

Travel. In the past I have traveled extensively and gave presentations on my work in a large number of domestic and foreign universities and in many international conferences. I presume this will continue throughout the years of my contract. In addition I hope to support some travel by my graduate students and postdoctoral fellows, and to support visits by my scientific collaborators to Toronto. I am requesting an amount of $9,000 per year for these purposes.

Books. Needs no explain.

Knot Theory as an Excuse

Discovery Grant Proposal References

[Al] J. W. Alexander, *Topological invariants of knots and links,* Trans. Amer. Math. Soc. **30** (1928) 275–306.

[AKKN1] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *The Goldman-Turaev Lie Bialgebra in Genus Zero and the Kashiwara-Vergne Problem,* Adv. Math. **326** (2018) 1–53, arXiv:1703.05813.

[AKKN2] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *Goldman-Turaev formality implies Kashiwara-Vergne,* Quant. Topol. **11-4** (2020) 657-–689, arXiv:1812.01159.

[BN] D. Bar-Natan, *Algebraic Knot Theory,* A talk given in Istanbul (2006), Jerusalem (2006), Uppsala (2006), Columbia U (2007), Buffalo (2007), Tokyo (2007), Aarhus (2007, 2013), Copenhagen (2008), Bogota (2009, 2011), Magnitogorsk (2014), Sydney U (2019), Macquarie U (2019), Melbourne (2019), Bonn (2020), Groningen (2020), and integrated into many other talks, lecture series, and classes. Video version: http://drorbn.net/syd2/.

[BD1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956.

[BD2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem,* Math. Ann. **367** (2017) 1517–1586, arXiv:1405.1955.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* Proc. Amer. Math. Soc. **147** (2019) 377–397, arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* arXiv:2109.02057.

[BV3] D. Bar-Natan and R. van der Veen, *A Perturbed-Alexander Invariant,* to appear in Quantum Topology, ωεβ/APAI, and arXiv:2206.12298.

[Jo] V. F. R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials,* Annals Math., **126** (1987) 335-388.

[Ka] R. Kashaev, *On Symmetric Matrices Associated with Oriented Link Diagrams,* in *Topology and Geometry, A Collection of Essays Dedicated to Vladimir G. Turaev,* EMS Press 2021, arXiv:1801.04632.

[Li] J. Liu, *A Proof of the Kashaev Conjecture for Signatures of Links,* in preparation.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Comm. Math. Phys. **169** (1995) 501–520.

[Oh] T. Ohtsuki, *On the 2–loop Polynomial of Knots,* Geom. Top. **11** (2007) 1357–1475.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* Ph.D. thesis, University of North Carolina, August 2013, ωεβ/Ov.

[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d Manifolds II*, Comm. Math. Phys. **175** (1996) 297–318, arXiv:hep-th/9403021.

[Ro3] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro4] L. Rozansky, *A Universal $U(1)$-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Wo] *Wolfram Language & System Documentation Center,* https://reference.wolfram.com/language/.

Knot Theory as an Excuse — Discovery Grant Proposal HQP Training Plan

My project clearly spreads in several directions. This means that there is ample room for advanced undergraduate students, for graduate students, and for postdoctoral fellows to take part in the research outlined in my proposal and/or in closely related research. This allowed me to participate in the training of many students and fellows in the past, and will continue to allow me to do the same in the future.

I share my significant use of computers as a tool for research, presentation and dissemination of knowledge with my students and postdoctoral fellows. I believe this adds major further quality to the training they receive.

Though frankly, I still don't know how to do the thing I'd really want to do.

For me, the best mathematics is the math that can be implemented on a computer. This ranges from the simplest, say Gaussian elimination or the Fibonacci sequence, and continues all the way to the fanciest and most abstract, be it a planar-algebra category-theory ultra-fast computation of Khovanov homology or a free-Lie-algebra meta-group-action-based computation of a non-commutative generalization of the Alexander polynomial or the implementation of the full portfolio of operations around the quantum universal enveloping algebras of solvable approximations of semisimple Lie algebras. I've implemented these, as well as a dozen other versions of the Alexander polynomial, and a dozen other knot invariants, and a very large number of other little things within knot theory, and a computer solution of the Rubik's cube, and a hyperbolic-geometry based algorithm for optimal camera motion, and I made computer generated pictures of various fancy links and surfaces and of steps within Arnold's resolution of Hilbert's 13th problem, and very many other things, big and small. (And most are on my web site).

For me, that's what keeps mathematics alive and sincere and believable (and when it comes to the graphics, sometimes also visually beautiful).

I wish I knew how to teach my students to actually compute (and draw) what they are talking about, and gain the benefit that that entails, and pass it on to their students later on. I wish they would do it routinely and often, and with joy. I think I've contributed some, and I hope to contribute further, to my students by sharing with them my love of the implementable (and teaching a bit of the how-to).

Knot Theory as an Excuse — Discovery Grant Relationship to Other Research Support

I am the lucky recipient of a C$221,000 grant from the Chu Family Foundation (NYC), used for a complete buy-out of my teaching and administrative duties in the academic years of 2022-23 and 2023-24, so I could concentrate on research. This grant cannot be used for any other purpose.

I am now applying for a further grant from the same source, which will allow me to bring to Toronto about 2 post-doctoral fellows for a period of about 3 years, to work on research related to the topics of this proposal. If approved, I expect that these funds would be restricted for use for this purpose only (hiring post-docs).

Knot Theory as an Excuse **Discovery Grant Most Significant Contributions**

A Perturbed-Alexander Invariant, with R. van der Veen, to appear in *Quantum Topology,* arXiv:2206.12298.
Abstract. In this note we give concise formulas, which lead to a simple and fast computer program that computes a powerful knot invariant. This invariant ρ_1 is not new, yet our formulas are by far the simplest and fastest: given a knot we write one of the standard matrices A whose determinant is its Alexander polynomial, yet instead of computing the determinant we consider a certain quadratic expression in the entries of A^{-1}. The proximity of our formulas to the Alexander polynomial suggest that they should have a topological explanation. This we don't have yet.

Perturbed Gaussian Generating Functions for Universal Knot Invariants, with R. van der Veen, arXiv:2109.02057.
Abstract. We introduce a new approach to universal quantum knot invariants that emphasizes generating functions instead of generators and relations. All the relevant generating functions are shown to be perturbed Gaussians of the form Pe^G, where G is quadratic and P is a suitably restricted "perturbation". After developing a calculus for such Gaussians in general we focus on the rank one invariant $Z_{\mathbb{D}}$ in detail. We discuss how it dominates the sl_2-colored Jones polynomials and relates to knot genus and Whitehead doubling. In addition to being a strong knot invariant that behaves well under natural operations on tangles $Z_{\mathbb{D}}$ is also computable in polynomial time in the crossing number of the knot. We provide a full implementation of the invariant and provide a table in an appendix.

Over then Under Tangles, with Z. Dancso and R. van der Veen, *Journal of Knot Theory and its Ramifications* **32-8** (2023), arXiv:2007.09828.
Abstract. Over-then-Under (OU) tangles are oriented tangles whose strands travel through all of their over crossings before any under crossings. In this paper we discuss the idea of *gliding*: an algorithm by which tangle diagrams could be brought to OU form. By analyzing cases in which the algorithm converges, we obtain a braid classification result, which we also extend to virtual braids, and provide a Mathematica implementation. We discuss other instances of successful "gliding ideas" in the literature – sometimes in disguise – such as the Drinfel'd double construction, Enriquez's work on quantization of Lie bialgebras, and Audoux and Meilhan's classification of welded homotopy links.

Handout Portfolio. I see lecturing and the assimilation of mathematical knowledge and the exposition of its beauty as one of my primary goals. I aim to polish my lectures to perfection; almost every lecture I give comes with a colourful handout summarizing the information in it, and with a web space with links to said handout, to relevant papers and programs, and almost always, with a link to a video recording of the talk itself. My 4th attached contribution is merely a reminder of that — an abridged version of my **Handout Portfolio** (the full version is at http://drorbn.net/hp).

Knot Theory as an Excuse **Discovery Grant 4 Samples of Research Contributions**

See ωεβ/APAI, ωεβ/PG, ωεβ/OU, and ωεβ/hp.

Knot Theory as an Excuse **Discovery Grant Proposal To Do List**

- Attach CCV.
- Reread instructions pages.
- Activate all the links.
- Clear this list and remove this page.