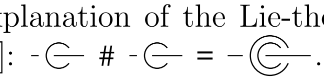
NSERC Research Proposal

Recent Progress.

Abstract Preliminaries. A *filtered vector space* is a vector space $V = V_0$ (usually infinite dimensional, usually one with elements whose study is hard yet desirable) along with a decreasing sequence of subspaces $V = V_0 \supset V_1 \supset V_2 \supset \dots$. I like to think of V as if it contains information that could be studied inductively, and of V_n as "that part of V that is irrelevant until day n of the study". Thus the quotient V/V_n is what we could or should have studied before day n. On day n we may study V_n, though we still don't care about V_{n+1}, so the quotient V_n/V_{n+1} is precisely what we need to study on that day. Thus the direct sum $\operatorname{gr} V := \bigoplus_n V_n/V_{n+1}$ (also known as "the *associated graded space of* V", or sometimes, "the *projectivization of* V") can be viewed as "V, sliced out for an inductive study". An *expansion* for V is a map $Z : V \to \operatorname{gr} V$ satisfying a simple non-degeneracy condition. An expansion may be thought of as a machine that breaks any element of V into a sequence of easier parts, with part n ready for study on day n. Often times the spaces V_n/V_{n+1} are much simpler than V, and so there is great interest in finding "good" expansions.

What's "good"? Often times V will come with various algebraic operations — various composition laws, or various other ways to transform its elements. Under mild conditions, these operations will themselves be amenable to inductive study — meaning that they induce similar operations on $\operatorname{gr} V$. A *homomorphic* ("good") expansion is an expansion $Z : V \to \operatorname{gr} V$ that intertwines the operations of V with the operations of $\operatorname{gr} V$. A homomorphic expansion allows for an even better inductive probing of V — not only the elements of V can be studied in the simplified inductive context, but so can the relations between the elements of V and the operations applied to these elements.

Generally speaking, mere expansions are cheap and easy to get. Homomorphic expansions, on the other hand, are expensive and valuable. They don't always exist and when they exist they are often hard to construct. Yet when they are available, they are often very useful.

Knots. The set $\mathcal{K}$ of knots ("**topology**", below), for example, can be made into a vector space by allowing formal linear combinations and can then be filtered using the "Vassiliev filtration" which will not be recalled here except by its essence, the formula "✕ → ✕ − ✕" (I was amongst the earliest contributors; see [BN1]). The resulting "$\operatorname{gr} V$" is the space $\mathcal{A}$ of chord diagrams ("**combinatorics**", below). Slightly generalizing to "parenthesized tangles", the construction of a homomorphic expansion Z turns out to depend mostly on the choice of one very special element $\Phi = Z(|\!/\!|)$ of $\mathcal{A}$, which has to satisfy some complicated equations whose origin is in category theory — mostly the "pentagon" and "hexagon" equations ("**high algebra**", below; I had a role in that too — see [BN2, BN3]). Further, it turns out that the space $\mathcal{A}$ can be re-interpreted as a space of formulas that make sense in any appropriate Lie algebra ("**low algebra**", below, [BN1]), and hence much that is done with and about Z and Φ has a Lie-theoretic and representation-theoretic meaning. A lovely example is the explanation of the Lie-theoretic Duflo isomorphism as the knot theoretic "$1 + 1 = 2$" [BLT]: .

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: \textbf{high algebra}}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{\textbf{low algebra}: pictures represent formulas}]{\text{given a ``Lie'' algebra } \mathfrak{g}} \text{``}\mathcal{U}(\mathfrak{g})\text{''}$$

$\mathcal{K}$ is some type of knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

The u, v, and w Stories. I found that the above pattern of **topology** leading to **combinatorics** by means of a natural filtration, leading to **low algebra** by interpreting the combinatorics as the combinatorics of formulas, and to **high algebra** by the study of expansions, persists for several other classes of knots. A quick summary is in the table below, in which "knots" are renamed to be "u-knots" ("u" for usual):

	u-Knots	v-Knots	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.

A more complete version of the above table would contain a few further rows, for quantum field theory, for configuration space integrals, for graph homology, and perhaps more. It may also contain some further columns (deformation quantization of Poisson structures "p" has entries in all rows except the topology row. I much want to know what "p-topology" would be). Also, the table fails to indicate that there are maps between the entities in the topology row — specifically, there are maps $u \to v \to w$. These maps have analogs or implications in all other rows, serving to explain the otherwise mysterious connections that exists between, say, Drinfel'd associators and solutions of the Kashiwara-Vergne problem [AT, BD3].

Thus my most significant scientific work over the last 5 years has been the assembly of the above table. Much of it is still unwritten. The written parts include:

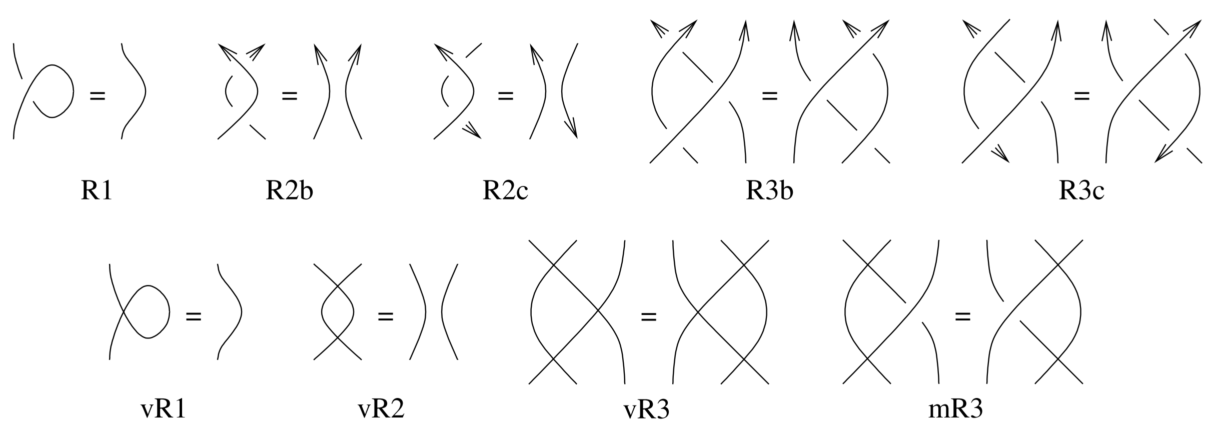
- My paper and series of videos [BD3] (with Dancso) contain the complete "w-story" from the topology of 2-dimensional knots in 4-dimensional space to the high algebra of Kashiwara-Vergne and Alekseev-Torossian and its relationship with Drinfel'd associators.

- [BD2] is about the u-column, describing the relationship between knotted trivalent graphs and associators.
- [BD1] lives at the intersection of u and high algebra, explaining (following Furusho) why under certain conditions, the pentagon equation implies the hexagon equation.
- [BHLR] contains strong computational evidence that some 18 variants of the combinatorics entry of the v column are "correct".

Many of the not-yet-written parts are available online as videotaped lectures accompanied by detailed summary handouts. With ω :=`http://www.math.toronto.edu/~drorbn/Talks/`, these include ω/Aarhus-0706 on potential applications to topology; ω/Trieste-0905, ω/Goettingen-1004, and ω/Caen-1206 on the whole table; ω/MSRI-0808, ω/Bonn-0908, and ω/Montpellier-1006 on the w column; ω/Toronto-11011 on the relationship between u and the Grothendieck-Teichmuller group; ω/Tennessee-1103 on the "configuration spaces" row that was omitted above; ω/SwissKnots-1105 on my hopes for the v column; ω/Sandbjerg-0810 on a "penultimate" Alexander invariant; ω/Chicago-1009 on the relationship between w and especially the "$ax+b$" Lie algebra and the Alexander polynomial; ω/Regina-1206 on an Alexander polynomial offshoot of the w story, which appears to be an "ultimate" Alexander invariant; and ω/Hamburg-1208 on a non-commutative generalization of the Alexander polynomial and its relationship with the BF quantum field theory.

Objectives. My primary objective over the next five years will be the continued study of the table above, especially in the w and v columns. I feel that the w column is nearly completed, though much remains to be written. In the v column much more remains to be done.

Most Important: The v-Column. What is clear is that virtual knotted objects[1] [Ka], $\mathcal{K}^v$, like knots, are made of "crossings" like ⤫ and ⤬ and of "virtual crossing" that look like ⨯, and that these are reduced modulo some relations, more or less the Reidemeister moves and the virtual Reidemeister moves shown here:



It is also clear that on the combinatorial, or "gr $\mathcal{K}^v$" level, the above relations lead to certain diagrammatic relations between "arrow diagrams" [Po, Hav, BHLR], which in themselves are the diagrammatic counterparts of the relations defining "Lie bialgebras" [Dr1].

[1]Topologically, virtual knots are knots drawn on surfaces modulo "stable equivalence", and algebraically they are "those most general things quantum knot invariants make sense on".

It is further clear, that the objects and equations needed in order to define a homomorphic expansion for virtual knotted objects are very closely related to objects that occur within the study of deformation quantization of Lie bialgebras [EK], hence within the foundations of quantum group theory: that Z(⤫) corresponds to a quantum Yang-Baxter element R, that the Reidemeister relations become the Yang-Baxter equation, and that if the study of $\mathcal{K}^v$ is extended to include virtually knotted graphs, then the trivalent vertex becomes related to "Drinfel'd twists" and the tetrahedral graph to associators.

Thus there is strong evidence that the "topology, combinatorics, low algebra, high algebra" story of the v-column of the table on page 2 is correct.

Yet there are still mismatches. There can be many competing definitions for virtual knots and it is not clear which one should work, or work best (see [BHLR] — do we use both b and c -type Reidemeister moves or only b-type? Do we include R1 or not? What is the right definition of framed v-knots? Etc.). Mismatches occur on the combinatorial level as well — some of the relations in $\operatorname{gr}\mathcal{K}^v$ do not have a Lie bi-algebra interpretation (most notably the XII relation of [BHLR]), it is not clear if "cyclic" arrow diagrams are to be allowed or not, etc. Similar mismatches and inconsistencies further arise when trivalent vertices are inserted into the picture. Thus much work remains.

It may be tempting to say "Dror just can't get his act together and sort out the little normalizations that he is getting wrong". But Dror's not a complete fool, and he worked hard and long on making things match. What's really happening is that **we don't really understand the topology that underlies the theory of quantum groups.** We know it is near virtual knot theory, but we don't know exactly where, and "near" is not good enough. My aim is to fix this.

Secondary project, yet promising. Recently (see [BN4]), by studying the w-column and especially by studying "knotted balloons and hoops in $\mathbb{R}^4$", I found that the (multi-variable) Alexander polynomial for tangles has an intriguing non-commutative extension ζ. After the change of variables $T_i \to e^{t_i}$, instead of taking values in the ring of power series in commuting variables t_i, it can be lifted to take values in formal linear combinations of cyclic words in non-commuting letters u_i. I'm still not sure what this means. While up to about half-way into the construction of ζ everything appears group-theoretical, I still do not understand the introduction of cyclic words in a group theoretic language; it ought to mean something! Finally, ζ appears to be a complete evaluation of the BF quantum field theory [CR] in the "ribbon" case. However this is not proven, and it is not known if and how ζ may be extended in the non-ribbon case. I plan to study all these questions.

Furthermore, when the above invariant ζ is reduced by declaring all the u_i to be commuting, the result β is in several ways much better than the original Alexander polynomial A — it is a highly-computable extension of the Alexander polynomial to tangles which takes values in a space of polynomial size (unlike other skein- or quantum-extensions of A, which are valued in spaces whose size grows exponentially in the number of components of the tangle involved). Also, every step within the computation of β is the invariant of some topological entity — this is unlike standard evaluations of A, in which a determinant must be computed by matrix manipulations that have no topological meaning. Finally, β is multiplicative (in the appropriate sense) under tangle composition and it has other "homomorphic" properties relative to several other tangle operations. I plan to study β further; especially, I plan to investigate whether there exists a knot homology theory generalizing β. Such a theory may share the excellent efficiency and composition properties that β has, and may thus be much more manageable than existing Alexander homologies.

Literature Review. I have little to add beyond the references cited above. The u column of the table on page 2 is "classical" and was mostly understood already around 1995 by myself [BN1, BN2, BN3], by Le-Murakami-Murakami-Ohtsuki [LM, LMMO, LMO] and by others (though the relationship between associators and knotted trivalent graphs is now better understood; see [CL, BD2]). By and large, the v and w columns of that table were assembled by myself over the last 5-6 years and are described in [BD3, BN4, BHLR] and in a large number of well-documented research talks (`http://drorbn.net/Talks/`), though several pieces of that story were studied before [AT, AET, HKS, HS, Hav, Ka, Po, Sa].

Methodology. The standard mathematical methodology is to stare for a long time at pieces of paper and think hard. Mine's the same, though in addition,

- My thinking is public. I believe in open science and my blackboard and even my personal handwritten notebook are open to be shared with my students and other researchers. See my "Academic Pensieve" at `http://drorbn.net/AcademicPensieve/`.
- I think with computers. Almost every bit of math that I do is computable and is immediately implemented. In fact, more often than not the implementation is a major part of doing and understanding the math, for me. All the programs I write are available as they are being written on my Academic Pensieve or elsewhere on my web site.
- Presentation is always a part of my thinking. See the large number of videos and colourful "handouts" on my "Talks" page, at `http://drorbn.net/Talks/`.

Impact. My main project on the v-column, if successful, will radically change the way deformation quantization is viewed, from a theory of deformations of certain algebraic objects, to a theory of "homomorphic expansions" of certain topological objects. I believe it will lead to a radical re-interpretation of the theory of quantum groups. My secondary project may lead to a wide-ranging extension of the most successful knot invariant, the Alexander polynomial, in several directions. In addition, I hope to continue to impact how math is done, presented, and disseminated, by being and remaining at the head of the technological curve.