Pensieve header: Testing the associativity of compositions in GDO. Based on GenericDoPeGDO.nb in pensieve://Talks/DaNang-1905/.

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We test that the composition law of \gdo\ is indeed associative, by defining it general and verifying associativity on random (and hence likely generic) morphisms. First, we define the composition law of two morphisms. The program first determines E_i, F_i, and G_i from Q_i ($i=1,2$) by taking partial derivatives, and then outputs the scalar ω and quadratic Q, with equations~\eqref{eq:gdocompositions} converted nearly literally into code (see also \cite[GDOCompositions.nb]{Self}):

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```
In[•]:= 𝕄A_→B_[ω1_, Q1_] // 𝕄B_→C_[ω2_, Q2_] := Module[{ζA, zC, E1, F1, G1, E2, F2, G2, I},
   ζA = Table[ζi, {i, A}]; zC = Table[zi, {i, C}]; I = IdentityMatrix@Length@B;
   E1 = Table[∂ζi,zj Q1, {i, A}, {j, B}]; E2 = Table[∂ζi,zj Q2, {i, B}, {j, C}];
   F1 = Table[∂ζi,ζj Q1, {i, A}, {j, A}]; F2 = Table[∂ζi,ζj Q2, {i, B}, {j, B}];
   G1 = Table[∂zi,zj Q1, {i, B}, {j, B}]; G2 = Table[∂zi,zj Q2, {i, C}, {j, C}];
   Expand /@ 𝕄A→C[ω1 ω2 Det[I - F2.G1]^(-1/2), ζA.E1.Inverse[I - F2.G1].E2.zC
      + 1/2 ζA.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).ζA +
      1/2 zC.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).zC]]
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Next we implement ``random morphisms'' (\verb"RM") by picking their quadratic parts to have small random integer coefficients. We also set M_1, M_2, and M_3 to be random morphisms in $\mor(\{1,2\}\to\{1,2,3\})$, $\mor(\{1,2,3\}\to\{1,2,3\})$, and $\mor(\{1,2,3\}\to\{1,2\})$, respectively:

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```
In[•]:= R𝕄A_→B_ := Module[{vs = Table[ζi, {i, A}] ⋃ Table[zi, {i, B}]},
     𝕄A→B[1, Sum[RandomInteger[{-3, 3}] vi vj, {vi, vs}, {vj, vs}]]];
   {M1 = R𝕄{1,2}→{1,2,3}, M2 = R𝕄{1,2,3}→{1,2,3}, M3 = R𝕄{1,2,3}→{1,2}} // Column
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Out[•]=

$$\mathbb{M}_{\{1,2\}\to\{1,2,3\}}\left[1,\ -z_1^2+4 z_1 z_2+z_2^2-z_1 z_3-2 z_2 z_3+2 z_3^2+4 z_1 \zeta_1+3 z_2 \zeta_1+2 z_3 \zeta_1+6 z_1 \zeta_2+z_2 \zeta_2+5 z_3 \zeta_2-2 \zeta_1 \zeta_2-\zeta_2^2\right]$$

$$\mathbb{M}_{\{1,2,3\}\to\{1,2,3\}}\left[1,\ z_1 z_2+3 z_2^2-z_1 z_3+5 z_2 z_3-z_3^2+2 z_1 \zeta_1-2 z_2 \zeta_1+\zeta_1^2-5 z_1 \zeta_2+3 z_2 \zeta_2+5 z_3 \zeta_2-3 \zeta_1 \zeta_2+2 \zeta_2^2-5 z_1 \zeta_3-2 z_2 \zeta_3-4 z_3 \zeta_3-\zeta_1 \zeta_3-2 \zeta_2 \zeta_3+\zeta_3^2\right]$$

$$\mathbb{M}_{\{1,2,3\}\to\{1,2\}}\left[1,\ -z_1^2+4 z_1 z_2-3 z_2^2+5 z_1 \zeta_1-z_2 \zeta_1+2 \zeta_1^2+2 z_1 \zeta_2-4 z_2 \zeta_2+2 \zeta_1 \zeta_2+\zeta_2^2+4 z_1 \zeta_3-z_2 \zeta_3+\zeta_1 \zeta_3+2 \zeta_3^2\right]$$

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Just to get an appreciation of what compositions look like, we compute $(M_1\act M_2)\act M_3$:

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In[ ]:= `(M1 // M2) // M3`

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Out[ ]= $$\mathbb{M}_{\{1,2\}\to\{1,2\}}\left[-\frac{1}{2\sqrt{655102}},\ -\frac{6526189\, z_1^2}{1310204}+\frac{4887535\, z_1\, z_2}{655102}-\frac{3883913\, z_2^2}{1310204}-\frac{258319\, z_1\, \zeta_1}{327551}-\frac{2762891\, z_2\, \zeta_1}{327551}-\frac{8260873\, \zeta_1^2}{2620408}-\frac{73313\, z_1\, \zeta_2}{93586}-\frac{867195\, z_2\, \zeta_2}{93586}-\frac{467207\, \zeta_1\, \zeta_2}{46793}-\frac{1189699\, \zeta_2^2}{187172}\right]$$

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Finally, we verify that composition is associative:

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In[ ]:= `((M1 // M2) // M3) == (M1 // (M2 // M3))`

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Out[ ]= `True`

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The last \verb”True” above is an in-practice proof of Theorem~\ref{thm:GDO},~(i).