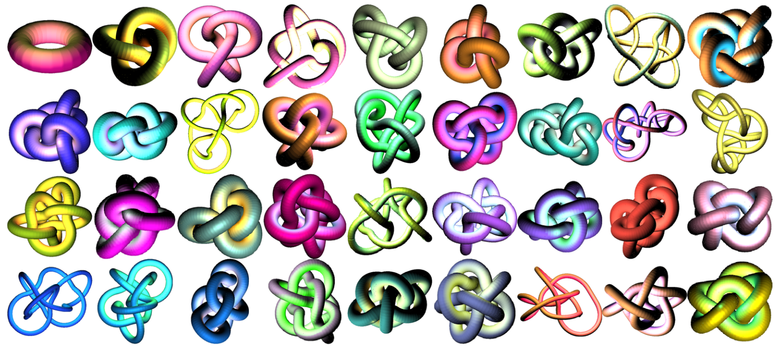
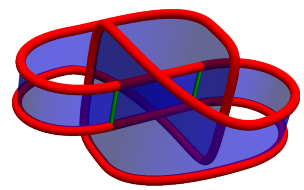
Quality Knot Invariants — A Non-Technical Summary of My Research Program



Knots are things like in the picture on the right (which I made in 2008). There are a lot of them! The picture shows the first 32, and all of those can be drawn in the plane as circles with at most 8 intersections. If we allowed ourself to double the complexity and draw circles with up to 16 intersections, the number of knots grows to over 1.7M. It is sometimes very difficult to tell these knots apart! In fact, even if we just look at the first two knots in this picture, are we really confident that they are different? Could it be that one can be deformed into the other without cutting or self-crossing? The key to answering these questions is the construction of "knot invariants" — numerical or symbolic quantities that are constructed from a picture of a knot and that aid in telling knots apart. Perhaps the most famous knot invariant is the "Jones polynomial", which earned Vaughan Jones (1952–2020) the Fields medal in 1990.

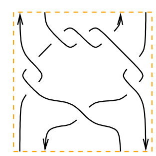


Beyond their knottedness, some knots may or may not have extra properties, like "being the boundary of a surface with certain allowed self-intersections" (picture on right, made 2017), or "being the boundary of a 2-dimensional disk in a 4-dimensional ball". Via "surgery", knots can be used to describe various alternative 3-dimensional and 4-dimensional spaces: roughly, you start with the standard Euclidean space, take a knotted tube out, put it back in in a different way, and you get an alternative 3D or 4D space. Thus it turns out that studying knots and their properties is a key to all of low dimensional topology. Sometimes knot invariants shed light on these extra properties that low dimensional topologists care about.

My own research is about finding *quality* invariants. More precisely,

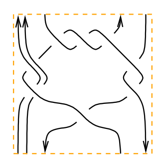
I seek strong, poly-time-computable, homomorphic knot and tangle invariants.

Let me quickly explain what these words mean:



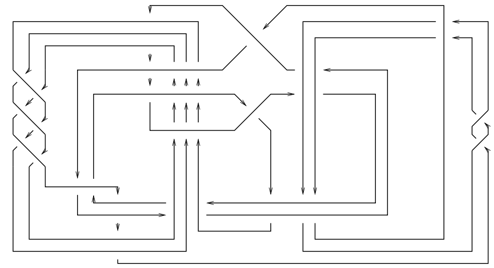
Knots and **invariants** I have already explained. **Tangles** are *incomplete knots*, or *knots with ends*. An example is on the right. Tangles are useful as an intermediate step towards studying knots.

Strong means "having high power to separate knots". There are millions of knots, and only the better invariants come close to telling each from each other.



Homomorphic is a bit hard to explain. There are certain operations one can do on knots, and more interestingly, on tangles. Two tangles can be placed side by side and connected together to make a bigger tangle, for example. Another example is the *strand doubling* operation: one may pick one of the components of a tangle, say the left-most component in the tangle from the previous picture, and *double* it to produce the tangle now shown on the right. A *homomorphic* invariant is an invariant that has a prescribed behaviour when knot and tangle operations are performed. Many interesting properties of knots and tangles can be described using these operations[1], and homomorphic invariants stand a chance to say something about these properties.

[1] Compare with, "a real number is non-negative if and only if it is the square of another real number". So positivity can be described once multiplication (and hence squaring) is understood.



Poly-time-computable should remind us of the famed $P = NP$ problem of computer science, which asks whether hard computations ("*NP*") can be reduced to what computers can compute in a reasonable amount of time ("*P*"). The *P* here stands for poly-time, and the invariants I seek should be computable in poly-time so they would be computable even on reasonably large knots. Many problems in low dimensional topology are related to problems on reasonably large knots, such as the 48-crossing knot on the right, which was drawn in 2010 by Gompf, Scharlemann, and Thompson.

Q. *Are there results already? Do "quality invariants" at all exist?* Yes. Along with Roland van der Veen we found some, and we know how to find more (see [BN1, BV]). There has to be a technical bit, so here it is: The fundamental operations of tangle theory (strand concatenation, doubling, and reversal) correspond to the fundamental operations in Hopf algebras (the multiplication m, the co-multiplication Δ, and the antipode S). But even if a Hopf algebra is finite dimensional, its tensor powers, where computations actually happen, have dimensions that rise exponentially and hence naively Hopf algebras do not yield poly-time computable invariants. We find that if one quantizes a variant of the Inonu-Wigner contraction of sl_2 along its lower Borel subalgebra one obtains an infinite-dimensional Hopf algebra whose fundamental tensors (m, Δ, S, and the crossing value R) can be described as "docile perturbed Gaussians". We then find that docile perturbed Gaussians can be composed using Feynman-diagrams techniques inspired by quantum field theory, and, oddly enough, these compositions can be computed in polynomial time, leading to poly-time computable knot invariants.

Q. *If it's already done, what's left to do?* It isn't fully done. We built a sophisticated machine, but we still have to document it. With luck, the work already done will turn into a very long and detailed article, or into a book, within my next two years of concentrated research.

Q. *There is promise, but are there already applications to topology?* More about the promise is e.g. at [BN2]. Yet largely, the applications aren't there yet. The subject is too new, and unfortunately, also too complicated. Roland van der Veen and I know how to simplify many parts, and we have leads on how to apply this to questions like the slice-ribbon conjecture of knot theory. Much remains to be done and we hope to make progress within the next two years of concentrated research.

Q. *How original is all this?* A few poly-time-computable knot polynomials were known before our work, but they are rather weak. Homomorphic invariants existed before, but were nearly impossible to compute and nobody thought to put their homomorphic properties to good use. Lev Rozansky considered the very same invariants that we consider, though using different techniques and without noting neither their poly-time nor their homomorphic properties.

Q. *Is this all that you do?* No; it's just my primary line. I'm also involved with several other projects with several other collaborators. With Zsuzsanna Dancso we study "w-knots" [BND]. With Itai Bar-Natan, Nancy Scherich, and Iva Halacheva we study "yarn ball knots" [BBHS]. I care for knot signatures and have ideas of things to say about them (e.g. [BN3]). I hope to continue working on these projects and few others that I have not listed over the next two years of concentrated research.

Q. *Is this really why you care about knot theory?* No. Much as number theory is a motivator and an excuse to study a wide range of other topics, from analysis through automorphic forms to arithmetic algebraic geometry, knot theory is really a motivator and an excuse to study a slew other things, including (but not limited to) algebraic topology, homological algebra, hyperbolic geometry, differential geometry, and quantum field theory. Personally, I feel that we don't understand the theory of quantum groups well enough and especially not how this theory arises within low dimensional topology. It is my dream to understand quantum groups better, one day. I hope to get at least a bit closer to this goal within my next two years of concentrated research.

References

[BN1] *Everything around sl_{2+}^{ϵ} is DoPeGDO. So what?*, talk given in Da Nang in 2019, ωεβ/DaNang.

[BN2] *Algebraic Knot Theory*, talk given in Sydney in 2019, ωεβ/Sydney.

[BN3] *Kashaev's Signature Conjecture*, talk given in a virtual CMS meeting in 2021, ωεβ/CMS.

[BBHS] With Itai Bar-Natan, Iva Halacheva, and Nancy Scherich, *Yarn Ball Knots and Faster Computations,* arXiv:2108.10923.

[BND] With Zsuzsanna Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956. *II: Tangles and the Kashiwara-Vergne Problem,* Math. Ann. **367** (2017) 1517–1586, arXiv:1405.1955. *III: Double Tree Construction,* in preparation.

[BV] With Roland van der Veen, *Perturbed Gaussian Generating Functions for Universal Knot Invariants,* submitted and arXiv:2109.02057.